A Coalgebraic View of ε-Transitions

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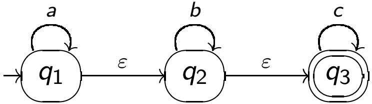
Refresher on ε-Transitions

Non-deterministic automaton for $a^*b^*c^*$:



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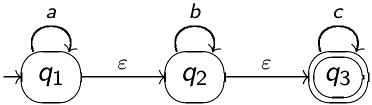

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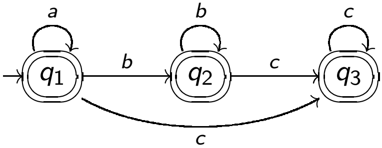

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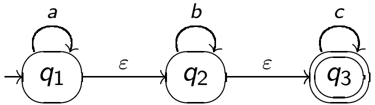

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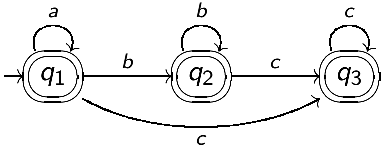

Can we treat ε-elimination $\alpha \mapsto \alpha^{\#}$ coalgebraically?

Outline

1. ε-Elimination for NDAs
2. Weighted Automata

Definition of NDA (with ε-Transitions)

NDA:

$$X \longrightarrow \wp(\, A \times X + 1\,)$$

NDA with ε-transitions:

$$X \longrightarrow \wp(\,(A + \{\varepsilon\})\, \times X + 1\,)$$

Semantics of NDAs

q **accepts** $w \equiv a_1 a_2 \cdots a_N$ if

$$q \xrightarrow{a_1} q_2 \xrightarrow{a_2} \cdots \xrightarrow{a_N} q_{N+1} \quad \text{and} \quad q_{N+1} \text{ is final}$$

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ε-Elimination for NDAs

Sought: NDA $\alpha^{\#}$ with

$$[\![-]\!]_{\alpha^{\#}} \;=\; [\![-]\!]^{\varepsilon}_{\alpha}$$

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Solution:

$$q \xrightarrow{a}_{\alpha^{\#}} r \qquad \Longleftrightarrow \qquad q \overset{\varepsilon}{\twoheadrightarrow} q' \xrightarrow{a}_{\alpha} r$$

$$q \text{ is final in } \alpha^{\#} \qquad \Longleftrightarrow \qquad q \overset{\varepsilon}{\twoheadrightarrow} q' \;\text{ and }\; q' \text{ is final in } \alpha$$

Definition of Trace

$$\mathrm{tr}_\alpha \colon X \longrightarrow \wp(\,\mathbb{N} \times (A \times X + 1)\,)$$

$$(n,(a,r)) \in \mathrm{tr}_\alpha(q) \quad\iff\quad q \xrightarrow{\varepsilon}_n q' \xrightarrow{a} r$$

$$(n,*) \in \mathrm{tr}_\alpha(q) \quad\iff\quad q \xrightarrow{\varepsilon}_n q' \;\text{ and }\; q' \text{ is final}$$

tr_α Versus $\alpha^\#$

$$b \in \alpha^\#(q) \quad \Longleftrightarrow \quad \exists n \in \mathbb{N}[\ (n, b) \in \mathrm{tr}_\alpha(q)\ ].$$

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$$b \in \alpha^\#(q) \quad \iff \quad \exists n \in \mathbb{N}[\ (n,b) \in \mathsf{tr}_\alpha(q)\ ].$$

More categorically, writing $B = A \times X + 1$

$$X \xrightarrow{\mathsf{tr}_\alpha} \wp(\,\mathbb{N} \cdot B\,) \xrightarrow{\wp(\nabla)} \wp(B), \qquad X \xrightarrow{\alpha^\#} \wp(B)$$

$\nabla \colon \mathbb{N} \cdot B \to B$ is the **codiagonal**: $\nabla(n,b) = b$.

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Recursive Description of tr_α

$$(0, b) \in \mathrm{tr}_\alpha(q) \iff b \in \alpha(q)$$
$$(n+1, b) \in \mathrm{tr}_\alpha(q) \iff \exists r \in X \left[\ q \xrightarrow{\varepsilon} r \ \wedge \ (n, b) \in \mathrm{tr}_\alpha(r)\ \right]$$

Recall, writing $B := A \times X + 1$:

$$\mathrm{tr}_\alpha \colon X \longrightarrow \wp(\mathbb{N} \cdot B)$$

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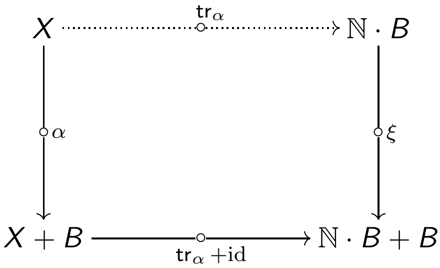
Recall, writing $B := A \times X + 1$:

$$\mathrm{tr}_\alpha \colon X \longrightarrow \wp(\mathbb{N} \cdot B)$$

In $\mathcal{K}\ell(\wp)$ — the Kleisli category of $\wp$:

$$\mathrm{tr}_\alpha \colon X \multimap \mathbb{N} \cdot B.$$

Universal Property of tr_α



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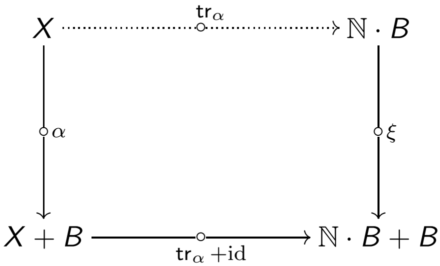

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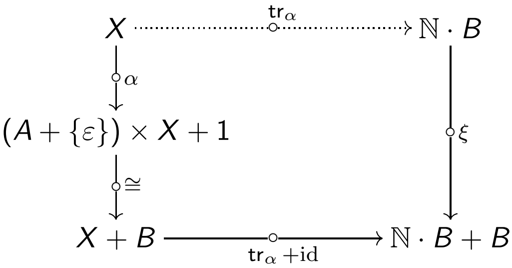

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Overview of ε-Elimination for NDAs

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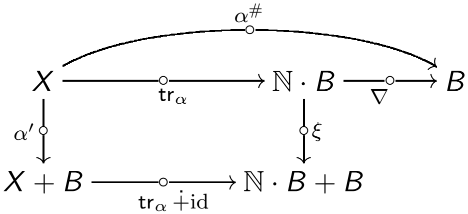

Here α is a NDA with ε-transitions with states X and alphabet A and

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Outline

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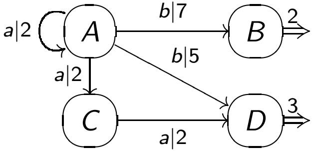
$\mathscr{M}$ is the **multiset monad**

$$\mathscr{M}(X) \;=\; \{\,\varphi\colon X \to S \mid \operatorname{supp}\varphi \text{ is finite}\,\}.$$

S is a semiring (such as $\mathbb{N}$ or $\mathbb{R}$)

Example of a Weighted Automaton



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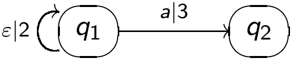

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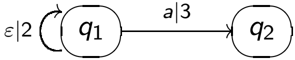

It is a map $\alpha\colon X \longrightarrow \mathscr{M}(\,(A + \{\varepsilon\}) \times X + 1\,)$.

$$\alpha(q_1)(a, q_2) \;=\; 3.$$

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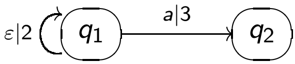

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$$\mathrm{tr}_\alpha(q_1)(n, (a, q_2)) \;=\; 2^n \cdot 3.$$

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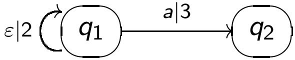

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$\mathrm{tr}_\alpha(q_1) \in \mathscr{M}(\cdots)$, but the support of $\mathrm{tr}_\alpha(q_1)$ is not finite!

The Monad $\mathscr{M}$

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To define $\mathscr{M}$, we need countable sums on our semiring S !

σ-Semirings

For **every** sequence $a_1, a_2, \ldots$ in S we should have a sum

$$\sum_{n=0}^{\infty} a_n$$

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Surprisingly, this entails S is **positive**:

$$a + b = 0 \implies a = 0 \text{ and } b = 0.$$

σ-Semirings are Positive

If $a + b = 0$, then

$$\begin{matrix} b & 0 & 0 & 0 & \cdots \\ a & b & 0 & 0 & \cdots \\ 0 & a & b & 0 & \cdots \\ 0 & 0 & a & b & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{matrix}$$

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Similarly, $a = 0$ **Conclusion**: $\mathbb{R}$ cannot be (part of) a σ-semiring

Problems with ε-Transitions in Weighted Automata over $\mathbb{R}$

Writing $\square$ for the empty word we have

$$[\![q_1]\!]^{\varepsilon}(\square) \;=$$

$$[\![q_2]\!]^{\varepsilon}(\square) \;=$$

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Writing $\Box$ for the empty word we have

$$[\![q_1]\!]^\varepsilon(\Box) \;=\; 1 + 0.5 + (0.5)^2 + \cdots \;=\; 2$$

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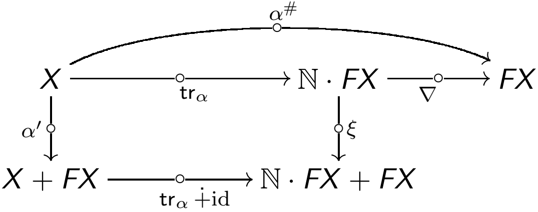
Overview of ε-Elimination for Weighted Automata over a σ-Semiring

In $\mathcal{K}\ell(\mathscr{M})$:



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Closing Remarks

We have seen:

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and also:

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Thank you!

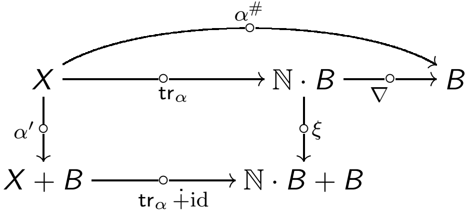
Questions?

Outline

3. Framework

Recap

We had the following diagram in $\mathcal{K}\ell(\wp)$.



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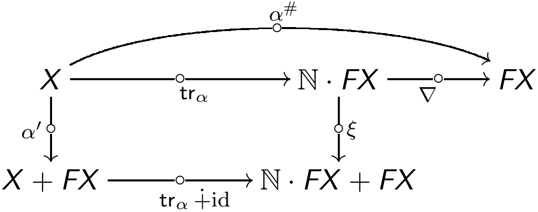

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Automata in General

Let **C** be a category which has countable colimits.
Let F be a functor on **C**.
Let T be a monad on **C** with Kleisli category $\mathcal{K}\ell(T)$.
Assume F is lifted to a functor $\overline{F}$ on $\mathcal{K}\ell(T)$ via a distributive law

$$\lambda\colon FT \longrightarrow TF.$$

Definition

An **automaton** *of type T, F is a morphism in $\mathcal{K}\ell(T)$ of the form*

$$\alpha\colon X \multimap\!\!\!\rightarrow \overline{F}X.$$

An ε-**automaton** of type T, F is a morphism of the form

$$\alpha\colon X \multimap\!\!\!\rightarrow X + \overline{F}X.$$

Automata in General

Assume that $\mathcal{K}\ell(T)$ has a final $\overline{F}$-coalgebra

$$\omega\colon \Omega \multimap F\Omega.$$

Writing $\overline{F}_\varepsilon := - + \overline{F}$, assume that $\mathcal{K}\ell(T)$ has a final $\overline{F}_\varepsilon$-coalgebra

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$$\omega_\varepsilon\colon \Omega_\varepsilon \multimap\!\!\longrightarrow F\Omega_\varepsilon.$$

Definition

Let $\alpha\colon X \multimap\!\!\longrightarrow FX$ be a automaton of type T, F. The **semantics** *of α is the unique homomorphism $[\![-]\!]_\alpha\colon X \multimap\!\!\longrightarrow \Omega$ from α to ω.*

Automata in General

Let B be an object of **C** Note that the morphism in $\mathcal{K}\ell(T)$

$$\iota_B \colon \mathbb{N} \cdot B + B \longrightarrow\!\!\circ\!\!\longrightarrow \mathbb{N} \cdot B$$

given by $\iota_B = [[\kappa_{n+1}]_{n \in \mathbb{N}}, \kappa_0]$ is the initial $- + B$-algebra.

Assume that $\xi_B := \iota_B^{-1}$ is the final $- + B$-coalgebra in $\mathcal{K}\ell(T)$.

Automata in General

Definition

Let $\alpha\colon X \multimap X + \overline{F}X$ be a ε-automaton of type T, F.

1. *The* **trace** *of α is the unique $- + \overline{F}X$-homomorphism*

$$\mathrm{tr}_\alpha\colon X \multimap \mathbb{N}\cdot\overline{F}X.$$

from α to the final $- + \overline{F}X$-coalgebra $\xi_{\overline{F}X}$.

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2. *The* **iterate** *of α is the map*

$$\alpha^{\#} \colon X \multimap \overline{F}X$$

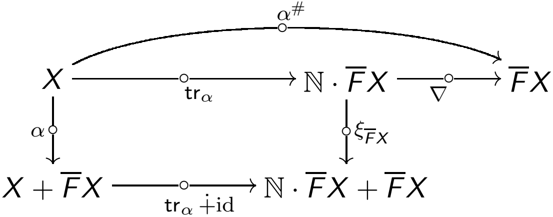
given by $\alpha^{\#} = \nabla \circ \mathrm{tr}_\alpha$ (composition in $\mathcal{K}\ell(R)$), where

$$\nabla \colon \mathbb{N} \cdot \overline{F}X \multimap \overline{F}X$$

is the **codiagonal** *given by $\nabla = [\mathrm{id}_{\overline{F}X}]_{n \in \mathbb{N}}$.*

Overview

We have the following diagram in $\mathcal{K}\ell(T)$.



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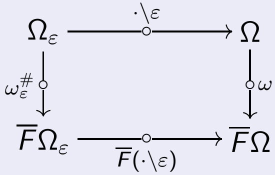
Semantics of ε-Automata

Recall that for an ε-NDA the semantics $[\![-]\!]^{\varepsilon}_{\alpha}$ is given by

$$[\![q]\!]^{\varepsilon}_{\alpha} \;=\; \{\, \tilde{w}\backslash\varepsilon \colon\; \tilde{w} \in [\![q]\!]_{\alpha} \,\} \;=\; \wp(-\backslash\varepsilon)([\![q]\!]_{\alpha}).$$

Definition

Let $-\backslash\varepsilon$ be the unique morphism in $\mathbf{C}$ *such that*



commutes. That is, $-\backslash\varepsilon = [\![-]\!]_{\omega_\varepsilon^{\#}}$.

Semantics of ε-Automata

Definition

Let $\alpha\colon X \longrightarrow\!\!\!\!\circ\; X + \overline{F}X$ *be an* ε*-automaton of type* T, F*.*
The **semantics** *of* α *is the map*

$$[\![-]\!]^{\varepsilon}_{\alpha}\colon X \multimap \Omega$$

given by $[\![-]\!]^{\varepsilon}_{\alpha} = -\backslash\varepsilon \circ [\![-]\!]_{\alpha}$ *(composition in* $\mathcal{K}\ell(T)$*).*

Semantics of ε-Automata

Theorem

Let $\alpha\colon X \longrightarrow\!\!\circ\!\!\longrightarrow X + \overline{F}X$ be an ε-automaton of type T, F. Then the semantics of α and $\alpha^{\#}$ coincide:

$$[\![-]\!]^{\varepsilon}_{\alpha} = [\![-]\!]_{\alpha^{\#}}.$$