A universal property for sequential measurement

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We study the *sequential product*[GN01,GG02,GG05], the operation $p * q = \sqrt{p}q\sqrt{p}$ on the set of effects, $[0,1]_{\mathscr{A}}$, of a von Neumann algebra $\mathscr{A}$ that represents sequential measurement of first p and then q. In[GL08] Gudder and Latémolière give a list of axioms based on physical grounds that completely determines the sequential product on a von Neumann algebra of type I, that is, a von Neumann algebra $\mathscr{B}(\mathscr{H})$ of all bounded operators on some Hilbert space $\mathscr{H}$. In this paper we give a list of axioms that completely determines the sequential product on all von Neumann algebras simultaneously, see Thm 4.

These axioms may be formulated in purely categorical terms (although we do not pursue this here, see also Remark 12). In this way this paper contributes to the larger program[Jac15,CJWW15b,CJWW15a] to identify structure in the category of von Neumann algebras with completely positive normal linear contractions to interpret the constructs in a programming language designed for a quantum computer: with the sequential product one can interpret measurement.[CJWW15b,CJWW15a]

Our axioms for the sequential product are based on the following observations. Given a von Neumann algebra $\mathscr{A}$ and $p \in [0,1]_{\mathscr{A}}$ the expression $\sqrt{p}a\sqrt{p}$ makes sense for all $a \in \mathscr{A}$ (and not only for $a \in [0,1]_{\mathscr{A}}$). The resulting map $\mathrm{asrt}_p \colon \mathscr{A} \to \mathscr{A}$ (so $\mathrm{asrt}_p(a) = \sqrt{p}a\sqrt{p}$) factors as

$$\mathscr{A} \xrightarrow{\pi\colon a \mapsto \lceil p \rceil a \lceil p \rceil} \lceil p \rceil \mathscr{A} \lceil p \rceil \xrightarrow{c\colon a \mapsto \sqrt{p}a\sqrt{p}} \mathscr{A}\ ,$$

where $\lceil p \rceil$ is the least projection above p.

(Roughly speaking, the von Neumann algebra $\lceil p \rceil \mathscr{A} \lceil p \rceil$ represents the subtype of $\mathscr{A}$ in which the predicate p holds. The map c is simply the restriction of asrt_p to $\lceil p \rceil \mathscr{A} \lceil p \rceil$, while π is the map which forgets that p holds. The map c is a more sharply typed version of sequential product than asrt_p — much in the same way that the absolute value on the reals is more sharply described as a map $\mathbb{R} \to [0,\infty)$ than as a map $\mathbb{R} \to \mathbb{R}$.)

The maps c and π have a universal property: c is a *compression of* p and π is a *corner of* $\lceil p \rceil$ (see Definition 2). Our first axiom for the sequential product $(p,q) \mapsto p * q$ will be that $p * (-) = \tilde{\pi} \circ \tilde{c}$ where $\tilde{\pi}$ is a corner of $\lceil p \rceil$ and $\tilde{c}$ is a compression of p. Somewhat to our surprise, while $\tilde{\pi}$ and $\tilde{c}$ are unique up to unique isomorphism, the composition $\tilde{\pi} \circ \tilde{c}$ is not uniquely determined. To mend this problem, we add three more axioms.

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Terminology 1. Although we assume the reader is familiar with the basics of the theory of von Neumann algebras,[Sak71] we have included the relevant definitions and a selection of useful results in the appendix.

For brevity, a linear map between von Neumann algebras, which is normal, completely positive, and contractive, will be called a **process**. (This generalizes the standard notion of *quantum process* between finite-dimensional Hilbert spaces to von Neumann algebras.)

Definition 2. Let $\mathscr{A}$ and $\mathscr{C}$ be a von Neumann algebras, and let $p \in \mathscr{A}$ with $0 \leq p \leq 1$ be given.

1. A map $\tilde{c} \colon \mathscr{C} \to \mathscr{A}$ is a **compression** of p if $\tilde{c}$ is a process with $\tilde{c}(1) \leq p$, and $\tilde{c}$ is final among such maps in the sense that for every von Neumann algebra $\mathscr{B}$ and process $f \colon \mathscr{B} \to \mathscr{A}$ with $f(1) \leq p$ there is a unique process $\bar{f} \colon \mathscr{B} \to \mathscr{C}$ such that $\tilde{c} \circ \bar{f} = f$.

2. A map $\tilde{\pi} \colon \mathscr{A} \to \mathscr{C}$ is a **corner** of p if $\tilde{\pi}$ is a process with $\tilde{\pi}(p) = \tilde{\pi}(1)$, and $\tilde{\pi}$ is initial among such maps in the sense that for every von Neumann algebra $\mathscr{B}$ and process $g \colon \mathscr{A} \to \mathscr{B}$ with $g(p) = g(1)$ there is a unique process $\bar{g} \colon \mathscr{C} \to \mathscr{B}$ with $\bar{g} \circ \tilde{\pi} = g$.

Definition 3. An **abstract sequential product** is a family of operations $\tilde{*} \colon [0,1]_{\mathscr{A}} \times [0,1]_{\mathscr{A}} \to [0,1]_{\mathscr{A}}$, where $\mathscr{A}$ ranges over all von Neumann algebras, which obeys the following axioms.

Ax.1 For every von Neumann algebra $\mathscr{A}$ and $p \in [0,1]_{\mathscr{A}}$, there is a compression $\tilde{c} \colon \mathscr{C} \to \mathscr{A}$ of p, and corner $\tilde{\pi} \colon \mathscr{A} \to \mathscr{C}$ of $\lceil p \rceil$ such that for all $q \in [0,1]_{\mathscr{A}}$,

$$p \,\tilde{*}\, q \;=\; \tilde{c}(\tilde{\pi}(q)).$$

Ax.2 $p \,\tilde{*}\, (p \,\tilde{*}\, q) = (p \,\tilde{*}\, p) \,\tilde{*}\, q$ for every von Neumann algebra $\mathscr{A}$ and all $p, q \in [0,1]_{\mathscr{A}}$.

Ax.3 $f(p \tilde{*} q) = f(p) \tilde{*} f(q)$ for every multiplicative process $f \colon \mathscr{A} \to \mathscr{B}$ and all $p, q \in [0,1]_{\mathscr{A}}$.

Ax.4 For every von Neumann algebra $\mathscr{A}$ and $p \in [0,1]_{\mathscr{A}}$, and projections $e_1, e_2 \in \mathscr{A}$,

$$p \,\tilde{*}\, e_1 \;\leq\; 1 - e_2 \qquad \iff \qquad p \,\tilde{*}\, e_2 \;\leq\; 1 - e_1.$$

Let us formulate the main result of this paper.

Theorem 4. *The sequential product, $*$, given by*

$$p * q \;=\; \sqrt{p}q\sqrt{p}$$

for every von Neumann algebra $\mathscr{A}$ and $p, q \in [0,1]_{\mathscr{A}}$, is the unique abstract sequential product (see Definition 3).

The proof of Theorem 4 spans the length of this paper.

I. CORNERS

Proposition 5. *Let $\mathscr{A}$ be a von Neumann algebra, and let $p \in [0,1]_{\mathscr{A}}$. Then $\pi\colon \mathscr{A} \to \lfloor p\rfloor \mathscr{A} \lfloor p\rfloor$, $a \mapsto \lfloor p\rfloor a \lfloor p\rfloor$ is a corner of p.*

Proof. Note that $\lfloor p\rfloor \mathscr{A} \lfloor p\rfloor$ is a von Neumann subalgebra of $\mathscr{A}$ (with unit $\lfloor p\rfloor$) by Corollary 42. Let us show that π is a process. To begin, π is normal and completely positive, because the map $\mathscr{A} \to \mathscr{A}$, $a \mapsto \lfloor p\rfloor a \lfloor p\rfloor$ is normal and completely positive by Lemma 41. Further, since $\|\lfloor p\rfloor\| \leq 1$, we have $\|\pi(a)\| = \|\lfloor p\rfloor a \lfloor p\rfloor\| \leq \|\lfloor p\rfloor\|^2\|a\| \leq \|a\|$, for all $a \in \mathscr{A}$, and so π is contractive. Hence π is a process. Further, $\pi(1) = \lfloor p\rfloor = \lfloor p\rfloor p \lfloor p\rfloor = \pi(p)$ by Proposition 43.

To prove that π is a corner of p it remains to be shown that π is initial in the sense that for every process $g\colon \mathscr{A} \to \mathscr{B}$ with $g(p) = g(1)$ there is a unique process $\bar{g}\colon \lfloor p\rfloor \mathscr{A} \lfloor p\rfloor \to \mathscr{B}$ with $\bar{g} \circ \pi = g$.

(Uniqueness) Let $\bar{g}_1, \bar{g}_2\colon \lfloor p\rfloor \mathscr{A} \lfloor p\rfloor \to \mathscr{B}$ be processes with $\bar{g}_1 \circ \pi = g = \bar{g}_2 \circ \pi$. We must show that $\bar{g}_1 = \bar{g}_2$.

Let $a \in \lfloor p\rfloor \mathscr{A} \lfloor p\rfloor$ be given. Then $a = \lfloor p\rfloor a \lfloor p\rfloor = \pi(a)$, and so $\bar{g}_1(a) = \bar{g}_1(\pi(a)) = g(a)$. Similarly $\bar{g}_2(a) = g(a)$, and so $\bar{g}_1(a) = \bar{g}_2(a)$. Hence $\bar{g}_1 = \bar{g}_2$.

(Existence) To begin, we will prove that $g(1-\lfloor p\rfloor) = 0$. Since $1 - \lfloor p\rfloor = \lceil 1-p\rceil$ is the supremum of $1 - p \leq (1-p)^{1/2} \leq (1-p)^{1/4} \leq \cdots$ (see Proposition 43) and g is normal, it suffices to show that

$$g(\ (1-p)^{1/2^n}\ ) \ = \ 0 \qquad \text{for all} \ \ n \in \mathbb{N}. \tag{1}$$

Note that $g(1-p) = 0$, so to prove (1) it suffices to show that $g(a) = 0$ entails $g(a^{1/2}) = 0$ for all $a \in \mathscr{A}$ with $a \geq 0$. Since g is 2-positive, we have (by Theorem 54), for all $b, c \in \mathscr{A}$,

$$\|g(b^*c)\|^2 \ \leq \ \|g(b^*b)\|\,\|g(c^*c)\|. \tag{2}$$

In particular, for $a \in \mathscr{A}_+$, we have

$$\|g(a^{1/2})\|^2 \ \leq \ \|g(1)\|\,\|g(a)\|.$$

So $g(a) = 0$ entails $\|g(a^{1/2})\|^2 = 0$, and $g(a^{1/2}) = 0$. Thus $g(1 - \lfloor p\rfloor) = 0$.

Recall that $\lfloor p\rfloor \mathscr{A} \lfloor p\rfloor$ is a von Neumann subalgebra of $\mathscr{A}$. Let $j\colon \lfloor p\rfloor \mathscr{A} \lfloor p\rfloor \to \mathscr{A}$ be the inclusion. Then j is a normal contractive $*$-homomorphism, and thus a process.

Define $\bar{g} := g \circ j\colon \lfloor p\rfloor \mathscr{A} \lfloor p\rfloor \to \mathscr{B}$. Then $\bar{g}$ is a process. To complete the proof, we must show that $\bar{g} \circ \pi = g$, that is, $g(\lfloor p\rfloor a \lfloor p\rfloor) = g(a)$ for all $a \in \mathscr{A}$.

Let $a \in \mathscr{A}$ be given. We show that $g(\lfloor p\rfloor a) = g(a)$. By the Cauchy–Schwarz inequality for 2-positive maps (see Statement (2)), we have,

$$\|g((1 - \lfloor p\rfloor)a)\|^2 \ \leq \ \|g(1 - \lfloor p\rfloor)\|\,\|g(a^*a)\|.$$

Since $g(1-\lfloor p\rfloor) = 0$, we have $\|g((1-\lfloor p\rfloor)a)\|^2 \leq 0$, and so $0 = g((1-\lfloor p\rfloor)a) = g(a) - g(\lfloor p\rfloor a)$. Thus $g(a) = g(\lfloor p\rfloor a)$.

Similarly, $g(a) = g(a\lfloor p\rfloor)$ and so $g(\pi(a)) = g(\lfloor p\rfloor a \lfloor p\rfloor) = g(\lfloor p\rfloor a) = g(a)$ for all $a \in \mathscr{A}$.

Hence π is a corner for p. □

II. COMPRESSIONS

Proposition 6. *Let $\mathscr{A}$ be a von Neumann algebra, and let $p \in [0,1]_{\mathscr{A}}$. Then $c\colon \lceil p\rceil \mathscr{A} \lceil p\rceil \to \mathscr{A}$, $a \mapsto \sqrt{p}a\sqrt{p}$ is a compression of p.*

Proof. Note that $\lceil p\rceil \mathscr{A} \lceil p\rceil$ is a von Neumann subalgebra of $\mathscr{A}$ with unit $\lceil p\rceil$ (see Corollary 42). Since therefore the inclusion $\lceil p\rceil \mathscr{A} \lceil p\rceil \to \mathscr{A}$ is a process, and the map $a \mapsto \sqrt{p}a\sqrt{p}\colon \mathscr{A} \mapsto \mathscr{A}$ is a process (see Lemma 41), it follows that c is a process. Further, note that $c(1) = p \leq p$.

To prove that c is a compression it remains to be shown that c is final in the sense that for every von Neumann algebra $\mathscr{B}$ and process $f\colon \mathscr{B} \to \mathscr{A}$ with $f(1) \leq p$ there is a unique $\bar{f}\colon \mathscr{B} \to \lceil p\rceil \mathscr{A} \lceil p\rceil$ such that $f = c \circ \bar{f}$.

(Existence) Note that if $\sqrt{p}$ is invertible in $\mathscr{A}$, then we can define $\bar{f}\colon \mathscr{B} \to \lceil p\rceil \mathscr{A} \lceil p\rceil$ by, for all $b \in \mathscr{B}$,

$$\bar{f}(b) \ = \ \sqrt{p}^{-1} f(b) \sqrt{p}^{-1},$$

and this does the job. Also, if $\sqrt{p}$ is pseudoinvertible — $q\sqrt{p} = \sqrt{p}q = \lceil p\rceil$ for some $q \in \mathscr{A}$ —, then $\bar{f}$ can be defined in a similar manner. However, p might not be pseudoinvertible.[Nota] Therefore, we will instead approximate the (possibly non-existent) pseudoinverse of $\sqrt{p}$ by a sequence $q_1, q_2, \ldots$ in $\mathscr{A}$ — much in the same way that an approximate identity in a C^*-algebra approximates a (possibly non-existent) unit —, and define, for $b \in \mathscr{B}$,

$$\bar{f}(b) \ = \ \operatorname*{uwlim}_{n\to\infty} q_n f(b) q_n. \tag{3}$$

By the Spectral Theorem (see Thm. 38 and Thm. 36), we may assume without loss of generality that $\mathscr{A}$ is a von Neumann subalgebra of the bounded operators $\mathscr{B}(L^2(X))$ on the Hilbert space $L^2(X)$ of square-integrable functions[Notb] on some measure space X, and that there is a real bounded integrable function $\hat{p}$ on X such that, for all $f \in L^2(X)$,

$$p(f) \ = \ \int \hat{p} \cdot f \, d\mu.$$

Let $\varrho\colon L^\infty(X) \to \mathscr{A}$ be given by $\varrho(g)(f) = \int f \cdot g \, d\mu$ for all $g \in L^\infty(X)$ and $f \in L^2(X)$, where $L^\infty(X)$ is the von Neumann algebra of bounded measurable functions[Notc] on X. Then ϱ is an injective normal $*$-homomorphism, and $\varrho(\hat{p}) = p$.

Note that $\sqrt{\hat{p}}$ might not be pseudoinvertible in $L^\infty(X)$, because the function $\hat{q}: X \to \mathbb{R}$ given by for $x \in X$,

$$\hat{q}(x) \ = \ \begin{cases} p(x)^{-1/2} & \text{if } p(x) \neq 0 \\ 0 & \text{if } p(x) = 0. \end{cases}$$

might not be (essentially) bounded. Nevertheless, $\sqrt{\hat{p}} \cdot \mathbf{1}_{Q_n}$ has $\hat{q} \cdot \mathbf{1}_{Q_n}$ as pseudoinverse in $L^\infty(X)$, where

$$Q_n \ = \ \{\, x \in X\colon \ \hat{p}(x) > {}^1\!/\!_n \,\} \ = \ \hat{p}^{-1}(\,({}^1\!/\!_n, 1]\,).$$

Define $q_n = \varrho(\hat{q}\cdot \mathbf{1}_{Q_n})$ for all $n\in\mathbb{N}$.

Let $b \in \mathscr{B}$ be given. We want to define $\bar{f}(b)$ by Equation (3), but for this, we must first show that $(q_n\, f(b)\, q_n)_n$ converges ultraweakly. It suffices to show that $(q_n\, f(b)\, q_n)_n$ is norm bounded, and ultraweakly Cauchy (see Proposition 40).

We only need to consider the case that $b \in [0,1]_{\mathscr{B}}$. Indeed, any $b\in\mathscr{B}$ can written as

$$b \;\equiv\; \|b\|\,(b_1 - b_2 + ib_3 - ib_4), \tag{4}$$

where $b_i \in [0,1]_{\mathscr{B}}$, and if $(q_n\, f(b_i)\, q_n)_n$ converges ultraweakly for each i, then so does $(q_n\, f(b)\, q_n)_n$.

Let $n\in\mathbb{N}$ be given. Since $f(b) \leq f(1) \leq p$, we have $q_n\, f(b)\, q_n \;\leq\; q_n\, p\, q_n$. Since $q_n = \varrho(\hat{q}\cdot\mathbf{1}_{Q_n})$, $p=\varrho(\hat{p})$, and $\hat{q}\cdot\mathbf{1}_{Q_n}$ is the pseudoinverse of $\sqrt{\hat{p}}\cdot\mathbf{1}_{Q_n}$, we get $q_n\, p\, q_n = \varrho(\mathbf{1}_{Q_n}) \leq 1$, and so $q_n\, f(b)\, q_n \leq 1$. Hence $\|q_n\, f(b)\, q_n\| \leq 1$, and so $(q_n\, f(b)\, q_n)_n$ is norm bounded.

Let $\varphi\colon \mathscr{A}\to\mathbb{C}$ be a normal state. To prove that $(q_n\, f(b)\, q_n)_n$ is ultraweakly Cauchy, we must show that $(\varphi(q_n\, f(b)\, q_n))_n$ is Cauchy.

For brevity, define for $n>m>0$,

$$\begin{aligned} S_{n,m} &\;=\; \hat{p}^{-1}(\,(1/n, 1/m]\,)\\ S_{\infty,m} &\;=\; \hat{p}^{-1}(\,(0,1/m]\,)\\ s_{n,m} &\;=\; \varrho(\,\hat{q}\cdot\mathbf{1}_{S_{n,m}}\,)\end{aligned}$$

Note that $S_{n,1}=Q_n$ and $s_{n,1}=q_n$. (We have not defined $s_{\infty,m}=\varrho(\hat{q}\cdot\mathbf{1}_{S_{\infty,m}})$, because $\hat{q}\cdot\mathbf{1}_{S_{\infty,m}}$ might not be bounded.) Note that

$$s_{n,m}\sqrt{p} \;=\; \sqrt{p}s_{n,m} \;=\; \varrho(\mathbf{1}_{S_{n,m}}). \tag{5}$$

Let $0<m<n$ be given. Since $(1/n,1]$ is the disjoint union of $(1/n,1/m]$ and $(1/m,1]$, Q_n is the disjoint union of $S_{n,m}$ and Q_m, and $q_n = s_{n,m}+q_m$, and

$$\begin{aligned} &q_n\, f(b)\, q_n \;-\; q_m\, f(b)\, q_m\\ &\quad=\; s_{n,m}\, f(b)\, s_{n,m} \;+\; s_{n,m}\, f(b)\, q_m \;+\; q_m\, f(b)\, s_{n,m}.\end{aligned}$$

Thus,

$$\begin{aligned} &|\,\varphi(\; q_n\, f(b)\, q_n - q_m\, f(b)\, q_m\;)\,|\\ &\quad\leq\; |\varphi(s_{n,m}\, f(b)\, s_{n,m})| + |\varphi(s_{n,m}\, f(b)\, q_m)|\\ &\qquad\quad + |\varphi(q_m\, f(b)\, s_{n,m})| \end{aligned} \tag{6}$$

Note that for $k<\ell$ and $m<n$, we have

$$\begin{aligned} &|\varphi(s_{\ell,k} f(b) s_{n,m})|^2\\ &=\; \left|\varphi(\;(\sqrt{f(b)}s_{\ell,k})^*\;\sqrt{f(b)}s_{n,m}\;)\right|^2\\ &\leq\; \varphi(s_{\ell,k}\, f(b)\, s_{\ell,k})\;\varphi(s_{n,m}\, f(b)\, s_{n,m}) && \text{by Ineq. (D1)}\\ &\leq\; \varphi(s_{\ell,k}\, p\, s_{\ell,k})\;\varphi(s_{n,m}\, p\, s_{n,m}) && \text{since } f(b)\leq p\\ &=\; \varphi(\varrho(\mathbf{1}_{S_{\ell,k}}))\;\varphi(\varrho(\mathbf{1}_{S_{n,m}})) && \text{by Eq. (5)}\\ &\leq\; \varphi(\varrho(\mathbf{1}_{S_{n,m}})) && \text{as } \mathbf{1}_{S_{\ell,k}}\leq\mathbf{1}\\ &\leq\; \varphi(\varrho(\mathbf{1}_{S_{\infty,m}})) && \text{as } S_{n,m}\subseteq S_{\infty,m}\end{aligned}$$

Thus using Eq. (6) and $q_n = s_{n,1}$ we get the bound

$$|\varphi(\;q_n\, f(b)\, q_n \;-\; q_m\, f(b)\, q_m\;)| \;\leq\; 3\sqrt{\varphi(\varrho(\mathbf{1}_{S_{\infty,m}}))}. \tag{7}$$

Since $(0,1]\supseteq(0,1/2]\supseteq(0,1/3]\supseteq\cdots$ and $\bigcap_m(0,1/m]=\varnothing$, we have $S_{\infty,1}\supseteq S_{\infty,2}\supseteq\ldots$ and $\bigcap_m S_{\infty,m}=\varnothing$. Then $\inf_m \mathbf{1}_{S_{\infty,m}}=0$, and so $\inf_m\varphi(\varrho(\mathbf{1}_{S_{\infty,m}}))=0$, because ϱ and φ are normal. Thus $(\,\varphi(\varrho(\mathbf{1}_{S_{\infty,m}}))\,)_m$ converges to 0, and so $(\,\sqrt{\varphi(\varrho(\mathbf{1}_{S_{\infty,m}}))}\,)_m$ converges to 0 as well.

Let $\varepsilon>0$ be given. There is $N\in\mathbb{N}$ such that for all $n>N$, we have $\sqrt{\varphi(\varrho(\mathbf{1}_{S_{\infty,m}}))}\leq\varepsilon/3$. Then given $n>m>N$, we have, by Equation (7),

$$|\varphi(\;q_n\, f(b)\, q_n \;-\; q_m\, f(b)\, q_m\;)| \;\leq\; \varepsilon. \tag{8}$$

Hence $(q_nf(b)q_n)_n$ is ultraweakly Cauchy and norm bounded, and must therefore converge ultraweakly. We may now (and do) define $\overline{f}(b)$ as in Equation (3).

Thus, $(q_nf(-)q_n)_n$ converges coordinatewise ultraweakly to $\overline{f}$. Note that the number N related to Inequality (8) depends on ε and φ, but does not depend on b. It follows that on $[0,1]_{\mathscr{B}}$ the sequence $(q_nf(-)q_n)_n$ converges uniformly ultraweakly to $\overline{f}$.

It is easy to see that $\overline{f}$ is linear and positive. It remains to be shown that $\overline{f}$ is contractive, normal, completely positive, $c\circ\overline{f}=f$, and $\overline{f}(\mathscr{B})\subseteq\lceil p\rceil\mathscr{A}\lceil p\rceil$.

($\overline{f}(\mathscr{B})\subseteq\lceil p\rceil\mathscr{A}\lceil p\rceil$) Let $b\in\mathscr{B}$ be given. We must show that $\overline{f}(b)\in\lceil p\rceil\mathscr{A}\lceil p\rceil$. By writing b as in Equation (4), the problem is easily reduced to the case that $b\in[0,1]_{\mathscr{B}}$.

Let $n\in\mathbb{N}$ be given. Since $b\leq 1$, we have $f(b)\leq f(1)\leq p$, and so $q_nf(b)q_n\leq q_npq_n=\varrho(\mathbf{1}_{Q_n})=\varrho(\mathbf{1}_{S_{\infty,1}})$. Since $\hat{p}^{-1}((0,1])=S_{\infty,1}$, and it is not hard to see that $\mathbf{1}_{\hat{p}^{-1}((0,1])}$ is the support of $\hat{p}$ in $L^\infty(X)$, it follows that $\varrho(\mathbf{1}_{S_{\infty,1}})=\lceil p\rceil$ (see Proposition 46). Thus $q_nf(b)q_n\leq\lceil p\rceil$ for all n, and so $\overline{f}(b)=\operatorname{uwlim}_n q_nf(b)q_n\leq\lceil p\rceil$. Corollary 28, gives us $\lceil p\rceil\overline{f}(b)\lceil p\rceil=\overline{f}(b)$, and so $\overline{f}(b)\in\lceil p\rceil\mathscr{A}\lceil p\rceil$.

($\overline{f}$ is contractive) It suffices to show that $\overline{f}(1)\leq 1$. Let $n\in\mathbb{N}$ be given. Since $f(1)\leq p$, we have

$$q_nf(1)q_n \;\leq\; q_npq_n \;=\; \varrho(\mathbf{1}_{Q_n}) \;\leq\; 1.$$

Thus $\overline{f}(1)=\operatorname{uwlim}_n q_nf(1)q_n\leq 1$.

($c\circ\overline{f}=f$) Let $b\in[0,1]_{\mathscr{B}}$. It suffices to show that $c(\overline{f}(b))=f(b)$. Since c is normal, we have

$$c(\overline{f}(b)) \;=\; \operatorname*{uwlim}_{n\to\infty}\sqrt{p}q_n\, f(b)\, q_n\sqrt{p}.$$

Thus we must show that $(\,\sqrt{p}q_n\, f(b)\, q_n\sqrt{p}\,)_n$ converges ultraweakly to $f(b)$.

Let $n\in\mathbb{N}$ be given. On the one hand we have $\sqrt{p}q_n=\varrho(\mathbf{1}_{Q_n})=\varrho(\mathbf{1}_{S_{n,1}})$ by definition of q_n. On the other hand we have $\lceil p\rceil f(b)\lceil p\rceil=f(b)$ and $\lceil p\rceil=\varrho(\mathbf{1}_{S_{\infty,1}})$. Thus, using $\mathbf{1}_{S_{\infty,1}}=\mathbf{1}_{S_{\infty,n}}+\mathbf{1}_{S_{n,1}}$, and writing $e_{k,\ell}=\varrho(\mathbf{1}_{S_{k,\ell}})$, we have

$$\begin{aligned} &f(b)\;-\;\sqrt{p}q_n\, f(b)\, q_n\sqrt{p}\\ &\qquad=\; e_{\infty,n}\, f(b)\, e_{\infty,n} \;+\; e_{n,1}\, f(b)\, e_{\infty,n}\\ &\qquad\quad+\; e_{\infty,n}\, f(b)\, e_{n,1}\end{aligned} \tag{9}$$

So to show that $(\sqrt{p}q_n\, f(b)\, q_n\sqrt{p}\,)_n$ converges ultraweakly to $f(b)$, it suffices to show that the terms on the right-hand side of Equation (9) converge ultraweakly to 0. Let $\varphi\colon \mathscr{A}\to\mathbb{C}$ be a normal state. Then

$$\begin{aligned} &|\varphi(\, e_{n,1}\, f(b)\, e_{\infty,n}\,)|^2 \\ &= |\varphi(\,(\sqrt{f(b)}e_{n,1})^*\,\sqrt{f(b)}e_{\infty,n}\,)|^2 \\ &\leq \varphi(e_{n,1}f(b)e_{n,1})\cdot\varphi(e_{\infty,n}f(b)e_{\infty,n}) && \text{by Ineq. (D1)} \\ &\leq \varphi(e_{n,1})\cdot\varphi(e_{\infty,n}) && \text{since } f(b)\leq 1 \\ &\leq \varphi(e_{\infty,n}) && \text{since } e_{n,1}\leq 1. \end{aligned}$$

Recall that $\varphi(e_{n,\infty})\equiv\varphi(\varrho(\mathbf{1}_{S_{\infty,n}}))$ converges to zero (because $\bigcap_n S_{\infty,n}=\varnothing$). It follows that $(\,e_{n,1}f(b)e_{\infty,n}\,)_n$ converges ultraweakly to 0.

By a similar reasoning, $(\,e_{\infty,n}f(b)e_{n,1}\,)_n$ and $(\,e_{\infty,n}f(b)e_{\infty,n}\,)_n$ converge ultraweakly to 0. Thus, by Equation (9), $(\sqrt{p}q_n\, f(b)\, q_n\sqrt{p}\,)_n$ converges ultraweakly to $f(b)$. Thus $c\circ\overline{f}=f$.

(f is normal) Since $(q_nf(-)q_n)_n$ converges uniformly ultraweakly on $[0,1]_{\mathscr{B}}$ to $\overline{f}$, and each $q_nf(-)q_n$ is normal (by Lemma 41), it follows that $\overline{f}$ is normal (by Corollary 49).

(f is completely positive) Since $(q_nf(-)q_n)_n$ converges coordinatewise ultraweakly to $\overline{f}$, and each $q_nf(-)q_n$ is completely positive (see Lemma 41), it follows that $\overline{f}$ is completely positive (by Corollary 51).

(Uniqueness) Let $g\colon\mathscr{B}\to\lceil p\rceil\mathscr{A}\lceil p\rceil$ be a process with $c\circ g=f$. We must show that $g=\overline{f}$.

Let $b\in[0,1]_{\mathscr{B}}$ be given. It suffices to show that $\overline{f}(b)=g(b)$. We have $\sqrt{p}g(b)\sqrt{p}=f(b)$. Let $n\in\mathbb{N}$ be given. We have $e_{n,1}g(b)e_{n,1}=q_n\sqrt{p}g(b)\sqrt{p}q_n=q_nf(b)q_n$ since $q_n\sqrt{p}=\varrho(1_{S_{n,1}})\equiv e_{n,1}$. On the one hand $(q_nf(b)q_n)_n$ converges ultraweakly to $\overline{f}(b)$ by definition of $\overline{f}(b)$. On the other hand $(e_{n,1}g(b)e_{n,1})_n$ converges ultraweakly to $g(b)$ as one can see with tricks that were used before. Hence $\overline{f}(b)=g(b)$. □

III. EXISTENCE

To show that the sequential product is an abstract sequential product, we use the following result, which (we think) is interesting in itself.

Lemma 7. *Let a be an element of a von Neumann algebra (or a unital C*-algebra) $\mathscr{A}$ with $a^*a\leq 1$. Then for projections $e_1,e_2\in\mathscr{A}$ the following are equivalent.*

1. $a^*e_1a\leq 1-e_2$
2. $ae_2a^*\leq 1-e_1$
3. $e_1ae_2=0$
4. $e_2a^*e_1=0$

Proof. (1 ⟹ 3) We must show that $e_1ae_2=0$. It suffices to show $e_2a^*e_1ae_2=0$, because $\|e_1ae_2\|^2=\|e_2a^*e_1ae_2\|$ by the C*-identity. Since $0\leq a^*e_1a\leq 1-e_2$, we have $0\leq e_2a^*e_1ae_2\leq e_2(1-e_2)e_2=0$, and so $e_2a^*e_1ae_2=0$.

(3 ⟹ 1) Since $e_1ae_2=0$, also $e_1a=e_1a(1-e_2)$, and $a^*e_1=(1-e_2)a^*e_1$. Then $a^*e_1a=(1-e_2)a^*e_1a(1-e_2)\leq 1-e_2$, because $a^*e_1a\leq a^*a\leq 1$.

(4 ⟺ 2) follows by the same reasoning as 1 ⟺ 3.

(3 ⟺ 4) follows by applying $(-)^*$. □

Proposition 8. *The sequential product $*$ (which is given by $p*q=\sqrt{p}q\sqrt{p}$) is an abstract sequential product.*

Proof. (Ax.1) Let $\mathscr{A}$ be a von Neumann algebra, and let $p,q\in[0,1]_{\mathscr{A}}$. Since $\lceil p\rceil\sqrt{p}=\sqrt{p}$ (by Prop. 43),

$$p*q \;=\; \sqrt{p}q\sqrt{p} \;=\; \sqrt{p}\lceil p\rceil q\lceil p\rceil\sqrt{p} \;=\; c(\pi p(q)),$$

where $\pi p\colon\mathscr{A}\to\lceil p\rceil\mathscr{A}\lceil p\rceil$ is the corner of $\lceil p\rceil$ from Proposition 5, and $c\colon\lceil p\rceil\mathscr{A}\lceil p\rceil\to\mathscr{A}$ is the compression of p from Proposition 6. Thus $*$ obeys Ax.1.

The proof of *(Ax.2)* and *(Ax.3)* is easy, and *(Ax.4)* follows from Lemma 7. □

IV. UNIQUENESS

We will need the following fact later on.

Lemma 9. *Let $f,g\colon V\to W$ be linear maps between complex vector spaces. Assume that for every $v\in V$, there is an $\alpha\in\mathbb{C}\backslash\{0\}$ with $f(v)=\alpha\cdot g(v)$.*

Then there is $\alpha_0\in\mathbb{C}\backslash\{0\}$ with $f=\alpha_0\cdot g$.

Proof. For the moment, assume f and g are injective. If $V=\{0\}$, then $\alpha_0\equiv 1$ works, so assume $V\neq\{0\}$. Pick any $v\in V$ with $v\neq 0$. Let $\alpha_0\in\mathbb{C}\backslash\{0\}$ be such that $f(v)=\alpha_0\cdot g(v)$. Let $w\in V$. We have to show that $f(w)=\alpha_0\cdot g(w)$. Now, either $g(v)$ and $g(w)$ are linearly dependent or not.

Suppose that $g(v)$ and $g(w)$ are linearly independent. Let $\beta\in\mathbb{C}\backslash\{0\}$ be such that $f(w)=\beta\cdot g(w)$, and let $\gamma\in\mathbb{C}\backslash\{0\}$ be such that $f(v+w)=\gamma\cdot g(v+w)$. Then

$$(\gamma-\alpha_0)\cdot g(v)\;+\;(\gamma-\beta)\cdot g(w)\;=\;0.$$

By linear independence, we have $\gamma-\alpha_0=0=\gamma-\beta$. Hence $\alpha_0=\beta$, and so $f(w)=\alpha_0\cdot g(w)$.

Suppose that $g(v)$ and $g(w)$ are linearly dependent. As $v\neq 0$ and g is injective, we have $g(v)\neq 0$. Thus $g(w)=\varrho\cdot g(v)$ for some $\varrho\in\mathbb{C}$. Then $g(w-\varrho\cdot v)=0$, and so $w=\varrho\cdot v$, since g is injective. We have

$$f(w)\;=\;\varrho\cdot f(v)\;=\;\varrho\cdot\alpha_0\cdot g(v)\;=\;\alpha_0\cdot g(w).$$

Thus we have $f(w)=\alpha_0g(w)$ whether $g(v)$ and $g(w)$ are linearly dependent or not.

We now return to the general case in which f and g might not be injective. Note that the kernels of f and g coincide, and so, writing $N\equiv\ker f=\ker g$, there are unique $t,s\colon V/N\to W$ such that $s\circ q=f$ and $t\circ q=g$, where $q\colon V\to V/N$ is the quotient map. Clearly, s and t are injective, and for every $v\in V/N$ there is $\alpha\in\mathbb{C}\backslash\{0\}$ with $s(v)=\alpha\cdot t(v)$. Thus, by the previous discussion, there is $\alpha_0\in\mathbb{C}\backslash\{0\}$ with $s=\alpha_0\cdot t$. Then $f=\alpha_0\cdot g$. □

Proposition 10. *For any abstract sequential product, $\tilde{*}$, we have $p \mathbin{\tilde{*}} q = \sqrt{p}q\sqrt{p}$, where $p, q \in [0,1]_{\mathscr{A}}$ and $\mathscr{A}$ is a von Neumann algebra.*

Proof. Let $\mathscr{A}$ be a von Neumann algebra, and $p \in [0,1]_{\mathscr{A}}$. By Ax.1 there is a corner $\tilde{\pi}$ of $\lceil p \rceil$ and a compression $\tilde{c}$ of p such that $p \mathbin{\tilde{*}} q = \tilde{c}(\tilde{\pi}(q))$ for all $q \in [0,1]_{\mathscr{A}}$.

Let $c\colon \lceil p\rceil \mathscr{A} \lceil p \rceil \to \mathscr{A}$ be the compression of p given by $c(a) = \sqrt{p}a\sqrt{p}$ for all $a \in \lceil p \rceil \mathscr{A} \lceil p \rceil$ (see Proposition 6). Since both c and $\tilde{c}$ are compressions of p it is easy to see that there is an invertible process ϑ such that $\tilde{c} = c \circ \vartheta$. In fact, ϑ is a $*$-isomorphism by Corollary 47.

Similarly, $\tilde{\pi} = \chi \circ \pi$ where χ is some $*$-isomorphism χ, and $\pi\colon \mathscr{A} \to \lceil p \rceil \mathscr{A} \lceil p \rceil$ is the corner of $\lceil p \rceil$ given by $\pi(a) = \lceil p \rceil a \lceil p \rceil$ for all $a \in \mathscr{A}$ (see Proposition 5).

Thus $p \mathbin{\tilde{*}} q = \sqrt{p}\,\psi(\,\lceil p \rceil q \lceil p \rceil\,)\sqrt{p}$ for all $q \in [0,1]_{\mathscr{A}}$, where $\psi = \vartheta \circ \chi$ is a $*$-automorphism of $\lceil p \rceil \mathscr{A} \lceil p \rceil$.

Roughly speaking, our goal is to prove $\psi = \mathrm{id}$. We will first consider the case that $\mathscr{A} = \mathscr{B}(\mathscr{H})$. Since $\lceil p \rceil \mathscr{B}(\mathscr{H}) \lceil p \rceil$ is a type I factor (i.e. $*$-isomorphic to $\mathscr{B}(\mathscr{K})$ for some Hilbert space $\mathscr{K}$), it is known[Kap52] that ψ must be an inner $*$-automorphism, that is, there is a unitary $u \in \lceil p \rceil \mathscr{B}(\mathscr{H}) \lceil p \rceil$ such that $\psi(a) = u^* a u$ for all $a \in \lceil p \rceil \mathscr{B}(\mathscr{H}) \lceil p \rceil$. Note that $\lceil p \rceil u = u$ since $u \in \lceil p \rceil \mathscr{B}(\mathscr{H}) \lceil p \rceil$. Thus we have, for all $b \in [0,1]_{\mathscr{B}(\mathscr{H})}$,

$$p \mathbin{\tilde{*}} b \;=\; \sqrt{p} u^* b u \sqrt{p}. \tag{10}$$

We aim to show that $u = 1$, or at least that $u = \alpha 1$ for some $\alpha \in \mathbb{C}$ with $|\alpha| = 1$.

Our first step is to prove that $up = pu$. To this end, we extract some information about u from Ax.4. First, note that for vectors $v, w \in \mathscr{H}$ with $\|w\| = 1$ and $\|v\| \leq 1$,

$$|v\rangle\langle v| \;\leq\; 1 - |w\rangle\langle w| \quad \text{if and only if} \quad \langle v, w\rangle \;=\; 0. \tag{11}$$

For any $v \in \mathscr{H}$ with $\|v\| = 1$,

$$p \mathbin{\tilde{*}} |v\rangle\langle v| \;=\; \sqrt{p} u^*\, |v\rangle\langle v|\, u \sqrt{p} \;=\; |\sqrt{p}u^* v\rangle\langle \sqrt{p} u^* v|\,. \tag{12}$$

For all $v, w \in \mathscr{H}$ with $\|v\| = \|w\| = 1$, the following are equivalent

$$\begin{aligned} \langle \sqrt{p} u^* v, w\rangle &= 0 \\ |\sqrt{p}u^* v\rangle\langle\sqrt{p}u^* v| &\leq 1 - |w\rangle\langle w| && \text{by (11)} \\ p \mathbin{\tilde{*}} |v\rangle\langle v| &\leq 1 - |w\rangle\langle w| && \text{by (12)} \\ p \mathbin{\tilde{*}} |w\rangle\langle w| &\leq 1 - |v\rangle\langle v| && \text{by Ax.4} \\ &\vdots \\ \langle \sqrt{p} u^* w, v\rangle &= 0 \\ \langle u\sqrt{p} v, w\rangle &= 0 \end{aligned}$$

Thus $\sqrt{p}u^* v$ and $u\sqrt{p}v$ are orthogonal to the same vectors, and so there is $\alpha \in \mathbb{C}\backslash\{0\}$ with

$$\sqrt{p} u^* v \;=\; \alpha \cdot u \sqrt{p} v.$$

By scaling it is clear that this statement is also true for all $v \in \mathscr{H}$ (and not just for v with $\|v\| = 1$).

Although a priori α might depend on v, we know by Lemma 9 that there is an $\alpha \in \mathbb{C}\backslash\{0\}$ such that $\sqrt{p}u^* = \alpha \cdot u\sqrt{p}$. It follows that $p = \sqrt{p}u^* u \sqrt{p} = \alpha \cdot u\sqrt{p} u \sqrt{p} = u\sqrt{p}\sqrt{p}u^* = upu^*$, and so $pu = up$. Then also $\sqrt{p}u = u\sqrt{p}$ (see Corollary 25), and thus $\sqrt{p}u^* = \alpha u \sqrt{p} = \alpha \sqrt{p} u$.

Note that $(\sqrt{p}u^*)^* = u\sqrt{p}$, and so $u\sqrt{p} = \alpha^* \sqrt{p} u^* = \alpha^* \alpha u \sqrt{p}$. Then if $u\sqrt{p} \neq 0$, we get $\alpha^*\alpha = 1$, and if $u\sqrt{p} = 0$, we can put $\alpha = 1$ and still have both $\sqrt{p}u^* = \alpha\sqrt{p}u$ and $\alpha^*\alpha = 1$. It follows that, for all $b \in \mathscr{B}(\mathscr{H})$,

$$c(u^* b u) = \sqrt{p} u^* b u \sqrt{p} = \sqrt{p} u b u^* \sqrt{p} = c(u b u^*),$$

where c is the compression of p from Proposition 6. By the universal property of c we get $u^*(-)u = u(-)u^*$, and thus $u^2 b = b u^2$ for all $b \in \mathscr{B}(\mathscr{H})$. Hence u^2 is central in $\mathscr{B}(\mathscr{H})$. Since $\mathscr{B}(\mathscr{H})$ is a factor, we get $u^2 = \lambda \cdot 1$ for some $\lambda \in \mathbb{C}$ with $|\lambda| = 1$.

Since p commutes with u, we easily get $p \mathbin{\tilde{*}} p = p^2$. Then from Ax.2 it follows that

$$\begin{aligned} p^2 \mathbin{\tilde{*}} q &= (p \mathbin{\tilde{*}} p) * q \\ &= p * (p \mathbin{\tilde{*}} q) \\ &= \sqrt{p}u^* \sqrt{p} u^*\, q\, u\sqrt{p}\, u \sqrt{p} \\ &= pqp. \end{aligned}$$

Thus, if we repeat the whole argument with p replaced by $\sqrt{p}$, we see that $p \mathbin{\tilde{*}} q = \sqrt{p} q \sqrt{p}$.

Let us now consider the general case in which $\mathscr{A}$ may not be $*$-isomorphic to $\mathscr{B}(\mathscr{H})$ for some Hilbert space $\mathscr{H}$, but is instead (without loss of generality) a von Neumann subalgebra of $\mathscr{B}(\mathscr{H})$ for some Hilbert space $\mathscr{H}$ (see Theorem 36). Let $q \in [0,1]_{\mathscr{A}}$. Since the inclusion $\varrho\colon \mathscr{A} \to \mathscr{B}(\mathscr{A})$ is a multiplicative process, we have $\varrho(p \mathbin{\tilde{*}} q) = \varrho(p) \mathbin{\tilde{*}} \varrho(q) = \sqrt{\varrho(p)}\varrho(q)\sqrt{\varrho(p)} = \varrho(\sqrt{p}q\sqrt{p})$. Since ϱ is injective, we conclude that $p \mathbin{\tilde{*}} q = \sqrt{p}q\sqrt{p}$. □

Proof of Theorem 4. By Proposition 8, the sequential product $*$ (given by $p * q = \sqrt{p}q\sqrt{p}$) is an abstract sequential product, and $*$ is the only abstract sequential product by Proposition 10 □

REMARKS

Remark 11. Gudder and Latémolière (G&L) showed in[GL08] that the sequential product on the effects of a Hilbert space $\mathscr{H}$ is the only binary operation $\tilde{*}$ that satisfies the following axioms. For all $a, b \in [0,1]_{\mathscr{B}(\mathscr{H})}$, and every density operator ϱ on $\mathscr{H}$,

GL1. $\mathrm{tr}[(a \mathbin{\tilde{*}} \varrho) b] = \mathrm{tr}[\varrho (a \mathbin{\tilde{*}} b)]$;

GL2. $a \mathbin{\tilde{*}} 1 = 1 \mathbin{\tilde{*}} a = a$;

GL3. $a \mathbin{\tilde{*}} (a \mathbin{\tilde{*}} b) = (a \mathbin{\tilde{*}} a) \mathbin{\tilde{*}} b = a^2 \mathbin{\tilde{*}} b$, and

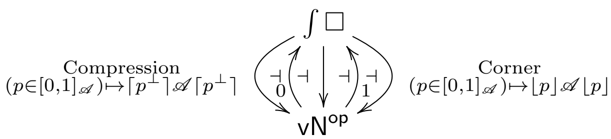
GL4. $a \mapsto a \mathbin{\tilde{*}} b$ is strongly continuous.

Let us compare their proof of uniqueness with our proof of uniqueness of the abstract sequential product. The broad strokes are similar: in both proofs it is shown

1. first that $p \mathbin{\tilde{*}} q = \sqrt{p}u^* q u \sqrt{p}$ for appropriate u;
2. then that $p^2 \mathbin{\tilde{*}} q = pqp$ using Ax.4 and GL1 resp.,
3. and finally $p \mathbin{\tilde{*}} q = \sqrt{p}q\sqrt{p}$ is obtained using GL3 and Ax.2 respectively.

However, the short strokes are quite different. For instance, while GL3 and Ax.2 clearly serve the same purpose in both proofs (enabling the third step mentioned above), the relation between GL1 and its analogue, Ax.4, is less clear: Ax.4 only comes into play at the second step, while GL1 is important in both the first and second steps. Also, the proof of G&L has a branch in the first step (case iii on page 9 of[GL08]), which has no companion in our proof.

Remark 12. The universal properties of the compression c (from Proposition 6) and of the corner π (from Proposition 5) may be cast into the following chain of adjunctions.



Here, vN is the category of von Neumann algebras and processes, and $\Box$ is the functor $\mathsf{vN}^{\mathsf{op}} \to \mathsf{Poset}^{\mathsf{op}}$ given by $\Box(\mathscr{A}) = [0,1]_{\mathscr{A}}$ and $\Box(f)(p) = f(p^\perp)^\perp$, and $\int \Box$ is its Grothendieck completion. Such chains appear in several other categories and provide a tool to study the sequential product in other settings (see[CJWW15b]).

Remark 13. We have shown that Ax.1, Ax.2, Ax.3, Ax.4 axiomatize the sequential product. A natural question is whether three of them would have sufficed. We will show that Ax.1, Ax.2, and Ax.4 cannot be dropped. We do not know whether Ax.3 is redundant.

(Of course, instead of being dropped, the axioms may also be weakened. For example, Ax.3 is used only with f a representation, and Ax.4 is only used with e_1, e_2 rank one projections.)

Ax.1 The operation $\tilde{*}$ given by $p \mathbin{\tilde{*}} q \equiv pqp$ satisfies Ax.2, Ax.3, and Ax.4, but not Ax.1.

Ax.2 Observe that if we pick for every effect p on a von Neumann algebra $\mathscr{A}$ a unitary u_p from $\lceil p \rceil \mathscr{A} \lceil p \rceil$, then we may form an operation $\tilde{*}$ on all effects by $p \mathbin{\tilde{*}} q \equiv \sqrt{p} u_p^* q u_p \sqrt{p}$, which satisfies Ax.1.

Further, note that if $u_p^2 = u_{p^2}$ for all p, then $\tilde{*}$ satisfies Ax.2; and if $f(u_p) = u_{f(p)}$ for any unital $*$-homomorphism f, then $\tilde{*}$ obeys Ax.3; and if every u_p is self-adjoint, then $\tilde{*}$ satisfies Ax.4.

Define u_p by $u_p = g(p)$, where $g\colon [0,1] \to \{-1, 1\}$ is any Borel function with $g(2/3) = 1$ and $g(4/9) = -1$. Then clearly $\tilde{*}$ (defined by u_p) satisfies Ax.1, Ax.3, and Ax.4. Also, $\tilde{*}$ does not satisfy Ax.2, because for $p = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 2/3 \end{smallmatrix}\right)$ in M_2 we have $u_p = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, while $u_{p^2} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, and so $(p \mathbin{\tilde{*}} p) \mathbin{\tilde{*}} q \neq p \mathbin{\tilde{*}} (p \mathbin{\tilde{*}} q)$, where $q = \frac{1}{\sqrt{2}}\left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right)$.

Ax.4 Pick a Borel function $g\colon [0,1] \to S^1$ such that $g(1/2) \neq 1$ and $g(\lambda)^2 = g(\lambda^2)$ for all $\lambda \in [0,1]$. Then $\tilde{*}$ given by $p \mathbin{\tilde{*}} q = \sqrt{p} g(p)^* \, q \, g(p) \sqrt{p}$ obeys Ax.1, Ax.2, and Ax.3, but not Ax.4.

Problem 14. *Do Ax.1, Ax.2 and Ax.4 imply Ax.3?*

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Sam Staton suggested we should look for a universal property for c from Proposition 6. Bart Jacobs noted the chain of adjunctions from Remark 12. Robert Furber pointed us towards the spectral theorem for the proof of Proposition 6. We thank them, and Kenta Cho for their friendly help.

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[Notb] Of course, the elements of $L^2(X)$ which are equal almost everywhere are identified.

[Notc] Of course, the elements of $L^\infty(X)$ which are equal almost everywhere are identified.

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Appendix A: C^*-algebras

Terminology 15. 1. A **C^*-algebra** $\mathscr{A}$ is a complete normed complex vector space endowed with a bilinear associative product and an antilinear map $(-)^*\colon \mathscr{A} \to \mathscr{A}$ such that $a^{**} = a$, $(ab)^* = b^*a^*$, $\|ab\| \leq \|a\|\|b\|$, and$\|a^*a\| = \|a\|^2$ for all $a, b \in \mathscr{A}$.

(The last equation is called the *C^*-identity*.)

2. An element a of a C^*-algebra $\mathscr{A}$ is called
   (a) **positive** if $a \equiv b^*b$ for some $b \in \mathscr{A}$;
   (b) **self-adjoint** if $a^* = a$;
   (c) a **projection** if $a^*a = a$;
   (d) **central** if $ab = ba$ for all $b \in \mathscr{A}$;
   (e) a **unit** if $ab = ba = b$ for all $b \in \mathscr{A}$.

   The set of positive elements of $\mathscr{A}$ is denoted by $\mathscr{A}_+$, and the set of self-adjoint elements of $\mathscr{A}$ by $\mathscr{A}_{\mathrm{sa}}$.

3. A C^*-algebra is partially ordered by as follows. For all $a, b \in \mathscr{A}$, we have $a \leq b$ iff $b - a$ is positive.

4. A C^*-algebra $\mathscr{A}$ is
   (a) **unital** if $\mathscr{A}$ contains a unit, 1;
   (b) **commutative** if $ab = ba$ for all $a, b \in \mathscr{A}$;
   (c) a **factor** if $\mathscr{S}$ is unital and all its central elements are of the form $\lambda \cdot 1$ where $\lambda \in \mathbb{C}$.

5. Let $\mathscr{A}$ and $\mathscr{B}$ be C^*-algebras. A linear map $f\colon \mathscr{A} \to \mathscr{B}$ is called
   (a) **bounded** if $\|f\| < \infty$, where

$$\|f\| \;=\; \sup\{\, \lambda \in [0, \infty)\colon\; \forall a \in \mathscr{A}[\; \|f(a)\| \leq \lambda\|a\|\;]\,\}.$$

   (b) **contractive** if $\|f\| \leq 1$;
   (c) a **$*$-homomorphism** if $f(ab) = f(a)f(b)$ and $f(a^*) = f(a)^*$ for all $a, b \in \mathscr{A}$;
   (d) a **$*$-isomorphism** if f is a bijective $*$-homomorphism;
   (e) **positive** if $f(a) \in \mathscr{B}_+$ for all $a \in \mathscr{A}_+$;
   (f) **unital** if $\mathscr{A}$ and $\mathscr{B}$ are unital, and $f(1) = 1$;
   (g) **normal** if for every directed subset D of self-adjoint elements of $\mathscr{A}$: if D has a supremum $\bigvee D$ in $\mathscr{A}_{\mathrm{sa}}$, then $f(\bigvee D)$ is the supremum of $\{f(d)\colon d \in D\}$ in $\mathscr{B}_{\mathrm{sa}}$;
   (h) a **process** if f is normal, completely positive and contractive.

6. Let $\mathscr{A}$ be a unital C^*-algebra. A **state** of $\mathscr{A}$ is a positive unital linear map $\varphi\colon \mathscr{A} \to \mathbb{C}$.

7. A **C^*-subalgebra** of a C^*-algebra $\mathscr{A}$ is a norm closed linear subspace S of $\mathscr{A}$ such that $ab \in \mathscr{S}$ and $a^* \in \mathscr{S}$ for all $a, b \in \mathscr{S}$. (Such a set $\mathscr{S}$ is itself a C^*-algebra in the obvious way.)

8. For every positive element a of a C^*-algebra $\mathscr{A}$ there is a unique positive $b \in \mathscr{A}$ with $a = b^2$ and $ba = ab$. We write $\sqrt{a} = b$.

Example 16. Let X be a compact Hausdorff space. The commutative unital *C^*-algebra of continuous functions on X* is the set $C(X)$ of continuous complex-valued functions on X endowed with the supremum norm and coordinatewise operations.

Theorem 17 (Gel'fand–Neumark)**.** *Every commutative unital C^*-algebra is $*$-isomorphic to a C^*-algebra of continuous functions on a compact Hausdorff space.*

Proof. Apply Theorem 2.1 of[Con90]. □

Example 18. Let $\mathscr{H}$ be a Hilbert space. The bounded operators on $\mathscr{H}$ form a unital C^*-algebra, $\mathscr{B}(\mathscr{H})$, in which the product is given by composition, $(-)^*$ is the adjoint, and the norm is the operator norm. Moreover, $\mathscr{B}(\mathscr{H})$ is a factor (of "type I"), and $A \in \mathscr{B}(\mathscr{H})$ is positive iff $0 \leq \langle x, Ax\rangle$ for all $x \in \mathscr{H}$.

A *C^*-algebra of bounded operators on $\mathscr{H}$* is a C^*-subalgebra $\mathscr{B}(\mathscr{H})$ of bounded operators on $\mathscr{H}$ (but need not be a factor).

Theorem 19 (Gel'fand–Neumark–Segal)**.** *Every unital C^*-algebra is $*$-isomorphic to a C^*-algebra of bounded operators on a Hilbert space.*

Proof. Unfold Theorem 5.17 of[Con90]. □

The norm determines the order:

Lemma 20. *Let $\mathscr{A}$ be a unital C^*-algebra, and $a \in \mathscr{A}_{\mathrm{sa}}$. Then $a \geq 0$ iff $\|\,\|a\| - a\,\| \;\leq\; \|a\|$.*

Proof. See VIII/Theorem 3.6 of[Con90]. □

Proposition 21. *Let $\mathscr{A}$ and $\mathscr{B}$ be unital C^*-algebras, and let $f\colon \mathscr{A} \to \mathscr{B}$ be a unital $*$-homomorphism.*

Then f is contractive, and $f(\mathscr{A})$ is norm closed and in fact a C^*-subalgebra of $\mathscr{B}$.

Moreover, if f is injective, then, for all $a \in \mathscr{A}$, we have $\|f(a)\| = \|a\|$, and $f(a) \geq 0$ iff $a \geq 0$.

Proof. Use Theorem VIII/4.8 of[Con90] and Lem. 20. □

If we apply the proposition above to the inclusion of a C^*-subalgebra, then we get the following desirable result.

Corollary 22. *Let $\mathscr{A}$ be a unital C^*-algebra, and let a be an element of a unital C^*-subalgebra $\mathscr{B}$ of $\mathscr{A}$.*

Then $\|a\|_{\mathscr{A}} = \|a\|_{\mathscr{B}}$, and $a \in \mathscr{A}_+$ iff $a \in \mathscr{B}_+$.

The order also determines the norm:

Corollary 23. *Let $\mathscr{A}$ be a unital C^*-algebra. Then*

$$\|a\| \;=\; \min\{\,\lambda \in [0,\infty)\colon\; -\lambda \leq a \leq \lambda\,\} \qquad \text{(A1)}$$

for any self-adjoint element a of $\mathscr{A}$.

Proof. Note that if $\mathscr{A} = C(X)$ for some compact Hausdorff space, then (A1) is evidently correct, because the norm on $C(X)$ is the supnorm. Thus, (A1) is also correct if $\mathscr{A}$ is commutative, since in that case $\mathscr{A}$ is $*$-isomorphic to some $C(X)$ by Theorem 17.

In general, however, $\mathscr{A}$ need not be commutative, but the C^*-subalgebra, $C^*(a)$, generated by a is commutative. Thus, since the order and the norm on $C^*(a)$ agree with the order and norm on $\mathscr{A}$ by Corollary 22, (A1) holds on $\mathscr{A}$ (because it holds on $C^*(a)$). □

Example 24. Let $\mathscr{A}$ be a C^*-algebra, and let $\mathscr{S}$ be a subset of $\mathscr{A}$. Then $\mathscr{S}^\square = \{\, a \in \mathscr{A}\colon\, \forall s \in \mathscr{S}\;[\,as = sa\,]\,\}$, the *commutant of $\mathscr{S}$*, is a C^*-subalgebra of $\mathscr{A}$ provided that $s^* \in \mathscr{S}$ for all $s \in \mathscr{S}$.

Corollary 25. *If an element, a, of a C^*-algebra commutes with $b \geq 0$, then a commutes with $\sqrt{b}$.*

Terminology 26. Let $\mathscr{A}$ be a C^*-algebra (of operators on a Hilbert space $\mathscr{H}$) and let $N \in \mathbb{N}$. By $M_N(\mathscr{A})$ we denote the set of $N \times N$-matrices over $\mathscr{A}$ which is itself a C^*-algebra (of operators on the Hilbert space $\mathscr{H}^{\oplus N}$).

Let $\mathscr{A}$ and $\mathscr{B}$ be C^*-algebras. Let $f\colon \mathscr{A} \to \mathscr{B}$ be a linear map. We say that f is N**-positive** if for every positive $N \times N$-matrix $(A_{ij})_{ij}$ over $\mathscr{A}$ the $N \times N$-matrix $(f(A_{ij}))_{ij}$ over $\mathscr{B}$ is positive in $M_N(\mathscr{B})$. f is **completely positive** if f is N-positive for all $N \in \mathbb{N}$.[Sti55]

Lemma 27. *Let a be an element and p a projection in a unital C^*-algebra $\mathscr{A}$. If $a^*a \leq p$, then $ap = a$.*

Proof. Follows from Lemma 7. □

Corollary 28. *Let $\mathscr{A}$ be a unital C^*-algebra. For every projection p in $\mathscr{A}$ and $a \in \mathscr{A}$ with $0 \leq a \leq p$, we have $ap = pa = a$.*

Corollary 29. *Let p, q be projections with $p + q \leq 1$ in a unital C^*-algebra. Then $pq = qp = 0$.*

Lemma 30. *For an element p of a unital C^*-algebra $\mathscr{A}$, the following are equivalent.*

1. *p is a projection.*
2. *$a \leq p$ and $a \leq 1 - p$ entails $a = 0$ for all $a \in \mathscr{A}_+$.*

Proof. (1⟹2) Let $a \in \mathscr{A}_+$ with $a \leq p$ and $a \leq 1 - p$ be given. Since $ap = a$ and $a(1-p) = a$ by Corollary 28, we get $a = ap + a(1-p) = 2a$, and so $a = 0$.

(2⟹1) We may assume that $\mathscr{A}$ is commutative (by considering the C^*-subalgebra generated by $\{a\}$ instead), and so $\mathscr{A} \cong C(X)$ for some compact Hausdorff space by Theorem 17.

Then $a \in C(X)$ given by $a(x) = \min\{p(x), 1-p(x)\}$ for all $x \in X$ is positive and below both p and $1-p$. Thus $a = 0$ by assumption. Then, for all $x \in X$, either $p(x) = 0$ or $1 - p(x) = 0$. Thus p takes only the values 0 and 1, and is therefore easily seen to be a projection. □

Corollary 31. *Let $f\colon \mathscr{A} \to \mathscr{B}$ be an invertible positive unital linear map between unital C^*-algebras, such that f^{-1} is positive. Then f preserves projections.*

Appendix B: Von Neumann Algebras

Terminology 32. A **von Neumann algebra** is a unital C^*-algebra $\mathscr{A}$ such that: *(I)* every bounded directed set of self-adjoint elements of $\mathscr{A}$ has a supremum in $\mathscr{A}_{\text{sa}}$, and *(II)* for every positive $a \in \mathscr{A}$: if $\varphi(a) = 0$ for every *normal* state φ of $\mathscr{A}$, then $a = 0$.[Kad56]

A **von Neumann subalgebra** of a von Neumann algebra $\mathscr{A}$ is a C^*-subalgebra $\mathscr{S}$ of $\mathscr{A}$ such that for every bounded directed set D of $\mathscr{S}_{\text{sa}}$ we have $\bigvee D \in \mathscr{S}$, where $\bigvee D$ is the supremum of D in $\mathscr{A}_{\text{sa}}$.

Terminology 33. Let $\mathscr{A}$ be a C^*-algebra. Given a net $(a_i)_i$ in $\mathscr{A}$ and $b \in \mathscr{A}$,—

1. $(a_i)_i$ converges **ultraweakly** to b if for every normal state φ of $\mathscr{A}$,
$$(\varphi(a_i))_i \quad \text{converges to} \quad \varphi(b);$$
2. and — provided $\mathscr{A}$ is a C^*-subalgebra of the space of bounded operators $\mathscr{B}(\mathscr{H})$ on a Hilbert space $\mathscr{H}$ — $(a_i)_i$ converges **weakly** to b (with respect to $\mathscr{H}$) if for all $x \in \mathscr{H}$,
$$(\langle a_i x, x\rangle)_i \quad \text{converges to} \quad \langle bx, x\rangle\,.$$

Example 34. Let $\mathscr{H}$ be a Hilbert space. Then $\mathscr{B}(\mathscr{H})$ is a von Neumann algebra.

Theorem 35 (Kadison)**.** *For a C^*-algebra of bounded operators on a Hilbert space, the following are equivalent.*

1. *$\mathscr{A}$ is a von Neumann subalgebra of $\mathscr{B}(\mathscr{H})$;*
2. *$\mathscr{A}$ is weakly closed in $\mathscr{B}(\mathscr{H})$.*

Proof. This follows from Lemma 1 of[Kad56]. □

Theorem 36 (Kadison)**.** *Any von Neumann algebra is $*$-isomorphic to a von Neumann subalgebra of $\mathscr{B}(\mathscr{H})$ for some Hilbert space $\mathscr{H}$.*

Moreover, $\mathscr{H}$ can be chosen in such a way that the ultraweak topology on $\mathscr{A}$ coincides with weak topology on $\mathscr{A}$ induced by $\mathscr{B}(\mathscr{H})$

Proof. That $\mathscr{A}$ is $*$-isomorphic to a von Neumann algebra of bounded operators on some Hilbert space $\mathscr{H}$ follows from Theorem 1 of[Kad56]. That the ultraweak topology on $\mathscr{A}$ coincides with the weak topology on $\mathscr{A}$ induced by $\mathscr{H}$ follows from the way the Hilbert space $\mathscr{H}$ is constructed in the first paragraph of the proof of Theorem 1[Kad56] (if we take $(\omega_\alpha)_{\alpha\in\Gamma}$ to be the collection of all normal states): for every normal state ω of $\mathscr{A}$ there is $x \in \mathscr{H}$ with $\omega(a) = \langle x, ax\rangle$ for all $a \in \mathscr{A}$. □

Example 37. Let X be a measure space. Then the C^*-algebra $L^\infty(X)$ of bounded measurable complex-valued functions on X (in which two such functions are identified when they are equal almost everywhere) is a commutative von Neumann algebra and the map $\varrho\colon L^\infty(X) \to \mathscr{B}(L^2(X))$ given by $\varrho(f)(g) = \int fgd\mu$ is an injective normal $*$-homomorphism, where $L^2(X)$ is the Hilbert space of square integrable complex-valued functions on X (in which two such functions are identified when they are equal almost everywhere).

Theorem 38 (Spectral Theorem)**.** *For every self-adjoint bounded operator A on a Hilbert space $\mathscr{H}$, there is a measure space X, an element a of $L^\infty(X)$, and a unitary $U\colon L^2(X) \to \mathscr{H}$, such that $U^*AU = \int a \cdot - d\mu$.*

Proof. See[Hal63]. □

Proposition 39. *Let D be a directed bounded set of self-adjoint elements of a von Neumann algebra $\mathscr{A}$.*

Let $b \in \mathscr{A}$. If b commutes with all $d \in D$, then b commutes with $\bigvee D$.

Proof. We may assume (by Theorem 36) without loss of generality that $\mathscr{A}$ is a von Neumann subalgebra of $\mathscr{B}(\mathscr{H})$ for some Hilbert space $\mathscr{H}$. Since $(d)_{d\in D}$ converges strongly to $\bigvee D$ (see Lemma 5.1.4 of[KR97]) we see that $(bd)_{d\in D}$ converges weakly to $b(\bigvee D)$. Since $bd = db$ for all $d \in D$, and $(db)_{d\in D}$ converges weakly to $(\bigvee D)b$ by a similar reasoning, we get $(\bigvee D)b = b(\bigvee D)$. □

Proposition 40. *Let $(a_i)_i$ be a net in a von Neumann algebra $\mathscr{A}$ such that*

1. *$(a_i)_i$ is* norm bounded*, that is $\sup_i \|a_i\| < \infty$, and*
2. *$(a_i)_i$ is* ultraweakly Cauchy*, that is, $(\varphi(a_i))_i$ is Cauchy for every normal state $\varphi\colon \mathscr{A} \to \mathbb{C}$.*

Then $(a_i)_i$ converges ultraweakly.

Proof. By Theorem 36, we may assume without loss of generality that $\mathscr{A}$ is a von Neumann algebra of bounded operators on some Hilbert space $\mathscr{H}$ such that the weak topology on $\mathscr{A}$ induced by $\mathscr{H}$ coincides with the ultraweak topology.

Let $x \in \mathscr{H}$ be given. Note that if $\|x\| = 1$, then $\langle x, -x\rangle : \mathscr{A} \to \mathbb{C}$ is a normal state, and so $(\langle x, a_i x\rangle)_i$ is Cauchy. It follows easily that $(\langle x, a_i x\rangle)_i$ is Cauchy for all $x \in \mathscr{H}$.

Let $x, y \in \mathscr{H}$ be given. Since for all $a \in \mathscr{A}$,

$$|\langle x, ay\rangle|^2 \;\leq\; \langle x, ax\rangle\,\langle y, ay\rangle\,,$$

we see that $(\langle x, a_i y\rangle)_i$ is Cauchy.

Since $(x, y) \mapsto \lim_i \langle x, a_i y\rangle$ gives a bilinear map on $\mathscr{H}$, which is bounded because $(a_i)_i$ is norm bounded, there is, by Riesz's representation theorem, a bounded operator a on $\mathscr{H}$ with $\langle ax, y\rangle = \lim_i \langle a_i x, y\rangle$ for all $x, y \in \mathscr{H}$.

Note that $(a_i)_i$ converges weakly to a. Thus $a \in \mathscr{A}$, because $\mathscr{A}$ is weakly closed by Theorem 35. Further, $(a_i)_i$ converges ultraweakly to a as well, because the weak and ultraweak topologies coincide on $\mathscr{A}$ by choice of $\mathscr{H}$. □

Lemma 41. *Let a be an element of a von Neumann algebra $\mathscr{A}$. Then the linear map $c\colon \mathscr{A} \to \mathscr{A}$, $b \mapsto a^*ba$ is normal and completely positive.*

Proof. (Normality) follows from Lemma 1.7.4 of[Sak71]. *(Complete positivity)* follows from Theorem 1 of[Sti55], but let us give an elementary proof.

Let $N \in \mathbb{N}$ be given. Let B be a positive $N\times N$-matrix over $\mathscr{A}$. We must show that $(a^*B_{ij}a)_{ij}$ is a positive $N \times N$-matrix over $\mathscr{A}$. Since B is positive, there is a $N \times N$-matrix C with $B = C^*C$. Note that

$$(a^*B_{ij}a)_{ij} \;=\; A^*BA \;\equiv\; A^*C^*CA \;=\; (CA)^*CA \;\geq 0,$$

where $A = (a)_{ij}$ is a diagonal $N \times N$-matrix. Thus c is completely positive. □

Corollary 42. *For every projection p of a von Neumann algebra $\mathscr{A}$, $p\mathscr{A}p$ is a von Neumann subalgebra of $\mathscr{A}$.*

Proof. Surely, $p\mathscr{A}p$ is a $*$-subalgebra of $\mathscr{A}$ with unit p. Since $\|pap-pbp\| \leq \|p\|\|a-b\|\|p\|$ for all $a, b \in \mathscr{A}$, we see that $p\mathscr{A}p$ is norm closed, and $p\mathscr{A}p$ is a C^*-subalgebra.

Let D be a bounded directed subset of $(p\mathscr{A}p)_{\mathrm{sa}}$. To prove that $p\mathscr{A}p$ is a von Neumann subalgebra, it suffices to show that the supremum $\bigvee D$ of D in $\mathscr{A}_{\mathrm{sa}}$ is in $p\mathscr{A}p$.

Since $a \mapsto pap$ is normal on $\mathscr{A}$ by Lemma 41, and we have $d = pdp$ for all $d \in D$, we see that $p(\bigvee D)p = \bigvee_{d\in D} pdp = \bigvee D$, and so $\bigvee D \in \mathscr{A}$. □

Proposition 43. *Let $\mathscr{A}$ be a von Neumann algebra. Let $a \in \mathscr{A}$ with $0 \leq a \leq 1$ be given.*

1. *There is a smallest projection, $\lceil a\rceil$, above a.*
2. *$\lceil a\rceil$ is the supremum of $a \leq a^{1/2} \leq a^{1/4} \leq a^{1/8} \leq \cdots$.*
3. *Then $ab = ba$ implies $\lceil a\rceil b = b\lceil a\rceil$ for all $b \in \mathscr{B}$.*

Proof. Let p be the supremum of a, $a^{1/2}$, $a^{1/4}$, … in $\mathscr{A}_{\mathrm{sa}}$. Let q be a projection in $\mathscr{A}$ with $a \leq q$. Then $aq = qa = a$ by Corollary 28, and so $a^{1/2}q = qa^{1/2}$ by Corollary 25. Since $a(1-q) = 0$, we have

$$\|\sqrt{a}(1-q)\|^2 \;=\; \|(1-q)a(1-q)\| = 0$$

by the C^*-identity, and so $\sqrt{a}(1-q) = 0$, and thus $\sqrt{a}q = \sqrt{a}$. Then $\sqrt{a} = \sqrt{a}q^2 = q\sqrt{a}q \leq q$. With a similar reasoning, we get $a^{1/4} \leq q$, and $a^{1/8} \leq q$, and so on. It follows that $p \leq q$, by definition of p.

Thus, to show that p is the least projection above a, we only need to show that p is a projection. Since $0 \leq p \leq 1$ (and thus $p^2 \leq p$) it suffices to show that $p \leq p^2$.

First note that any $b \in \mathscr{A}$ that commutes with a, commutes with $a^{1/2}$, and with $a^{1/4}$, etc., and thus b commutes with p by Proposition 39.

In particular, since each $a^{1/2^n}$ commutes with a, we see that $a^{1/2^n}$ commutes with p. Then, by Lemma 41,

$$\begin{aligned} p^2 &= \sqrt{p}p\sqrt{p} \\ &= \bigvee_n \sqrt{p}\, a^{1/2^n} \sqrt{p} \\ &= \bigvee_n a^{1/2^{n+1}}\; p\; a^{1/2^{n+1}} \\ &= \bigvee_n \bigvee_m a^{1/2^{n+1}} a^{1/2^m} a^{1/2^{n+1}}. \end{aligned}$$

Thus $p^2 \geq a^{1/2^k}$ for every $k \in \mathbb{N}$, and so $p^2 \geq p$.

Hence p is a projection. □

Proposition 44. *Let $f\colon \mathscr{A} \to \mathscr{B}$ be a positive linear contraction between von Neumann algebras. Let $a \in \mathscr{A}$. Then $f(\lceil a \rceil) \leq \lceil f(a) \rceil$, and $\lceil f(\lceil a \rceil)\rceil = \lceil f(a) \rceil$.*

Proof. Since $\lceil a \rceil = \bigvee_n a^{1/2^n}$ by Proposition 43, and f is normal, we have

$$f(\lceil a \rceil) \;=\; \bigvee_n f(a^{1/2^n}) \;\overset{(*)}{\leq}\; \bigvee_n f(a)^{1/2^n} \;=\; \lceil f(a) \rceil.$$

To justify Inequality $(*)$ we claim that $f(\sqrt{b}) \leq \sqrt{f(b)}$ for all $b \in \mathscr{B}_+$. Since $\sqrt{-}$ is order preserving[Ped72], it suffices to show that $f(\sqrt{b})^2 \leq f(\sqrt{b}^2)$, and this has been done in Theorem 1 of[Kad52].

Let prove that $\lceil f(\lceil a \rceil)\rceil = \lceil f(a) \rceil$. On the one hand, we have $\lceil f(\lceil a \rceil)\rceil \geq \lceil f(a) \rceil$, because $\lceil a \rceil \geq a$. On the other hand, since $\lceil f(a) \rceil$ is a projection, and we have just shown that $f(\lceil a \rceil) \leq \lceil f(a) \rceil$, we get $\lceil f(\lceil a \rceil)\rceil \leq \lceil f(a) \rceil$ by definition of $\lceil f(a) \rceil$. □

Theorem 45 (Gardner)**.** *For a positive linear map $f\colon \mathscr{A} \to \mathscr{B}$ between unital C*-algebras, the following are equivalent.*

(ii) $f(1) \cdot f(ab) = f(a) \cdot f(b)$ *for all* $a, b \in \mathscr{A}$.

(iii)′ f *is 2-positive, and for all* $a, b \in \mathscr{A}_+$ *with* $ab = 0$ *we have* $f(a)f(b) = 0$.

Proof. See Theorem 2 of[Gar79]. □

Proposition 46. *For a 2-positive normal unital linear map $f\colon \mathscr{A} \to \mathscr{B}$ between von Neumann algebras the following are equivalent.*

1. *f is a $*$-homomorphism.*
2. *f preserves projections.*
3. *$\lceil f(a) \rceil = f(\lceil a \rceil)$ for every $a \in [0,1]_{\mathscr{A}}$.*

Proof. (1 $\Longrightarrow$ 2) Easy.

(2 $\Longrightarrow$ 3) Let $a \in [0,1]_{\mathscr{A}}$ be given. By Proposition 44 we have $\lceil f(a) \rceil = \lceil f(\lceil a \rceil)\rceil = f(\lceil a \rceil)$, where the latter equality follows from the fact that $f(\lceil a \rceil)$ is a projection.

(3 $\Longrightarrow$ 1) Let $a, b \in \mathscr{A}_+$ with $ab = 0$ be given. To prove that f is multiplicative, it suffices to show that $f(a)f(b) = 0$ by Theorem 45 (since $f(1) = 1$).

If either a or b is zero, we are done, so we may assume that $a \neq 0$ and $b \neq 0$. We may also assume that $a, b \leq 1$ (by replacing them by $a/\|a\|$ and $b/\|b\|$ if necessary).

It suffices to show that $\lceil f(a) \rceil \lceil f(b) \rceil = 0$, because then $f(a)f(b) = f(a)\lceil f(a) \rceil \lceil f(b) \rceil f(b) = 0$, where we used that $f(a) = f(a)\lceil f(a) \rceil$ (see Proposition 43).

Note that a and b commute, because $ba = b^*a^* = (ab)^* = 0 = ab$. Then $\sqrt{a}$ and $\sqrt{b}$ commute as well, and so $\sqrt{a}b\sqrt{a} = ab = 0$. Then $0 \leq \sqrt{a}\lceil b \rceil\sqrt{a} \leq \lceil \sqrt{a}b\sqrt{a} \rceil = 0$, and so $a\lceil b \rceil = 0$. By repeating this argument, we see that $\lceil a \rceil \lceil b \rceil = 0$.

It follows that $\lceil a \rceil + \lceil b \rceil$ is a projection, and

$$\lceil f(a) \rceil \,+\, \lceil f(b) \rceil \;=\; f(\lceil a \rceil) \,+\, f(\lceil b \rceil) \;\leq\; f(1) \;=\; 1.$$

Thus Corollary 29 implies that $\lceil f(a) \rceil \lceil f(b) \rceil = 0$. □

Corollary 47. *Let f be an invertible process between von Neumann algebras such that f^{-1} is a process as well. Then f is a $*$-isomorphism.*

Proof. Since $f(1) \leq 1 = f(f^{-1}(1))$ we have $1 \leq f^{-1}(1) \leq 1$, and so $f^{-1}(1) = 1$. Thus both f and f^{-1} are unital. Then f preserves projections by Corollary 31, and is thus a $*$-homomorphism by Proposition 46.

Hence f is a $*$-isomorphism. □

Appendix C: Ultraweak limits of maps

Lemma 48. *For a positive linear map $f\colon \mathscr{A} \to \mathscr{B}$ between von Neumann algebras the following are equivalent.*

1. *f is normal.*
2. *f is ultraweakly continuous.*
3. *The restriction of f to a map $[0,1]_{\mathscr{A}} \to \mathscr{B}$ is ultraweakly continuous.*

Proof. (1 $\Longrightarrow$2) Let $\varphi\colon \mathscr{B} \to \mathbb{C}$ be a normal state. To prove that f is ultraweakly continuous we must show that $\varphi \circ f\colon \mathscr{A} \to \mathbb{C}$ is continuous with respect to the ultraweak topology on $\mathscr{A}$ and the standard topology on $\mathbb{C}$. It suffices to show that $\varphi \circ f$ is normal, which indeed it is, as both φ and f are normal.

(2 $\Longrightarrow$3) is trivial.

(3 $\Longrightarrow$1) Let D be a bounded directed set of self-adjoint elements of $\mathscr{A}$ with supremum, $\bigvee D$. Then as f is positive, $\{\, f(d) \colon d \in D \,\}$ is directed and bounded by $f(\bigvee D\,)$, and thus has a supremum, $\bigvee_{d \in D} f(d)$. To show that f is normal, we must prove that $f(\bigvee D\,) = \bigvee_{d \in D} f(d)$. Since f is linear, we may assume without

loss of generality that $D \subseteq [0,1]_{\mathscr{A}}$. Let $\varphi\colon \mathscr{B} \to \mathbb{C}$ be a normal state. It suffices to show that

$$\varphi(f(\bigvee D)) \;=\; \varphi(\bigvee_{d\in D} f(d)). \tag{C1}$$

Note that D (as net) converges ultraweakly to $\bigvee D$ in $\mathscr{A}$, and thus in $[0,1]_{\mathscr{A}}$ as well. Since the restriction of f to $[0,1]_{\mathscr{A}}$ is ultraweakly continuous, the net $(f(d))_{d\in D}$ converges ultraweakly to $f(\bigvee D)$ in $\mathscr{B}$. So $(\varphi(f(d)))_{d\in D}$ converges to $\varphi(f(\bigvee D))$. Since $(\varphi(f(d)))_{d\in D}$ is directed, $\varphi(f(\bigvee D))$ is in fact its supremum. Finally, since φ is normal, $\varphi(\bigvee_{d\in D} f(d)) = \bigvee_{d\in D}\varphi(f(d)) = \varphi(f(\bigvee D))$. We have proven Statement (C1), so f is normal. □

Corollary 49. *Let $f\colon \mathscr{A}\to\mathscr{B}$ be a positive linear map between von Neumann algebras. Let $(f_\alpha)_{\alpha\in D}$ be a net of normal positive linear maps from $\mathscr{A}$ to $\mathscr{B}$ which converges uniformly on $[0,1]_{\mathscr{A}}$ ultraweakly to f.*

Then f is normal.

Proof. The uniform limit of continuous functions is continuous. In particular, since the f_α (being normal and hence ultraweakly continuous) converge uniformly on $[0,1]_{\mathscr{A}}$ to f, we see that the restriction of f to $[0,1]_{\mathscr{A}}$ is ultraweakly continuous, and thus f is normal by Lemma 48. □

Lemma 50. *Let $\mathscr{B}$ be a C^*-algebra of operators on a Hilbert space $\mathscr{H}$. Let $\mathscr{A}$ be a C^*-algebra. Let $(f_\alpha)_{\alpha\in D}$ be a net of completely positive linear maps from $\mathscr{A}$ to $\mathscr{B}$ which converges pointwise weakly to a linear map $f\colon\mathscr{A}\to\mathscr{B}$. Then f is completely positive.*

Proof. Let A be a positive $N\times N$-matrix over $\mathscr{A}$ for some $N\in\mathbb{N}$. We must show that $(f(A_{ij}))_{ij}$ is a positive $N\times N$-matrix over $\mathscr{B}$. Note that the $N\times N$-matrices over $\mathscr{B}$ can be considered a C^*-subalgebra of operators on $\mathscr{H}^{\oplus N}$. To prove that $(f(A_{ij}))_{ij}$ is positive, we will show that $(f_\alpha(A_{ij}))_{ij}$ converges to $(f(A_{ij}))_{ij}$ weakly with respect to $\mathscr{H}^{\oplus N}$. (This is sufficient, because the weak limit of positive operators is positive, and each $(f_\alpha(A_{ij}))_{ij}$ is positive.)

Let $x,y\in\mathscr{H}^{\oplus N}$ be given. To show that $(f_\alpha(A_{ij}))_{ij}$ converges to $(f(A_{ij}))_{ij}$ in the weak operator topology we must show that

$$\begin{aligned}&\langle\, (f(A_{ij})-f_\alpha(A_{ij}))_{ij}\; x,\; y\,\rangle\\ &\qquad\equiv\; \textstyle\sum_{i,j}\langle\,(f(A_{ij})-f_\alpha(A_{ij}))\;x_j,\; y_i\,\rangle\end{aligned} \tag{C2}$$

converges to 0 as $\alpha\to\infty$. Let $i,j\in\{1,\dots,N\}$ be given. Since f_α converges pointwise in the weak operator topology to f, $\langle (f_\alpha(A_{ij})-f(A_{ij}))x_j, y_i\rangle$ converges in $\mathbb{C}$ to 0. Thus the right-hand side of Equality (C2), being a finite sum of such terms, converges to 0 as $\alpha\to\infty$. Thus f is completely positive. □

Corollary 51. *Let $\mathscr{A}$ and $\mathscr{B}$ be von Neumann algebras. Let $(f_\alpha)_{\alpha\in D}$ be a net of completely positive linear maps from $\mathscr{A}$ to $\mathscr{B}$ which converges pointwise ultraweakly to a linear map $f\colon\mathscr{A}\to\mathscr{B}$. Then f is completely positive.*

Appendix D: Cauchy–Schwarz for 2-Positive Maps

The classical form of the Cauchy–Schwarz inequality is that for any vectors x and y in a complex vector space $\mathscr{X}$ with semi-inner product $\langle -,-\rangle$ we have

$$\left|\langle x,y\rangle\right|^2 \;\leq\; \langle x,x\rangle\;\langle y,y\rangle\,.$$

Since any positive functional φ on a C^*-algebra $\mathscr{A}$ gives a semi-inner product on $\mathscr{A}$ by $\langle a,b\rangle = \varphi(a^*b)$,

$$\left|\varphi(a^*b)\right|^2 \;\leq\; \varphi(a^*a)\,\varphi(b^*b). \tag{D1}$$

This is known as *Kadison's inequality*. We need the following generalization. Given a 2-positive linear map $\varphi\colon\mathscr{A}\to\mathscr{B}$ we have, for all $a,b\in\mathscr{A}$,

$$\|\varphi(a^*b)\|^2 \;\leq\; \|\varphi(a^*a)\|\,\|\varphi(b^*b)\|. \tag{D2}$$

Since it is an exercise in[Pau02] and seems not to be mentioned elsewhere we have included a proof of Inequality (D2) in this subsection (see Theorem 54).

Recall that a linear map $\varphi\colon\mathscr{A}\to\mathscr{B}$ is 2-positive whenever $\left[\begin{smallmatrix}\varphi(a) & \varphi(b)\\ \varphi(c) & \varphi(d)\end{smallmatrix}\right]$ is positive for every positive matrix $\left[\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right]$ with $a,b,c,d\in\mathscr{A}$. The trick behind the proof of Inequality (D2) is to analyze which 2×2 matrices of operators on a Hilbert space are positive (see Lemma 53). Let us first recall the situation for 2×2-matrices over $\mathbb{C}$.

Lemma 52. *Let $T\equiv\left[\begin{smallmatrix}p & a\\ a^* & q\end{smallmatrix}\right]$ be a self-adjoint 2×2 matrix over $\mathbb{C}$. The following are equivalent.*

1. *T is positive;*
2. *T has positive eigenvalues;*
3. *T has positive determinant and positive trace;*
4. *$p,q\geq 0$ and $|a|^2\leq pq$.*

Proof. We leave this to the reader. □

Lemma 53. *Let $T\equiv\left[\begin{smallmatrix}P & A\\ A^* & Q\end{smallmatrix}\right]$ be a self-adjoint 2×2 matrix of bounded operators on a Hilbert space $\mathscr{H}$. The following are equivalent.*

1. *T is positive.*
2. *$P,Q\geq 0$, and for all $x,y\in\mathscr{H}$,*

$$\left|\langle Ay,x\rangle\right|^2 \;\leq\; \langle Px,x\rangle\;\langle Qy,y\rangle\,. \tag{D3}$$

Moreover, if T is positive, then:

3. *$A^*A\leq\|P\|\,Q$*
4. *$AA^*\leq\|Q\|\,P$*
5. *$\|A\|^2\leq\|P\|\,\|Q\|$*

Proof. *(1 $\implies$ 2)* Let $x, y \in \mathscr{H}$ be given. Let us consider $T' := \left[\begin{smallmatrix} \langle Px,x\rangle & \langle Ay,x\rangle \\ \langle A^*x,y\rangle & \langle Qy,y\rangle \end{smallmatrix}\right]$. Since T is self-adjoint, T' is self-adjoint. Further, given $\lambda, \mu \in \mathbb{C}$ we have

$$\left\langle \begin{bmatrix} P & A \\ A^* & Q \end{bmatrix} \begin{bmatrix} \lambda x \\ \mu y \end{bmatrix}, \begin{bmatrix} \lambda x \\ \mu y \end{bmatrix} \right\rangle = \left\langle \begin{bmatrix} \langle Px,x\rangle & \langle Ay,x\rangle \\ \langle A^*x,y\rangle & \langle Qy,y\rangle \end{bmatrix} \begin{bmatrix} \lambda \\ \mu \end{bmatrix}, \begin{bmatrix} \lambda \\ \mu \end{bmatrix} \right\rangle.$$

From this we see that as T is positive, T' is positive. Then by Lemma 52 we get $\langle Px,x\rangle \geq 0$, $\langle Qy,y\rangle \geq 0$, and $|\langle Ay,x\rangle|^2 \leq \langle Px,x\rangle \, \langle Qy,y\rangle$. Hence P and Q are positive, and Inequality (D3) holds.

(2 $\implies$ 1) We must show that T is positive. Note that T is self-adjoint since both P and Q are self-adjoint. Given $x, y \in \mathscr{H}$ we have

$$\left\langle \begin{bmatrix} P & A \\ A^* & Q \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} x \\ y \end{bmatrix} \right\rangle = \left\langle \begin{bmatrix} \langle Px,x\rangle & \langle Ay,x\rangle \\ \langle A^*x,y\rangle & \langle Qy,y\rangle \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\rangle.$$

So to show that T is positive, it suffices to show that $T' := \left[\begin{smallmatrix} \langle Px,x\rangle & \langle Ay,x\rangle \\ \langle A^*x,y\rangle & \langle Qy,y\rangle \end{smallmatrix}\right]$ is positive. By Lemma 52 we must show that $\langle Px,x\rangle \geq 0$, $\langle Qy,y\rangle \geq 0$, and $|\langle Ay,x\rangle|^2 \leq \langle Px,x\rangle \, \langle Qy,y\rangle$. The latter statement is Inequality (D3) and holds by assumption. The other two statements follow from $P \geq 0$ and $Q \geq 0$.

(3) Assume that T is positive. Let $y \in \mathscr{H}$ be given. We must show that

$$\langle A^*Ay, y\rangle \;\leq\; \|P\| \, \langle Qy, y\rangle \,. \tag{D4}$$

Note that $0 \leq \langle A^*Ay,y\rangle = \langle Ay,Ay\rangle = |\langle Ay,Ay\rangle|$. So

$$\begin{aligned} |\langle Ay,Ay\rangle|^2 \;&\leq\; \langle PAy,Ay\rangle \, \langle Qy,y\rangle \\ &\qquad \text{by Ineq. (D3) with } x = Ay \\ &\leq\; \|P\| \, \langle Ay,Ay\rangle \, \langle Qy,y\rangle \\ &\qquad \text{since } P \leq \|P\| \text{ and } 0 \leq Q. \end{aligned}$$

So either $\langle A^*Ay,y\rangle = 0$ — in which case Inequality (D4) holds trivially — or $\langle A^*Ay,y\rangle \neq 0$ in which case we get

$$\langle A^*Ay,y\rangle = \langle Ay,Ay\rangle \;\leq\; \|P\| \, \langle Qy,y\rangle \,.$$

Thus $A^*A \leq \|P\| \, Q$.

(4) follows by a similar reasoning as in 3.

(5) We have $\|A\|^2 = \|A^*A\| \leq \|\,\|P\|Q\,\| = \|P\| \, \|Q\|$ since $A^*A \leq \|P\|Q$ by 3. ☐

Theorem 54 (Cauchy–Schwarz for 2-positive maps)**.** *Let $f\colon \mathscr{A} \to \mathscr{B}$ be a 2-positive map between C^*-algebras. Then we have, for all $a, b \in \mathscr{A}$:*

1. $f(b^*a)\, f(a^*b) \;\leq\; \|f(a^*a)\| \, f(b^*b)$
2. $f(a^*b)\, f(b^*a) \;\leq\; \|f(b^*b)\| \, f(a^*a)$
3. $\|f(a^*b)\|^2 \;\leq\; \|f(a^*a)\| \, \|f(b^*b)\|$

Proof. We may assume that $\mathscr{B}$ is a C^*-subalgebra of the space of bounded linear operators $\mathscr{B}(\mathscr{H})$ on Hilbert space $\mathscr{H}$. Since $\left[\begin{smallmatrix} a^*a & a^*b \\ b^*a & b^*b \end{smallmatrix}\right] \equiv \left[\begin{smallmatrix} a^* & 0 \\ b^* & 0 \end{smallmatrix}\right] \left[\begin{smallmatrix} a & b \\ 0 & 0 \end{smallmatrix}\right]$ is positive and f is 2-positive we get that $T := \left[\begin{smallmatrix} f(a^*a) & f(a^*b) \\ f(b^*a) & f(b^*b) \end{smallmatrix}\right]$ is positive in $M_2(\mathscr{B})$, and thus T is positive in $M_2(\mathscr{B}(\mathscr{H}))$.

Now apply Lemma 53. ☐