LATTICE VALUATIONS, A GENERALISATION OF MEASURE AND INTEGRAL

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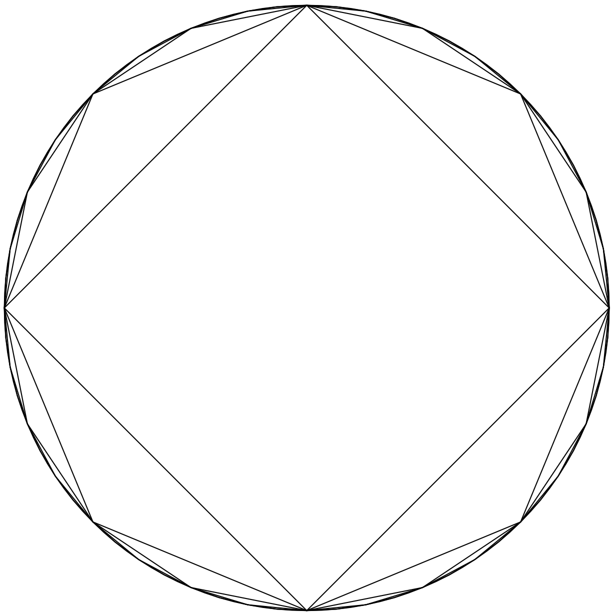

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Abstract. Measure and integral are two closely related, but distinct objects of study. Nonetheless, they are both real-valued *lattice valuations*: order preserving real-valued functions φ on a lattice L which are *modular*, i.e.,

$$\varphi(x) + \varphi(y) \;=\; \varphi(x \wedge y) + \varphi(x \vee y) \qquad (x, y \in L).$$

We unify measure and integral by developing a theory for lattice valuations. We allow these lattice valuations to take their values from the reals, or any suitable ordered Abelian group.

dedicated to my father
Henk Westerbaan
† june, 2012

Preface

In the summer of 2009 Bas Westerbaan and I worked out an overly general approach to the introduction of the Lebesgue measure and the Lebesgue integral with the help of dr. A.C.M. van Rooij. The theory that is presented in this thesis is based on the work done in that summer.

Since I was fortunate enough to be offered a Ph.D.-position, this thesis was written under time constraints. Hence the text is not nearly as polished as I would like it to be, and the proofs of some statements have been left to the reader. I hope the reader will be able to ignore the rough edges and enjoy this fresh view on the old subject of measure and integration.

I would like to thank all my teachers for showing me the beauty of mathematics. In particular, I thank dr. Mai Gehrke for showing me its elegance, dr. Wim Veldman for showing me its content, and dr. Henk Barendregt for showing me how it is written. Furthermore, I am most grateful to dr. A.C.M. van Rooij for his never relenting willingness to answer my questions and note my errors.

Contents

The theory of integration, because of its central rôle in mathematical analysis and geometry, continues to afford opportunities for serious investigation.

— M.H. Stone, 1948, [4]

1. Introduction

There are many ways (some more popular than others) to introduce the Lebesgue measure and the Lebesgue integral. For the purposes of this introduction, we define the Lebesgue measure and integral in such a way that the similarity between them is obvious. This similarity is the basis of this thesis. We leave it to the reader to compare the definitions below to those that are familiar to him/her.

Definition 1. The **Lebesgue measure** $\mu_{\mathcal{L}}\colon \mathcal{A}_{\mathcal{L}} \to \mathbb{R}$ is the smallest[1] $\mu\colon \mathcal{A} \to \mathbb{R}$ where $\mathcal{A}$ is a subset of $\wp(\mathbb{R})$ that has the following properties.

(i) Let $a, b \in \mathbb{R}$ with $a \leq b$. Then $[a,b] \in \mathcal{A}$ and $(a,b) \in \mathcal{A}$, and
$$\mu(\,[a,b]\,) \;=\; \mu(\,(a,b)\,) \;=\; b-a.$$
(ii) *(Monotonicity)* Let $A, B \in \mathcal{A}$. Then $\mu(A) \leq \mu(B)$ when $A \subseteq B$.

(iii) *(Modularity)* Let $A, B \in \mathcal{A}$. Then $A \cap B \in \mathcal{A}$ and $A \cup B \in \mathcal{A}$, and
$$\mu(\,A \cap B\,) \,+\, \mu(\,A \cup B\,) \;=\; \mu(A) \,+\, \mu(B).$$
(iv) *(Π-Completeness)* Let $A_1 \supseteq A_2 \supseteq \cdots$ from $\mathcal{A}$ be given.
Assume that the set $\{\,\mu(A_1),\, \mu(A_2),\, \dots\}$ has an infimum, $\bigwedge_n \mu(A_n)$.
Then we have $\bigcap_n A_n \in \mathcal{A}$. Moreover,
$$\mu(\,\textstyle\bigcap_n A_n\,) \;=\; \textstyle\bigwedge_n \mu(A_n).$$
(v) *(Σ-Completeness)* Let $A_1 \subseteq A_2 \subseteq \cdots$ from $\mathcal{A}$ be such that $\bigvee_n \mu(A_n)$ exists.
Then we have $\bigcup_n A_n \in \mathcal{A}$. Moreover,
$$\mu(\,\textstyle\bigcup_n A_n\,) \;=\; \textstyle\bigvee_n \mu(A_n).$$
(vi) *(Convexity)* Let $A \subseteq Z \subseteq B$ be subsets of $\mathbb{R}$.
Assume that $A, B \in \mathcal{A}$ and $\mu(A) = \mu(B)$.
Then we have $Z \in \mathcal{A}$ and $\mu(A) = \mu(Z) = \mu(B)$.

Definition 2. The **Lebesgue integral** $\varphi_{\mathcal{L}}\colon F_{\mathcal{L}} \to \mathbb{R}$ is the smallest $\varphi\colon F \to \mathbb{R}$ where F is a subset of $[-\infty, +\infty]^{\mathbb{R}}$ that has the following properties.

(i) Let $a, b, \lambda \in \mathbb{R}$ with $a \leq b$. Then $\lambda \cdot \mathbf{1}_{[a,b]} \in F$ and $\lambda \cdot \mathbf{1}_{(a,b)} \in F$, and
$$\varphi(\,\lambda \cdot \mathbf{1}_{[a,b]}\,) \;=\; \varphi(\,\lambda \cdot \mathbf{1}_{(a,b)}\,) \;=\; \lambda \cdot (b-a).$$
(ii) *(Monotonicity)* Let $f, g \in F$. Then $\varphi(f) \leq \varphi(g)$ when $f \leq g$.

(iii) *(Modularity)* Let $f, g \in F$. Then $f \wedge g \in F$ and $f \vee g \in F$, and
$$\varphi(\,f \wedge g\,) \,+\, \varphi(\,f \vee g\,) \;=\; \varphi(f) \,+\, \varphi(g).$$
(iv) *(Π-Completeness)* Let $f_1 \geq f_2 \geq \cdots$ from F be such that $\bigwedge_n \varphi(f_n)$ exists.
Then we have $\bigwedge_n f_n \in F$. Moreover,
$$\varphi(\,\textstyle\bigwedge_n f_n\,) \;=\; \textstyle\bigwedge_n \varphi(f_n).$$
Here $\bigwedge_n f_n$ is the infimum of $\{\,f_1,\, f_2,\, \dots\}$ in $[-\infty, +\infty]^{\mathbb{R}}$; more concretely, it is the *pointwise infimum*, i.e., $(\bigwedge_n f_n)(x) = \bigwedge_n f_n(x)$ for all $x \in \mathbb{R}$.

(v) *(Σ-Completeness)* Let $f_1 \leq f_2 \leq \cdots$ from F be such that $\bigvee_n \varphi(f_n)$ exists.
Then we have $\bigvee_n f_n \in F$. Moreover,
$$\varphi(\,\textstyle\bigvee_n f_n\,) \;=\; \textstyle\bigvee_n \varphi(f_n).$$
(vi) *(Convexity)* Let $f \leq z \leq g$ be $[-\infty, +\infty]$-valued functions on $\mathbb{R}$.
Assume that $f, g \in F$ and $\varphi(f) = \varphi(g)$.
Then we have $z \in F$ and $\varphi(f) = \varphi(z) = \varphi(g)$.

[1] "Smallest" with respect to the following order. We say that μ_1 *is extended by* μ_2 where $\mu_i\colon \mathcal{A}_i \to \mathbb{R}$ and $\mathcal{A}_i \subseteq \wp(\mathbb{R})$ provided that $\mathcal{A}_1 \subseteq \mathcal{A}_2$, and $\mu_1(A) = \mu_2(A)$ for all $A \in \mathcal{A}_1$.

In this thesis we present an abstract theory based on the properties (Monotonicity), (Modularity), (Π-Completeness), (Σ-Completeness) and (Convexity) and we try to fit some of the results of measure and integration theory in this framework.

1.1. **Valuations.** We begin by considering (Monotonicity) and (Modularity).

Maps with these two properties are called *(lattice) valuations*. More precisely, let L be a lattice, and let E be an ordered Abelian group (e.g. $\mathbb{R}$, see Appendix A). A map $\varphi\colon L \to E$ is a *valuation* if it is order preserving and *modular*, i.e.,

$$\varphi(x) + \varphi(y) = \varphi(x \wedge y) + \varphi(x \vee y) \qquad (x, y \in L).$$

Of course, the Lebesgue measure $\mu_{\mathcal{L}}$ and the Lebesgue integral $\varphi_{\mathcal{L}}$ are valuations, and there are many more examples. We study valuation in Section 2.

1.2. **Valuation Systems.** Let us now look at (Π-Completeness). For the Lebesgue measure it involves intersections, "$\bigcap_n A_n$", i.e., infima in $\wp(\mathbb{R})$. Similarly, for the Lebesgue integral it involves pointwise infima, "$\bigwedge_n f_n$", i.e., infima in $[-\infty, +\infty]^{\mathbb{R}}$. In order to generalise the notion of (Π-Completeness) to any valuation $\varphi\colon L \to E$ we involve a 'surrounding' lattice, V. That is, we will define what it means for an object of the following shape to be Π-*complete* (see Definition 77).

$$V \supseteq L \xrightarrow{\varphi} E$$

We call these objects *valuation systems*, and we study them in Section 4.

The Lebesgue measure and the Lebesgue integral give us valuation systems:

$$\wp\mathbb{R} \supseteq \mathcal{A}_{\mathcal{L}} \xrightarrow{\mu_{\mathcal{L}}} \mathbb{R} \qquad \text{and} \qquad [-\infty, +\infty]^{\mathbb{R}} \supseteq F_{\mathcal{L}} \xrightarrow{\varphi_{\mathcal{L}}} \mathbb{R}.$$

Of course these valuation systems are Π-complete by (Π-Completeness).
They are also Σ-*complete*, which is a generalisation of (Σ-Completeness).

Finally, (Convexity) can easily be generalised to valuation systems as well. We will define what it means for a valuation system to be *convex* in Definition 82. We study these convex valuation systems in Subsection 4.4.

Now that we have introduced the main objects of study, valuations and valuation systems, let us spend some words on the theorems that we will prove.

1.3. **Completion and Convexification.** Recall that we defined the Lebesgue measure $\mu_{\mathcal{L}}$ as the smallest map $\mu\colon \mathcal{A} \to \mathbb{R}$ that has properties (i)–(vi). It is important to note that it is not obvious at all that such a map exists. While it is relatively easy to see that if there is a map $\mu\colon \mathcal{A} \to \mathbb{R}$ that has properties (i)–(vi), then there is also a smallest one, it takes quite some effort to prove that there is any map $\mu\colon \mathcal{A} \to \mathbb{R}$ with properties (i)–(vi) to begin with.

One could call this statement the *Extension Theorem for the Lebesgue measure.* Similarly, to define $\varphi_{\mathcal{L}}$, we need an *Extension Theorem for the Lebesgue integral.*

We will generalise (a part) of these two theorems to the setting of valuations. To see how we could do this, note that to prove the Extension Theorem for the Lebesgue measure, one could take the following three steps.

(i) *Find the smallest map* $\mu_{\mathrm{S}}\colon \mathcal{A}_{\mathrm{S}} \to \mathbb{R}$ *that has properties (i)–(iii).*

This is not too difficult. Let $\mathcal{S}$ be the family of subsets of $\mathbb{R}$ of the form $[a, b]$ or (a, b) where $a, b \in \mathbb{R}$ with $a \leq b$. Let $\mathcal{A}_{\mathrm{S}}$ be the set of all unions of finite disjoint subsets of $\mathcal{S}$, and let $\mu_{\mathrm{S}}\colon \mathcal{A}_{\mathrm{S}} \to \mathbb{R}$ be given by

$$\mu_{\mathrm{S}}(\, I_1 \cup \cdots \cup I_N \,) = |I_1| + \cdots + |I_N|,$$

where $I_1, \ldots, I_N \in \mathcal{S}$ with $I_n \cap I_m = \varnothing$ when $n \neq m$.

Of course, it requires some calculations to see that such a map μ_{S} exists, and that μ_{S} will have the properties (i)–(iii) (see Example 10).

(ii) *Extend* μ_{S} *to the smallest map* $\overline{\mu_{\mathrm{S}}}\colon \overline{\mathcal{A}_{\mathrm{S}}} \to \mathbb{R}$ *that has properties (i)–(v).*

This is the most interesting and the most difficult step. To give an idea of how one could try obtain such $\overline{\mu_{\mathrm{S}}}$, consider the following 'algorithm'.

> Let $\mu\colon \mathcal{A} \to \mathbb{R}$ be a variable. To begin, set $\mu := \mu_{\mathrm{S}}$.
> (∗) For all $A_1, A_2, \dots$ from $\mathcal{A}$ do the following.
> - If $A_1 \supseteq A_2 \supseteq \cdots$ and $\bigwedge_n \mu(A_n)$ exists and $\bigcap_n A_n \notin \mathcal{A}$, then add $\bigcap_n A_n$ to $\mathcal{A}$, and set $\mu(\bigcap_n A_n) := \bigwedge_n \mu(A_n)$.
> - If $A_1 \subseteq A_2 \subseteq \cdots$ and $\bigvee_n \mu(A_n)$ exists and $\bigcup_n A_n \notin \mathcal{A}$, then add $\bigcup_n A_n$ to $\mathcal{A}$, and set $\mu(\bigcup_n A_n) := \bigvee_n \mu(A_n)$.
>
> If μ was changed, repeat (∗).

There are many problems with this 'algorithm'. Perhaps the most serious problem is, loosely speaking, that the same set A may be obtained in several ways and it is not clear that $\mu(A)$ would be given the same value each time.

Note that the 'algorithm' resembles the definition of the Borel sets. In fact, $\overline{\mu_{\mathrm{S}}}$ will be the family of all Borel subsets of $\mathbb{R}$ with finite measure.

(iii) *Extend* $\overline{\mu_{\mathrm{S}}}$ *to the smallest map* $\mu_{\mathcal{L}}\colon \mathcal{A}_{\mathcal{L}} \to \mathbb{R}$ *that has properties (i)–(vi).*

This is straightforward. Simply define $\mathcal{A}_{\mathcal{L}}$ to be the family of all subsets of $\mathbb{R}$ that are 'sandwiched' between elements of $\overline{\mathcal{A}_{\mathrm{S}}}$, that is, all $Z \in \wp(\mathbb{R})$ for which there are $A, B \in \overline{\mathcal{A}_{\mathrm{S}}}$ such that $A \subseteq Z \subseteq B$ and $\overline{\mu_{\mathrm{S}}}(A) = \overline{\mu_{\mathrm{S}}}(B)$.

Now, define $\mu_{\mathcal{L}}\colon \mathcal{A}_{\mathcal{L}} \to \mathbb{R}$ by $\mu_{\mathcal{L}}(Z) = \overline{\mu_{\mathrm{S}}}(A)$ for Z and A as above.

We have sketched how to get the Lebesgue measure $\mu_{\mathcal{L}}\colon \mathcal{A}_{\mathcal{L}} \to \mathbb{R}$ in three steps,

$$\overset{\text{(i)}}{\dashrightarrow} \mu_{\mathrm{S}} \overset{\text{(ii)}}{\dashrightarrow} \overline{\mu_{\mathrm{S}}} \overset{\text{(iii)}}{\dashrightarrow} \mu_{\mathcal{L}}.$$

We will generalise step (ii) and step (iii) to the setting of valuations. More precisely:

(i) Let $V \supseteq L \xrightarrow{\varphi} E$ be a valuation system. We will give a necessary and sufficient condition, namely $V \supseteq L \xrightarrow{\varphi} E$ is *extendible* (see Definition 141), for the existence of a smallest valuation $\overline{\varphi}\colon \overline{L} \to E$ which extends φ where $\overline{L}$ is a sublattice of V such that the valuation system $V \supseteq \overline{L} \xrightarrow{\overline{\varphi}} E$ is both Π-complete and Σ-complete (see Lemma 142 and Proposition 148).
We will call $\overline{\varphi}$ the *completion* of φ (relative to V).

(ii) Let $V \supseteq L \xrightarrow{\varphi} E$ be a valuation sytem. We will prove the following.
There is smallest valuation $\varphi^\bullet\colon L^\bullet \to E$ extending φ with $L^\bullet$ a sublattice of V such that $V \supseteq L^\bullet \xrightarrow{\varphi^\bullet} E$ is convex (see Propisition 85).
Moreover, $V \supseteq L^\bullet \xrightarrow{\varphi^\bullet} E$ is Π-complete and Σ-complete provided that $V \supseteq L \xrightarrow{\varphi} E$ is Π-complete and Σ-complete (see Proposition 88).
We will call $\varphi^\bullet$ the *convexification* of φ (relative to V).

By the discussion above we see that the Lebesgue measure $\mu_{\mathcal{L}}$ is the convexification of the completion of μ_{S} relative to $\wp(\mathbb{R})$:

$$\mu_{\mathrm{S}} \overset{\text{completion}}{\dashrightarrow} \overline{\mu_{\mathrm{S}}} \overset{\text{convexification}}{\dashrightarrow} \mu_{\mathcal{L}}$$

Similarly, the Lebesgue integral $\varphi_{\mathcal{L}}$ is the convexification of the completion of φ_{S} relative to $[-\infty, +\infty]^{\mathbb{R}}$, where $\varphi_{\mathrm{S}}\colon F_{\mathrm{S}} \to \mathbb{R}$ is the obvious valuation on the set of *step functions* F_{S} (see Example 15). So we get the following diagram.

$$\varphi_{\mathrm{S}} \overset{\text{completion}}{\dashrightarrow} \overline{\varphi_{\mathrm{S}}} \overset{\text{convexification}}{\dashrightarrow} \varphi_{\mathcal{L}}$$

Let us note that $\overline{\varphi_{\mathrm{S}}}\colon \overline{F_{\mathrm{S}}} \to \mathbb{R}$ will be the restriction of the Lebesgue integral to the set $\overline{F_{\mathrm{S}}}$ of Lebesgue integrable *Baire functions*. We will not prove this.

We belief that the completion is the most important step, and that the convexification is mere decoration. In line with this believe, we spend most words on the completion, and we leave it to the reader to think about the convexification.

1.4. **Closedness under Operations.** We have found an abstract method to get the Lebesgue measure $\mu_\mathcal{L}$ and the Lebesgue integral $\varphi_\mathcal{L}$. However, such a method is nothing but a curiosity if we cannot use it to derive some basic properties of $\mu_\mathcal{L}$ and $\varphi_\mathcal{L}$. One such property might be:

$$\left[\begin{array}{l} \text{If } f, g \in \mathbb{R}^\mathbb{R} \text{ are Lebesgue integrable,} \\ \text{then } f+g \text{ is Lebesgue integrable,} \\ \text{and } \varphi_\mathcal{L}(f+g) = \varphi_\mathcal{L}(f) + \varphi_\mathcal{L}(g). \end{array}\right.$$

So, roughly speaking, $\varphi_\mathcal{L}$ is *closed under the operation "+"*. Instead of this, we will prove that $\overline{\varphi_\mathrm{S}}$ is closed under the operation "+". We leave it to the reader to use this to prove that the convexification of $\overline{\varphi_\mathrm{S}}$, i.e. $\varphi_\mathcal{L}$, is closed under "+" as well.

More generally, in Section 6 we will prove some statements of the following shape. If $V \supseteq L \xrightarrow{\varphi} E$ is a valuation system, and φ is closed under some operation in some sense, then the completion $\overline{\varphi}$ is closed under the same operation as well.

1.5. **Convergence Theorems.** An important part of the theory of integration is that of the convergence theorems. So we have studied whether these make sense in the setting of valuations. We will show in Subsection 3.3 that it is possible to formulate and prove the Lemma of Fatou and Lebesgue's Dominated Convergence Theorem for complete valuation systems. Interestingly, the surrounding lattice V will play no role. This leads to the study of *complete valuations* (as opposed to complete valuation systems), see Section 3 for more details.

1.6. **Fubini's Theorem.** Another important part of the theory of integration is Fubini's Theorem. Unfortunately, it seems that that it not possible to make sense of Fubini's Theorem in the general setting of valuations.

Nevertheless, in Section 9 we will split the proof of Fubini's Theorem for the Lebesgue integral into two parts. The first part concerns step functions and is specific to the Lebesgue integral, while the second part is a consequence of a general extension theorem for valuations (see Theorem 199).

1.7. **Extendibility.** We have remarked that a valuation system $V \supseteq L \xrightarrow{\varphi} E$ has a completion if and only if φ is *extendible*. As the reader will see in Subsection 5.5 the definition of "φ is extendible" is rather involved.

Fortunately, the situation is simpler for some choices of E. We say that E is *benign* if for every valuation system $V \supseteq L \xrightarrow{\varphi} E$ we have that φ is extendible iff

$$\left[\begin{array}{l} \text{Let } a_1 \geq a_2 \geq \cdots \text{ in } L \text{ with } \bigwedge_n \varphi(a_n) \text{ exists be given.} \\ \text{Let } b_1 \leq b_2 \leq \cdots \text{ in } L \text{ with } \bigvee_n \varphi(b_n) \text{ exists be given.} \\ \text{Then we have the following implication.} \\ \qquad \bigwedge_n a_n \leq \bigvee_n b_n \quad \Longrightarrow \quad \varphi(\bigwedge_n a_n) \leq \varphi(\bigvee_n b_n), \\ \text{Here, } \bigwedge_n a_n \text{ is the infimum of } a_1 \geq a_2 \geq \cdots \text{ in } V, \\ \text{and } \bigvee_n b_n \text{ is the supremum of } b_1 \leq b_2 \leq \cdots \text{ in } V. \end{array}\right.$$

We will prove that $\mathbb{R}$ is benign. More generally, we will prove in Section 8 any ordered Abelian group E that has a suitable unformity (see Def. 168) is benign.

1.8. **Attribution.** Some work of others has been included in this master's thesis.

(i) An early version of the theory in Section 8 has been developed together with Bas Westerbaan, and some of his work is undoubtedly still there.
(ii) The proof of the Borel Hierarchy Theorem in Subsection 5.4 is an adaptation of the work by Wim Veldman [6].
(iii) Countless improvements were suggested by dr. A.C.M. van Rooij. Most notably, he strengthened Lemma 179 to its current form, and he suggested that I should restrict the theory to lattice valuations.

Aside from the things mentioned above, and that which is common knowledge, and unless stated otherwise, every definition and proof in this thesis is my own.

Nevertheless, I do not want to claim that any part of my work is original as well, because that would be mere gambling. Indeed, recently I discovered an article [1] on the foundation of integration in which valuations are used as well.

1.9. **Prerequisites.** We have tried to keep this text as accessible as possible.
We assume that the reader is familiar with the ordinal numbers and is comfortable with the basic notions of order theory (suprema, infima, lattices, etc., see [3]).
We have attached some material on *ordered Abelian groups*, in Appendix A.
While some knowledge about measure, integral, topology, uniform spaces, Borel sets, and Riesz spaces will helpful as well, we hope this will not be necessary.

1.10. **Notation.** Let us take this opportunity to fix some notation.

(i) We write $\mathbb{N} = \{1, 2, \dots\}$ and $\omega = \{0, 1, 2, \dots\}$.
(ii) We will use the symbol "$\bigvee$" for suprema, and the symbol "$\bigwedge$" for infima, as the symbol "$\sum$" is used for sums, and the symbol "$\prod$" for products.
(iii) Given $x \in \mathbb{R}$ we say that x is *positive* when $x \geq 0$.
Given $x \in \mathbb{R}$ we say that x is *strictly positive* when $x > 0$.

2. Valuations

Both the Lebesgue measure and Lebesgue integral are *valuations*. In fact they are both *complete valuations* (see Section 3). While the better part of this thesis involves complete valuations, much can already be said about valuations.

In this section we study the elementary properties of valuations. We start with some examples in Subsection 2.1. We study the distance induced by a valuation φ,

$$d_\varphi(x,y) \;=\; \varphi(x \vee y) \;-\; \varphi(x \wedge y),$$

in Subsection 2.2. Finally, we study the equivalence induced by this distance,

$$x \approx y \qquad \iff \qquad d_\varphi(x,y) = 0,$$

in Subsection 2.3. The notion of distance is especially important.

We end the section some exotic examples (in Subsection 2.4).

2.1. Introduction.

Definition 3. Let L be a lattice. Let E an ordered Abelian group (see Section A). Let $\varphi\colon L \to E$ be a map. We say that

(i) φ is **modular** provided that

$$\varphi(a \wedge b) \;+\; \varphi(a \vee b) \;=\; \varphi(a) \;+\; \varphi(b) \qquad (a, b \in L);$$

(ii) φ is a **valuation** provided that φ is modular and order preserving.

Example 4. Let $\mathcal{F}$ be the set of finite subsets of $\mathbb{N}$, and for each $A \in \mathcal{F}$, let $\#(A)$ be the number of elements of A. Then we have

$$\#(A \cap B) \;+\; \#(A \cup B) \;=\; \#(A) \;+\; \#(B) \qquad (A, B \in \mathcal{F}),$$

so obviously the map $\mathcal{F} \to \mathbb{N}$ given by by $A \mapsto \#(A)$ is a valuation.

Example 5. Let $\mathcal{A}_\mathcal{L}$ be the set of Lebesgue measurable subsets of $\mathbb{R}$ with finite Lebesgue measure. Then $\mathcal{A}_\mathcal{L}$ is a lattice of subsets of $\mathbb{R}$. Given $A \in \mathcal{A}_\mathcal{L}$, let $\mu_\mathcal{L}(A)$ denote the Lebesgue measure of A. Then $A \subseteq B \implies \mu_\mathcal{L}(A) \leq \mu_\mathcal{L}(B)$, and

$$\mu_\mathcal{L}(A \cap B) \;+\; \mu_\mathcal{L}(A \cup B) \;=\; \mu_\mathcal{L}(A) \;+\; \mu_\mathcal{L}(B),$$

where $A, B \in \mathcal{A}_\mathcal{L}$. So $\mu_\mathcal{L}$ is a valuation.

Remark 6. Valuations have been known for a long time, see [2].

Example 7. Let $F_\mathcal{L}$ be the set of Lebesgue integrable functions on $\mathbb{R}$. When we write "function on $\mathbb{R}$" we mean a map $f\colon \mathbb{R} \to [-\infty, +\infty]$. Allowing the infinite values $+\infty$ and $-\infty$ might make the story a bit more complicated in the short run, but it will turn out to be a convenient choice later on (see Remark 38).

The set $F_\mathcal{L}$ is a lattice under pointwise ordering, and $F_\mathcal{L} \cap \mathbb{R}^\mathbb{R}$ is a lattice ordered Abelian group. The assignment $f \mapsto \int f$ yields an order preserving map

$$\varphi_\mathcal{L}\colon F_\mathcal{L} \longrightarrow \mathbb{R}$$

that is group homomorphism restricted to $F_\mathcal{L} \cap \mathbb{R}^\mathbb{R}$.

It takes some work see that $\varphi_\mathcal{L}$ is modular (and hence a valuation).

First, note that for $x, y \in \mathbb{R}$, we have

$$\min\{x,y\} \;+\; \max\{x,y\} \;=\; x \;+\; y.$$

So given $f, g \in F_\mathcal{L} \cap \mathbb{R}^\mathbb{R}$, we have $f \wedge g + f \vee g = f + g$, and hence

$$\begin{aligned} \varphi_\mathcal{L}(f \wedge g) \;+\; \varphi_\mathcal{L}(f \vee g) \;&=\; \varphi_\mathcal{L}(f \wedge g + f \vee g) \\ &=\; \varphi_\mathcal{L}(f + g) \\ &=\; \varphi_\mathcal{L}(f) \;+\; \varphi_\mathcal{L}(g). \end{aligned} \tag{1}$$

So we see that $\varphi_\mathcal{L}$ is modular on $F_\mathcal{L} \cap \mathbb{R}^\mathbb{R}$.

To see that $\varphi_{\mathcal{L}}$ is modular on $F_{\mathcal{L}}$, we need some observations.

(i) Let $f \in F_{\mathcal{L}}$. Then the set of $x \in \mathbb{R}$ such that $f(x) = +\infty$ or $f(x) = -\infty$ is negligible. Define $f_{\mathbb{R}} \colon \mathbb{R} \to \mathbb{R}$ by, for $x \in \mathbb{R}$,

$$f_{\mathbb{R}}(x) \;=\; \begin{cases} f(x) & \text{if } f(x) \in \mathbb{R}, \\ 0 & \text{otherwise.} \end{cases}$$

Then $f(x) = f_{\mathbb{R}}(x)$ for almost all $x \in \mathbb{R}$.

(ii) Let $f_1, f_2 \in F_{\mathcal{L}}$ be given and assume $f_1(x) = f_2(x)$ for almost all $x \in \mathbb{R}$. (We denote this by $f_1 \approx f_2$.) Then we have $\varphi_{\mathcal{L}}(f_1) = \varphi_{\mathcal{L}}(f_2)$.

(iii) Let $f_1, f_2 \in F_{\mathcal{L}}$ with $f_1 \approx f_2$ be given, and let $g \in F_{\mathcal{L}}$. Then $f_1 \wedge g \approx f_2 \wedge g$ and $f_1 \vee g \approx g_2 \vee g$.

Now, let $f, g \in F_{\mathcal{L}}$ be given. To prove that $\varphi_{\mathcal{L}}$ is modular, we must show that

$$\varphi_{\mathcal{L}}(f) + \varphi_{\mathcal{L}}(g) \;=\; \varphi_{\mathcal{L}}(f \wedge g) + \varphi_{\mathcal{L}}(f \vee g).$$

Indeed, we have:

$$\begin{aligned} \varphi_{\mathcal{L}}(f) + \varphi_{\mathcal{L}}(g) &= \varphi_{\mathcal{L}}(f_{\mathbb{R}}) + \varphi_{\mathcal{L}}(g_{\mathbb{R}}) && \text{by (i) and (ii)} \\ &= \varphi_{\mathcal{L}}(f_{\mathbb{R}} \wedge g_{\mathbb{R}}) + \varphi_{\mathcal{L}}(f_{\mathbb{R}} \vee g_{\mathbb{R}}) && \text{by Statement (1)} \\ &= \varphi_{\mathcal{L}}(f \wedge g_{\mathbb{R}}) + \varphi_{\mathcal{L}}(f \vee g_{\mathbb{R}}) && \text{see (iii)} \\ &= \varphi_{\mathcal{L}}(f \wedge g) + \varphi_{\mathcal{L}}(f \vee g) \end{aligned}$$

Hence the Lebesgue integral $\varphi_{\mathcal{L}} \colon F_{\mathcal{L}} \to \mathbb{R}$ is a valuation.

Example 8. Let C be a chain, i.e. a totally ordered set. Then C is a lattice with

$$a \wedge b = \min\{a, b\}, \qquad a \vee b = \max\{a, b\}.$$

One quickly sees that *any map $f \colon C \to E$ to an ordered Abelian group is modular.*

Example 9. Let X be a set and let $\mathcal{A}$ be a ring of subsets of X. That is,

$$A \cap B, \qquad A \cup B, \qquad A \backslash B$$

are in $\mathcal{A}$ for all $A, B \in \mathcal{A}$. Then clearly $\mathcal{A}$ is a lattice.

Let E be an ordered Abelian group and let $\mu \colon \mathcal{A} \to E$ be a map. Recall that μ is **additive** if $\mu(A) + \mu(B) = \mu(A \cup B)$ for all $A, B \in \mathcal{A}$ with $A \cap B = \varnothing$.

If μ is additive, then μ is modular. Indeed, let $A, B \in \mathcal{A}$ be given. We need to prove that $\mu(A) + \mu(B) = \mu(A \cap B) + \mu(A \cup B)$ assuming μ is additive. We have

$$\begin{aligned} \mu(A) + \mu(B) &= \mu(\, A \cap B \;\cup\; A \backslash B \,) + \mu(B) \\ &= \mu(A \cap B) + \mu(A \backslash B) + \mu(B) && \text{since } A \cap B \;\cap\; A \backslash B = \varnothing \\ &= \mu(A \cap B) + \mu(A \backslash B \;\cup\; B) && \text{since } A \backslash B \;\cap\; B = \varnothing \\ &= \mu(A \cap B) + \mu(A \cup B). \end{aligned}$$

Recall that μ is **positive** whenever $\mu(A) \in E^+$ for all $A \in \mathcal{A}$.

If μ is additive and positive, then μ is a valuation. Since μ is additive, μ is modular. It remains to be shown (see Definition 3) that μ is order preserving. Let $A \subseteq B$ from $\mathcal{A}$ be given in order to prove $\mu(A) \leq \mu(B)$. We have

$$(B \backslash A) \cup A \;=\; B, \qquad (B \backslash A) \cap A \;=\; \varnothing.$$

So by additivity, $\mu(B) = \mu(B \backslash A) + \mu(A)$. Then $\mu(B) \geq \mu(A)$, since $\mu(B \backslash A) \geq 0$.

Example 10. We describe a ring of subsets of $\mathbb{R}$ and a positive and additive map $\mu_{\mathrm{S}} \colon \mathcal{A}_{\mathrm{S}} \to \mathbb{R}$ that will eventually lead to the Lebesgue measure.

Let $\mathcal{S}$ be the set of all subsets of $\mathbb{R}$ of the form, with $a \leq b$ from $\mathbb{R}$,

$$(a, b) \qquad \text{or} \qquad [a, b].$$

Let $\mathcal{A}_{\mathrm{S}}$ be the ring generated by $\mathcal{S}$. Every element A of $\mathcal{A}_{\mathrm{S}}$ is of the form

$$I_1 \cup \cdots \cup I_N$$

where $I_1, \dots, I_N \in \mathcal{S}$ are disjoint. Let $\mu_{\mathrm{S}}(A)$ be given by

$$\mu_{\mathrm{L}}(A) \;:=\; |I_1| + \cdots + |I_N|.$$

One can verify that the number $\mu_{\mathrm{S}}(A)$ only depends on A and not on the choice of $I_1, \dots, I_N$. Hence we obtain a map $\mu_{\mathrm{S}} \colon \mathcal{A}_{\mathrm{S}} \to \mathbb{R}$. Almost by definition μ_{S} is additive and positive. Hence $\mu_{\mathrm{S}} \colon \mathcal{A}_{\mathrm{S}} \to \mathbb{R}$ is a valuation (see Example 9).

In Example 7, we saw a group homomorphism that is modular, namely the Lebesgue integral $\varphi_{\mathcal{L}}$ restricted to $\mathbb{R}^{\mathbb{R}}$. In fact any group homomorphism on a lattice ordered Abelian group is modular (see Corollary 13(ii)).

Example 11. Let R be a lattice ordered Abelian group. Then the identity map id_R is a valuation. Indeed, id_R is modular by Lemma 212, and clearly order preserving.

Lemma 12. *Suppose we have the following situation.*

$$L' \xrightarrow{\;f\;} L \xrightarrow{\;\varphi\;} E \xrightarrow{\;g\;} E',$$

where L, L' are lattices, E, E' are ordered Abelian groups, f is a lattice homomorphism, φ a map, and g is a group homomorphism. Then

(i) $g \circ \varphi \circ f$ is modular provided that φ is modular;

(ii) $g \circ \varphi \circ f$ is a valuation provided that φ is a valuation and g is positive.

Proof. (i) Suppose φ is modular. Let $a, b \in L'$ be given. Writing $\varphi' = g \circ \varphi \circ f$, we need to prove that $\varphi'(a \wedge b) + \varphi'(a \vee b) = \varphi'(a) + \varphi'(b)$. We have

$$\begin{aligned}
\varphi'(a) + \varphi'(b) &= g(\varphi(f(a))) \;+\; g(\varphi(f(b))) \\
&= g(\;\varphi(f(a)) + \varphi(f(b))\;) \\
&= g(\;\varphi(f(a) \wedge f(b)) + \varphi(f(a) \vee f(b))\;) \\
&= g(\;\varphi(f(a \wedge b)) \;+\; \varphi(f(a \vee b))\;) \\
&= g(\varphi(f(a \wedge b))) \;+\; g(\varphi(f(a \vee b))) \\
&= \varphi'(a \wedge b) + \varphi'(a \vee b)
\end{aligned}$$

(ii) Suppose φ is a valuation and g is positive. We need to prove that $\varphi' := g \circ \varphi \circ f$ is a valuation. By part (i) we know that φ' is modular. It remains to be shown that φ' is order preserving. This is easy: g, φ, and f are all order preserving. So $\varphi' = g \circ \varphi \circ f$ must be order preserving too. □

Corollary 13. *Let R be a lattice ordered Abelian group.*

(i) Let L be a lattice. Any lattice homomorphism $f \colon L \to R$ is a valuation.

(ii) Let E be an ordered Abelian group and $g \colon R \to E$ a group homomorphism. Then g is modular. Moreover, if g is positive, then g is a valuation.

Proof. Apply Lemma 12 to the following situations.

$$L \xrightarrow{\;f\;} R \xrightarrow{\;\mathrm{id}_R\;} R \xrightarrow{\;\mathrm{id}_R\;} R \qquad\qquad R \xrightarrow{\;\mathrm{id}_R\;} R \xrightarrow{\;\mathrm{id}_R\;} R \xrightarrow{\;g\;} E$$

(Recall that id_R is a valuation, see Example 11.) □

Example 14. Let X be a set. We say that $F \subseteq \mathbb{R}^X$ is *Riesz space of functions* if

$$f \vee g, \qquad f \wedge g, \qquad f + g, \qquad \lambda \cdot f$$

are all in F where $f, g \in F$ and $\lambda \in \mathbb{R}$. Then F is a lattice ordered Abelian group.

Let E be an ordered Abelian group and let $\varphi \colon F \to E$ be a positive linear map. We see that φ is a valuation by Corollary 13(ii).

Example 15. We describe a Riesz space of functions F_{S} on $\mathbb{R}$ and a positive linear map $\varphi_{\mathrm{S}}\colon F_{\mathrm{S}} \to \mathbb{R}$ that will eventually lead to the Lebesgue integral.

A *step function* is a function $f\colon \mathbb{R} \to \mathbb{R}$ for which there are $s_1 < s_2 < \cdots < s_N$ in $\mathbb{R}$ such that f is constant on each (s_n, s_{n+1}) and f is zero outside $[s_1, s_N]$.

Let F_{S} be the set of step functions. One can easily see that F_{S} is a Riesz space of functions. Let $f \in F_{\mathrm{S}}$. Let $s_1 < s_2 < \cdots < s_N$ be such that f is constant, say $c_n \in \mathbb{R}$, on (s_n, s_{n+1}) and f is zero outside $[s_1, s_N]$. One can prove that

$$\sum_{n=1}^{N-1} c_n \cdot (s_{n+1} - s_n) \tag{2}$$

does not depend on the choice of $s_1 < s_2 < \cdots < s_N$. So Expression (2) gives a map $\varphi_{\mathrm{S}}\colon F_{\mathrm{S}} \to \mathbb{R}$. This map is easily seen to be linear.

Consequently, $\varphi_{\mathrm{S}}\colon F_{\mathrm{S}} \to \mathbb{R}$ is a valuation (see Example 14).

We end this subsection with some tame examples of valuations we need later on.

Example 16. Let $I = \{1, 2\}$. For each $i \in I$, let L_i be a lattice, E_i an ordered Abelian group, and $\varphi_i\colon L_i \to E_i$ a valuation. Then the map

$$\varphi_1 \times \varphi_2\colon\ L_1 \times L_2 \longrightarrow E_1 \times E_2,$$

given by $(\varphi_1 \times \varphi_2)(a_1, a_2) = (\varphi_1(a), \varphi_2(b))$ for all $a_i \in L_i$, is a valuation.

We call the valuation $\varphi_1 \times \varphi_2$ the *product* of φ_1 and φ_2. Of course, one can similarly define a product of an I-indexed family of valuations for any set I.

Example 17. Let L be a lattice. If we reverse the order on L, i.e., consider the partial order on L given by $a \leq_{L^{\mathrm{op}}} b \iff a \geq_L b$, then if a subset $S \subseteq L$ has a supremum, $\bigvee S$, then $\bigvee S$ is the *infimum* of S with respect to $\leq^{\mathrm{op}}$. So we see that $\leq^{\mathrm{op}}$ gives us a lattice, L^{op}. (The *opposite* lattice.)

Let E be an ordered Abelian group. If we reverse the order on E, we obtain an ordered Abelian group E^{op} with the same group structure, but whose positive elements, $(E^{\mathrm{op}})^+$, are precisely the negative elements of E.

Let $\varphi\colon L \to E$ be a modular map (see Definition 3). Then one quickly sees that φ is also modular considered as a map $L^{\mathrm{op}} \to E$. However, $\varphi\colon L^{\mathrm{op}} \to E$ is a valuation (that is, also order preserving) if and only if $\varphi\colon L \to E$ is order *reversing*, i.e., $a \leq b \implies \varphi(a) \geq \varphi(b)$ for all $a, b \in L$.

Of course, if $\varphi\colon L \to E$ is a valuation, then φ is a valuation $L^{\mathrm{op}} \to E^{\mathrm{op}}$.

2.2. **Distance Induced by a Valuation.** In this subsection, we derive some facts concerning the following notion of distance induced by a valuation.

Definition 18. Let E be an ordered Abelian group. Let L be a lattice.
Let $\varphi\colon L \to E$ be a valuation. Define $d_\varphi\colon L \times L \to E$ by

$$d_\varphi(a, b) \;=\; \varphi(a \vee b) - \varphi(a \wedge b) \qquad (a, b \in L).$$

To give the name "distance" for d_φ some credibility, we will prove that d_φ is a pseudometric (see Lemma 21). After that, we turn our attention to the following fact, which we will use often. Given $a \in L$, the map $x \mapsto a \wedge x$ is a *contraction*, i.e.,

$$d_\varphi(\, a \wedge x,\, a \wedge y\,) \;\leq\; d_\varphi(x, y) \qquad (x, y \in L).$$

In fact, we will prove the following, stronger, statement (see Lemma 23).

$$d_\varphi(\, a \wedge x,\, a \wedge y\,) \;+\; d_\varphi(\, a \vee x,\, a \vee y\,) \;\leq\; d_\varphi(x, y) \qquad (x, y \in L).$$

Before we do all this, let us consider some examples.

Example 19. Let E be an ordered Abelian group. Let F be a Riesz space of functions, and let $\varphi\colon F \to E$ be a positive and linear map (see Example 14).

Let $f, g \in F$ be given. The distance between f and g is the usual one,

$$d_\varphi(f,g) \;=\; \|f-g\|_1 \;:=\; \varphi(\,|f-g|\,).$$

To see this, note that since φ is linear, we have

$$d_\varphi(f,g) \;=\; \varphi(f\vee g) - \varphi(f\wedge g) \;=\; \varphi(f\vee g - f\wedge g).$$

Further, since we have the identity $\max\{x,y\} - \min\{x,y\} = |x-y|$ for reals x, y, we have the identity $f\vee g - f\wedge g = |f-g|$ for functions.

Example 20. Let E be an ordered Abelian group. Let $\mathcal{A}$ be a ring of sets, and let $\mu\colon \mathcal{A} \to E$ be a positive additive map (see Example 9). Let $A, B \in \mathcal{A}$. We have

$$d_\mu(A,B) \;=\; \mu(A\ominus B),$$

where $A\ominus B := A\backslash B \,\cup\, B\backslash A$ is the *symmetric difference* of A and B. To see this, note that $A\cup B$ is the disjoint union of $A\ominus B$ and $A\cap B$. So since μ is additive,

$$\mu(A\cup B) \;=\; \mu(A\ominus B) + \mu(A\cap B).$$

Lemma 21. *Let E be an ordered Abelian group.*
Let L be a lattice, and let $\varphi\colon L \to E$ be a valuation.
Let $a, b, z \in L$ be given. We have:

(i) $d_\varphi(a,b) \;\geq\; 0$
(ii) $d_\varphi(a,a) \;=\; 0$
(iii) $d_\varphi(a,b) \;=\; d_\varphi(b,a)$
(iv) $d_\varphi(a,b) \;\leq\; d_\varphi(a,z) \;+\; d_\varphi(z,b)$

Proof. Only point (iv) requires some work. Let $a, b, z \in L$ be given. We want to show that $d_\varphi(a,b) \leq d_\varphi(a,z) + d_\varphi(z,b)$. In other words:

$$\varphi(a\vee b) \;+\; \varphi(a\wedge z) \;+\; \varphi(z\wedge b) \;\leq\; \varphi(a\vee z) \;+\; \varphi(z\vee b) \;+\; \varphi(a\wedge b). \qquad (3)$$

By modularity, the left-hand side equals

$$\varphi(a\vee b) \;+\; \varphi(\,(a\wedge z)\vee(b\wedge z)\,) \;+\; \varphi(a\wedge b\wedge z).$$

On the other hand, using modularity the right-hand side of Inequality (3) becomes

$$\varphi(a\vee b\vee z) \;+\; \varphi(\,(a\vee z)\wedge(b\vee z)\,) \;+\; \varphi(a\wedge b).$$

Note that $a\vee b \,\leq\, a\vee b\vee z$, and $(a\wedge z)\vee(b\wedge z) \,\leq\, z \,\leq\, (a\vee z)\wedge(b\vee z)$, and $a\wedge b\wedge z \leq a\wedge b$, so that the monotonicity of φ yields Inequality (3). □

It is possible that $d_\varphi(a,b) = 0$ while $a \neq b$ (see Example 27). So in general, d_φ is not a metric (but merely a *pseudo*metric). Those φ for which d_φ is a metric turn out to be useful. So let us give them a name.

Definition 22. Let L be a lattice. Let E be an ordered Abelian group.
Let $\varphi\colon L \to E$ be a valuation. We say φ is **Hausdorff** provided that

$$d_\varphi(a,b) = 0 \quad\Longrightarrow\quad a = b \qquad\qquad (a, b \in L).$$

We return to Hausdorff valuations in Subsection 2.3.

Lemma 23. *Let E be an ordered Abelian group.*
Let L be a lattice, and $\varphi\colon L \to E$ a valuation. We have, for $a, b, z \in L$,

$$d_\varphi(a\wedge z,\, b\wedge z) \;+\; d_\varphi(a\vee z, b\vee z) \;\leq\; d_\varphi(a,b).$$

Proof. By expanding Definition 18, we see that we need to prove that

$$\begin{aligned}\varphi(a \vee b \vee z) \;+\; \varphi(\,(a \wedge z) \vee (b \wedge z)\,) \;+\; \varphi(a \wedge b) \\ \leq\; \varphi(a \vee b) + \varphi(\,(a \vee z) \wedge (b \vee z)\,) \;+\; \varphi(a \wedge b \wedge z)\end{aligned} \tag{4}$$

By modularity, the left-hand side equals

$$\varphi(a \vee b \vee z) \;+\; \varphi(\,(a \wedge z) \vee (b \wedge z) \vee (a \wedge b)\,) \;+\; \varphi(\,a \wedge b \wedge ((a \wedge z) \vee (b \wedge z))\,).$$

To simplify the above expression, we prove that $a \wedge b \wedge ((a \wedge z) \vee (b \wedge z) = a \wedge b \wedge z$. To this end, note that $a \wedge z \;\leq\; (a \wedge z) \vee (b \wedge z) \leq z$ so that

$$a \wedge b \wedge z \;=\; a \wedge b \wedge (a \wedge z) \;\leq\; a \wedge b \wedge ((a \wedge z) \vee (b \wedge z)) \;\leq\; a \wedge b \wedge z.$$

Hence the left-hand side of Inequality (4) equals

$$\varphi(a \vee b \vee z) \;+\; \varphi(\,(a \wedge z) \vee (b \wedge z) \vee (a \wedge b)\,) \;+\; \varphi(\,a \wedge b \wedge z\,).$$

In a similar fashion, one can show that the right-hand side of Inequality (4) equals

$$\varphi(a \vee b \vee z) \;+\; \varphi(\,(a \vee z) \wedge (b \vee z) \wedge (a \vee b)\,) \;+\; \varphi(\,a \wedge b \wedge z\,).$$

So in order to prove Inequality (4), we must show that

$$\varphi(\,(a \wedge z) \vee (b \wedge z) \vee (a \wedge b)\,) \;\leq\; \varphi(\,(a \vee z) \wedge (b \vee z) \wedge (a \vee b)\,).$$

Since φ is order preserving, it suffices to show that

$$(a \wedge z) \vee (b \wedge z) \vee (a \wedge b) \;\leq\; (a \vee z) \wedge (b \vee z) \wedge (a \vee b).$$

Writing $c_1 = a$, $c_2 = b$, and $c_3 = z$, we must prove that

$$\textstyle\bigvee_{i \neq j} c_i \wedge c_j \;\leq\; \bigwedge_{k \neq \ell} c_k \vee c_\ell.$$

That is, we must show that $c_i \wedge c_j \leq c_k \vee c_\ell$ for given $i \neq j$ and $k \neq \ell$. Now, note

$$\#(\{i,j\} \cap \{k,\ell\}) + \#\{i,j,k,\ell\} \;=\; \#\{i,j\} + \#\{k,\ell\} \;=\; 4.$$

Since $\#\{i,j,k,\ell\} \leq 3$, we see that $\#\{i,j\} \cap \{k,\ell\} \geq 1$. So pick $m \in \{i,j\} \cap \{k,\ell\}$. Then $c_i \wedge c_j \leq c_m \leq c_k \vee c_\ell$. □

Lemma 24. *Let E be an ordered Abelian group.*
Let L be a lattice, and $\varphi\colon L \to E$ a valuation. Then we have

$$d_\varphi(a \wedge w, b \wedge z) \;+\; d_\varphi(a \vee w, b \vee z) \;\leq\; d_\varphi(a,b) \;+\; d_\varphi(w,z),$$

where $a, b, w, z \in L$.

Proof. By the triangle inequality (point (iv) of Lemma 21), we have

$$\begin{aligned} d_\varphi(a \wedge w,\, b \wedge z) \;&\leq\; d_\varphi(a \wedge w,\, b \wedge w) \;+\; d_\varphi(b \wedge w,\, b \wedge z), \\ d_\varphi(a \vee w,\, b \vee z) \;&\leq\; d_\varphi(a \vee w,\, b \vee w) \;+\; d_\varphi(b \vee w,\, b \vee z). \end{aligned} \tag{5}$$

On the other hand, Lemma 23 gives us

$$\begin{aligned} d_\varphi(a \wedge w, b \wedge w) + d_\varphi(a \vee w, b \vee w) \;&\leq\; d_\varphi(a,b), \\ d_\varphi(b \wedge w, b \wedge z) + d_\varphi(b \vee w, b \vee z) \;&\leq\; d_\varphi(w,z). \end{aligned} \tag{6}$$

The sum of the right-hand sides of Equation (5) equals the sum of the left-hand sides of Equation (6). Hence $d_\varphi(a \wedge w, b \wedge z) + d_\varphi(a \vee w, b \vee z) \leq d_\varphi(a,b) + d_\varphi(w,z)$. □

2.3. **Equivalence Induced by a Valuation.** In measure theory two functions are considered equivalent if they are equal almost everywhere.

In this subsection, we extend this notion of equivalence to valuations.

Definition 25. Let E be an ordered Abelian group. Let L be a lattice. Let $\varphi\colon L \to E$ be a valuation. We define $\approx$ to be the binary relation on L given by

$$a \approx b \qquad \iff \qquad d_\varphi(a,b) = 0 \qquad\qquad (a, b \in L).$$

Remark 26. φ is Hausdorff (see Definition 22) iff $a \approx b \iff a = b$.

Example 27. We consider the Lebesgue integral $\varphi_\mathcal{L}\colon F_\mathcal{L} \to \mathbb{R}$ (see Example 7).

Let $f, g \in F_\mathcal{L} \cap \mathbb{R}^\mathbb{R}$ be given. By Example 19 we know that

$$f \approx g \qquad \iff \qquad \varphi_\mathcal{L}(\,|f - g|\,) = 0. \tag{7}$$

In fact, Statement (7) holds for all $f, g \in F_\mathcal{L}$, as the reader can verify using the remarks made in Example 7.

Now, for any $h \in F_\mathcal{L}$ with $h \geq 0$, we have that

$$\varphi_\mathcal{L}(h) \;=\; 0 \qquad \iff \qquad h(x) = 0 \quad \text{for almost all } x.$$

So we see that we have, for $f, g \in F_\mathcal{L}$,

$$f \approx g \qquad \iff \qquad f(x) = g(x) \quad \text{for almost all } x.$$

So "$\approx$" is *equality almost everywhere* when $\varphi = \varphi_\mathcal{L}$, as was intended.

Proposition 28. *Let E be an ordered Abelian group. Let L be a lattice. Let $\varphi\colon L \to E$ be a valuation. Let $\approx$ be as in Definition 25.*

(i) The relation $\approx$ is an equivalence.

(ii) Let $a_1, a_2 \in L$ with with $a_1 \approx a_2$ be given. Then $\varphi(a_1) = \varphi(a_2)$.

(iii) Let $a_1, a_2 \in L$ with $a_1 \approx a_2$, and let $b_1, b_2 \in L$ with $b_1 \approx b_2$ be given. Then

$$a_1 \wedge b_1 \;\approx\; a_2 \wedge b_2 \qquad \text{and} \qquad a_1 \vee b_1 \;\approx\; a_2 \vee b_2.$$

(iv) Let $a_1, a_2 \in L$ with $a_1 \approx a_2$, and let $b_1, b_2 \in L$ with $b_1 \approx b_2$ be given. Then

$$d_\varphi(a_1, b_1) \;=\; d_\varphi(a_2, b_2).$$

Proof. (i) The relation $\approx$ is clearly reflexive and symmetric. So to prove $\approx$ is an equivalence relation, we will only show that $\approx$ is transitive. Let $a, b, c \in L$ with $a \approx b \approx c$ be given. We must show that $a \approx c$. Or in other words, $d_\varphi(a, c) = 0$.

By Lemma 21, points (i) and (iv), we get

$$0 \;\leq\; d_\varphi(a, c) \;\leq\; d_\varphi(a, b) + d_\varphi(b, c). \tag{8}$$

But $d_\varphi(a, b) = 0$ and $d_\varphi(b, c) = 0$, since $a \approx b$ and $b \approx c$, respectively.

So we see that Statement (8) implies $d_\varphi(a, c) = 0$. Hence $a \approx c$.

(ii) Let $a_1, a_2 \in L$ with $a_1 \approx a_2$ be given. We must prove $\varphi(a_1) = \varphi(a_2)$.

Let $i \in \{1, 2\}$ be given. Note that $a_1 \wedge a_2 \leq a_i \leq a_1 \vee a_2$. So we have

$$\varphi(a_1 \wedge a_2) \;\leq\; \varphi(a_i) \;\leq\; \varphi(a_1 \vee a_2). \tag{9}$$

Since $d_\varphi(a_1, a_2) = 0$, we know that $\varphi(a_1 \vee a_2) = \varphi(a_1 \wedge a_2)$. So Statement (9) implies that $\varphi(a_1 \vee a_2) = \varphi(a_i) = \varphi(a_1 \wedge a_2)$. Hence $\varphi(a_1) = \varphi(a_2)$.

(iii) Let $a_1, a_2 \in L$ with $a_1 \approx a_2$ be given. Let $b_1, b_2 \in L$ with $b_1 \approx b_2$ be given. We will only show that $a_1 \wedge b_1 \approx a_2 \wedge b_2$; the proof of $a_1 \vee b_1 \approx a_2 \vee b_2$ is similar.

Note that we have the following inequalities by Lemma 21(i) and Lemma 24.

$$0 \;\leq\; d_\varphi(a_1 \wedge b_1, a_2 \wedge b_2) \;\leq\; d_\varphi(a_1, b_1) + d_\varphi(a_2, b_2) \tag{10}$$

Since $a_1 \approx a_2$ and $b_1 \approx b_2$, we have $d_\varphi(a_1, b_1) = 0$ and $d_\varphi(a_2, b_2) = 0$, respectively. Hence Statement (10) implies $d_\varphi(a_1 \wedge b_1, a_2 \wedge b_2) = 0$. So $a_1 \wedge b_1 \approx a_2 \wedge b_2$.

(iv) Let $a_1, a_2 \in L$ with $a_1 \approx a_2$ be given. Let $b_1, b_2 \in L$ with $b_1 \approx b_2$ be given. We must prove that $d_\varphi(a_1, b_1) = d_\varphi(a_2, b_2)$. Note that by point (iii) we have

$$a_1 \wedge b_1 \approx a_2 \wedge b_2 \qquad \text{and} \qquad a_1 \vee b_1 \approx a_2 \vee b_2.$$

So by point (ii) of this lemma, we get

$$\varphi(a_1 \wedge b_1) = \varphi(a_2 \wedge b_2) \qquad \text{and} \qquad \varphi(a_1 \vee b_1) = \varphi(a_2 \vee b_2).$$

So if we unfold Definition 18, we see that

$$\begin{aligned} d_\varphi(a_1, b_1) \;&=\; \varphi(a_1 \vee b_1) - \varphi(a_1 \vee b_1) \\ &=\; \varphi(a_2 \vee b_2) - \varphi(a_2 \vee b_2) \;=\; d_\varphi(a_2, b_2). \quad \square \end{aligned}$$

When studying the Lebesgue integrable functions, $F_\mathcal{L}$, it is sometimes convenient to consider the space $\mathrm{L}^1 = F_\mathcal{L}/{\approx}$ of integrable functions modulo equality almost everywhere (see Example 27). Of course, one can consider the space $L/{\approx}$ for any valuation $\varphi\colon L \to E$. We list some of the properties of $L/{\approx}$ in Proposition 29.

Proposition 29. *Let E be an ordered Abelian group.*
Let L be a lattice. Let $\varphi\colon L \to E$ be a valuation. Let $\approx$ be as in Definition 25.
Let $L/{\approx}$ denote the quotient set, and let $q\colon L \to L/{\approx}$ be the quotient map. Then:

(i) The set $L/{\approx}$ is lattice if the operations are given by

$$qa \wedge qb \;=\; q(a \wedge b), \qquad qa \vee qb \;=\; q(a \vee b) \qquad (a, b \in L).$$

Then, in particular, $q\colon L \to L/{\approx}$ is a lattice homomorphism.

(ii) There is a unique map $\varphi/{\approx}\colon L/{\approx} \to E$ such that

$$(\varphi/{\approx})(q(a)) \;=\; \varphi(a) \qquad (a \in L).$$

Moreover, the map $\varphi/{\approx}$ is a valuation.

(iii) We have $d_{\varphi/\approx}(\,qa,\, qb\,) = d_\varphi(a, b)$ for all $a, b \in L$.

(iv) We have $d_{\varphi/\approx}(\mathfrak{a}, \mathfrak{b}) = 0 \implies \mathfrak{a} = \mathfrak{b}$ for all $\mathfrak{a}, \mathfrak{b} \in L/{\approx}$.

Proof. Follows from Proposition 28. We leave the verification to the reader. $\square$

Remark 30. Note that $\varphi/{\approx}$ is Hausdorff (see Definition 22) by Proposition 29(iv).

2.4. **More Examples.** Valuations also appear outside measure theory.
We begin with an example from elementary number theory.

Example 31. Recall that *Euler's totient function* φ is given by, for $n \in \mathbb{N}$,

$$\varphi(n) \;=\; \#\{\; x \in \{1, \ldots, n\}\colon\;\; \gcd\{x, n\} = 1 \;\}.$$

We will prove that φ is a valuation 'with respect to the division order'.

More precisely, we consider φ to be a map

$$\varphi\colon \mathbb{N} \longrightarrow \mathbb{Q}^\circ,$$

where $\mathbb{Q}^\circ$ is the set of strictly positive rational numbers. Write, for $q, r \in \mathbb{Q}^\circ$,

$$q \preccurlyeq r \qquad \iff \qquad \exists n \in \mathbb{N}\,[\; q \cdot n \;=\; r \;].$$

We order the sets $\mathbb{N}$ and $\mathbb{Q}^\circ$ by "$\preccurlyeq$". They are both lattices with

$$a \wedge b \;=\; \gcd\{a, b\}, \qquad a \vee b \;=\; \operatorname{lcm}\{a, b\}.$$

Moreover, $\mathbb{Q}^\circ$ is an ordered Abelian group under the normal multiplication "$\cdot$".

Before we prove that φ is a valuation, we make a useful observation: for $n \in \mathbb{N}$,

$$\varphi(n) \;=\; \#\mathbb{Z}_n^*. \tag{11}$$

Here $\mathbb{Z}_n$ is the set of integers modulo n, and $\mathbb{Z}_n^*$ are the invertible elements of $\mathbb{Z}_n$.

To see that Equation (11) holds, note that for $x \in \mathbb{Z}$, we have

$$\begin{aligned} \gcd\{x,n\} = 1 \quad &\iff \quad \exists a,b \in \mathbb{Z},\quad ax + bn = 1 \qquad \text{by Bézout's Lemma} \\ &\iff \quad \exists a,b \in \mathbb{Z},\quad ax = 1 - bn \\ &\iff \quad \exists a \in \mathbb{Z},\quad [a]_n \cdot [x]_n = 1 \\ &\iff \quad [x]_n \in \mathbb{Z}_n^*, \end{aligned}$$

where $[-]_n \colon \mathbb{Z} \to \mathbb{Z}_n$ is the quotient map.

Let us now prove that φ is a valuation. We first prove that φ is order preserving. Let $m, n \in \mathbb{N}$ with $m \preccurlyeq n$ be given. We must show that $\varphi(m) \preccurlyeq \varphi(n)$.

Note that there is a unique ring homomorphism $h \colon \mathbb{Z}_n \to \mathbb{Z}_m$ given by, for $x \in \mathbb{Z}$,

$$h(\,[x]_n\,) = [x]_m.$$

Note that h is surjective, and that $[x]_n$ if invertible iff $[x]_m$ is invertible for $x \in \mathbb{Z}$. So we see that if we restrict h to $\mathbb{Z}_n^*$, we get a surjective group homomorphism

$$\tilde{h} \colon \mathbb{Z}_n^* \longrightarrow \mathbb{Z}_m^*.$$

By Lagrange's Theorem we know that

$$\#\mathbb{Z}_m^* \cdot \#\ker(\tilde{h}) = \#\mathbb{Z}_n^*,$$

where $\ker(\tilde{h}) := \{\, a \in \mathbb{Z}_n \colon\ \tilde{h}(a) = 0 \,\}$ is the kernel of $\tilde{h}$. Thus

$$\varphi(m) = \#\mathbb{Z}_m^* \preccurlyeq \#\mathbb{Z}_n^* = \varphi(n).$$

Hence φ is order preserving.

It remains to be shown that φ is modular. That is, for $m, n \in \mathbb{N}$,

$$\varphi(\,\gcd\{m,n\}\,) \cdot \varphi(\,\mathrm{lcm}\{m,n\}\,) = \varphi(m) \cdot \varphi(n). \tag{12}$$

We first prove a special case, namely, that for $m, n \in \mathbb{N}$ with $\gcd\{m,n\} = 1$,

$$\varphi(m \cdot n) = \varphi(m) \cdot \varphi(n). \tag{13}$$

By the Chinese Remainder Theorem we have the following isomorphism of rings.

$$\mathbb{Z}_m \times \mathbb{Z}_n \cong \mathbb{Z}_{m\cdot n}$$

As a consequence, we get the following isomorphism of groups.

$$\mathbb{Z}_m^* \times \mathbb{Z}_n^* \cong \mathbb{Z}_{m\cdot n}^*$$

If we count the number of elements in the above groups we see that

$$\#\mathbb{Z}_m^* \cdot \#\mathbb{Z}_n^* = \#\mathbb{Z}_{m\cdot n}^*.$$

Hence Equation (13) holds (see Equation (11)).

Let $m, n \in \mathbb{N}$ be given. We prove that Equation (12) holds.

By the Fundamental Theorem of Arithmatic, we have

$$m = \prod_{p\in\mathbb{P}} p^{w(p)}, \qquad\qquad n = \prod_{p\in\mathbb{P}} p^{v(p)},$$

where $v, w \colon \mathbb{P} \to \{0,1,2,\dots\}$ have finite support, and $\mathbb{P}$ are the primes. Hence

$$\varphi(m) = \prod_{p\in\mathbb{P}} \varphi(p^{w(p)}), \qquad \varphi(n) = \prod_{p\in\mathbb{P}} \varphi(p^{v(p)}),$$

by Equation (13), because $\gcd\{p_1^{k_1}, p_2^{k_2}\} = 1$ for all $p_1 \neq p_2$ from $\mathbb{P}$ and $k_1, k_2 \in \mathbb{N}$.

Let $p \in \mathbb{P}$ be given. Note that either $w(p) \leq v(p)$ or $v(p) \leq w(p)$. Hence,

$$\varphi(p^{w(p)}) \cdot \varphi(p^{v(p)}) = \varphi(\,p^{\min\{w(p),v(p)\}}\,) \cdot \varphi(\,p^{\max\{w(p),v(p)\}}\,). \tag{14}$$

This gives us the following equality.

$$m \cdot n = \prod_{p\in\mathbb{P}} \varphi(\,p^{\min\{w(p),v(p)\}}\,) \cdot \prod_{p\in\mathbb{P}} \varphi(\,p^{\max\{w(p),v(p)\}}\,).$$

Note that $\gcd\{m,n\} = \prod_{p\in\mathbb{P}} p^{\min\{w(p),v(p)\}}$, so we have

$$\varphi(\gcd\{m,n\}) = \prod_{p\in\mathbb{P}} \varphi(p^{\min\{w(p),v(p)\}}).$$

Similarly, we have

$$\varphi(\operatorname{lcm}\{m,n\}) = \prod_{p\in\mathbb{P}} \varphi(p^{\max\{w(p),v(p)\}}).$$

If we apply the above equalities to Equation (14) we get

$$\varphi(m)\cdot\varphi(n) = \varphi(\gcd\{m,n\})\cdot\varphi(\operatorname{lcm}\{m,n\}).$$

So φ is modular. Hence Euler's totient function φ is a valuation.

Up to this point we have only seen valuations on distributive lattices. We will now give an example of a valuation on a non-distributive lattice.

Example 32. Let W be a vector space. Let L be the set of finite-dimensional linear subspaces of W ordered by inclusion. Then L is a lattice, and for all $A, B \in L$,

$$A \wedge B = A \cap B, \qquad A \vee B = \langle A \cup B \rangle,$$

where $\langle S \rangle$ denotes the smallest linear subspace containing S. We have

$$\dim(A \wedge B) + \dim(A \vee B) = \dim A + \dim B \qquad (A, B \in L).$$

To see this, apply the dimension theorem to the map $f\colon A \times B \to A \vee B$ given by $(a, b) \mapsto a + b$. Hence the assignment $A \mapsto \dim A$ gives a valuation $\dim\colon L \to \mathbb{N}$.

The lattice L might be distributive. For instance, if $W = \{0\}$. This occurs only seldom: if W contains two linearly independent vectors, then L is non-distributive.

Indeed, let $v_1, v_2 \in W$ be linearly independent vectors and consider $w := v_1 + v_2$. One can verify that v_i, w are linearly independent too. So $\langle v_i \rangle \cap \langle w \rangle = \{0\}$. Hence

$$\langle w \rangle \wedge (\langle v_1 \rangle \vee \langle v_2 \rangle) = \langle v_1, v_2 \rangle \neq \{0\} = (\langle w \rangle \wedge \langle v_1 \rangle) \vee (\langle w \rangle \wedge \langle v_2 \rangle).$$

It is interesting to note that there are some 'connections' between modular maps (see Definition 3) and *modular lattices*. Recall that a lattice L is modular if

$$\ell \vee (a \wedge u) = (\ell \vee a) \wedge u$$

for all $\ell, u, a \in L$ with $\ell \leq u$. One such connection is given by the following lemma.

Lemma 33. *Let E be an ordered Abelian group. Let L be a lattice. Let $\varphi\colon L \to E$ be a modular map. Let $\ell, u \in L$ with $\ell \leq u$ be given. We have*

$$\varphi(\ell \vee (a \wedge u)) = \varphi((\ell \vee a) \wedge u) \qquad (a \in L). \tag{15}$$

Proof. The trick is to consider the expression $\varphi(\ell) + \varphi(a) + \varphi(u)$. On the one hand,

$$\begin{aligned} \varphi(\ell) + \varphi(a) + \varphi(u) &= \varphi(\ell \wedge a) + \varphi(\ell \vee a) + \varphi(u) \\ &= \varphi(\ell \wedge a) + \varphi((\ell \vee a) \wedge u) + \varphi(a \vee u), \end{aligned}$$

where we have used modularity twice. On the other hand,

$$\begin{aligned} \varphi(\ell) + \varphi(a) + \varphi(u) &= \varphi(\ell) + \varphi(a \wedge u) + \varphi(a \vee u) \\ &= \varphi(\ell \wedge a) + \varphi(\ell \vee (a \wedge u)) + \varphi(a \vee u). \end{aligned}$$

The difference, $\varphi((\ell \vee a) \wedge u) - \varphi(\ell \vee (a \wedge u))$, must be zero. □

3. Complete Valuations

We now turn to the study of *complete* valuations (see Definition 35). Among all valuations the complete valuations resemble the Lebesgue measure and the Lebesgue integral most closely. To support this claim, we will prove generalisations of some of the classical convergence theorems of integration in Subsection 3.3.

But first, we give some examples of complete valuations in Subsection 3.1.

After that, we study a notion of completeness for an ordered Abelian group E, called *R-completeness*, in Subsection 3.2, which will be useful later on.

The notion of complete valuation is not at the end of the road. We will study the slightly more sophisticated *valuation systems* (see Definition 72) and *complete valuation systems* (see Definition 77) in Section 4.

3.1. **Introduction.**

Definition 34. Let E be an ordered Abelian group.
Let L be a lattice, and let $\varphi\colon L \to E$ be a valuation.

Consider a sequence $a_1 \geq a_2 \geq \cdots$ from L. We say

$$a_1 \geq a_2 \geq \cdots \text{ is } \boldsymbol{\varphi}\textbf{-convergent} \qquad \text{if} \qquad \textstyle\bigwedge_n \varphi(a_n) \text{ exists.}$$

Similarly, if $b_1 \leq b_2 \leq \cdots$ is a sequence in L, then

$$b_1 \leq b_2 \leq \cdots \text{ is } \boldsymbol{\varphi}\textbf{-convergent} \qquad \text{if} \qquad \textstyle\bigvee_n \varphi(b_n) \text{ exists.}$$

Definition 35. Let E be an ordered Abelian group. Let L be a lattice.
Let $\varphi\colon L \to E$ be a valuation. We say φ is **Π-complete** if

$$a_1 \geq a_2 \geq \cdots \varphi\text{-convergent} \quad \Longrightarrow \quad \textstyle\bigwedge_n a_n \text{ exists,} \quad \text{and} \quad \varphi(\,\bigwedge_n a_n\,) = \bigwedge_n \varphi(a_n).$$

We say φ is **Σ-complete** if

$$b_1 \leq b_2 \leq \cdots \varphi\text{-convergent} \quad \Longrightarrow \quad \textstyle\bigvee_n b_n \text{ exists,} \quad \text{and} \quad \varphi(\,\bigvee_n b_n\,) = \bigvee_n \varphi(b_n).$$

We say φ is **complete** if φ is both Π-complete and Σ-complete.

Example 36. The Lebesgue measure $\mu_{\mathcal{L}}$ (see Example 5) is a complete valuation. We must show that $\mu_{\mathcal{L}}$ is both Π-complete and Σ-complete (see Definition 35).

Let us prove $\mu_{\mathcal{L}}$ is Σ-complete. Let $B_1 \subseteq B_2 \subseteq \cdots$ in $\mathcal{A}_{\mathcal{L}}$ be $\mu_{\mathcal{L}}$-convergent. We must prove that $\bigvee_n B_n$ exists in $\mathcal{A}_{\mathcal{L}}$ and that $\mu_{\mathcal{L}}(\bigvee_n B_n) = \bigvee_n \mu_{\mathcal{L}}(B_n)$.

Note that $\bigcup_n B_n$ is Lebesgue measurable, and that $\bigcup_n B_n$ has (finite) Lebesgue measure $\bigvee_n \mu_{\mathcal{L}}(B_n)$. Hence we have $\bigcup_n B_n \in \mathcal{A}_{\mathcal{L}}$, and,

$$\mu_{\mathcal{L}}(\,\textstyle\bigcup_n B_n\,) \;=\; \bigvee_n \mu_{\mathcal{L}}(B_n). \tag{16}$$

So we are done if we prove that $\bigcup_n B_n = \bigvee_n B_n$. Since $\bigcup_n B_n$ is the smallest subset of $\mathbb{R}$ containing all B_n (i.e. $\bigcup_n B_n$ is the supremum of the B_n in $\wp\mathbb{R}$), $\bigcup_n B_n$ is also the smallest subset *of finite Lebesgue measure* containing all B_n (i.e. $\bigcup_n B_n$ is the supremum of the B_n in $\mathcal{A}_{\mathcal{L}}$). So $\bigcup_n B_n = \bigvee_n B_n$. Hence $\mu_{\mathcal{L}}$ is Σ-complete.

Using an easier reasoning one can prove that $\mu_{\mathcal{L}}$ is Π-complete.

Example 37. The Lebesgue integral $\varphi_{\mathcal{L}}$ (see Example 7) is a complete valuation. We must show that $\varphi_{\mathcal{L}}$ is both Π-complete and Σ-complete (see Definition 35).

Let us prove $\varphi_{\mathcal{L}}$ is Σ-complete. Let $f_1 \leq f_2 \leq \cdots$ in $F_{\mathcal{L}}$ be $\varphi_{\mathcal{L}}$-convergent. We must prove that $\bigvee_n f_n$ exists in $F_{\mathcal{L}}$, and that

$$\varphi_{\mathcal{L}}(\textstyle\bigvee_n f_n) \;=\; \bigvee_n \varphi_{\mathcal{L}}(f_n).$$

Of course, this follows immediately from Levi's Monotone Convergence Theorem; the supremum $\bigvee_n f_n$ in $F_{\mathcal{L}}$ is simply the pointwise supremum (which is the supremum of $f_1 \leq f_2 \leq \cdots$ in $[-\infty, +\infty]^{\mathbb{R}}$). So we see $\varphi_{\mathcal{L}}$ is Σ-complete.

With a similar argument one can see that $\varphi_{\mathcal{L}}$ is Π-complete.

Remark 38. Note that the restriction of $\varphi_{\mathcal{L}}$ to $F_{\mathcal{L}} \cap \mathbb{R}^{\mathbb{R}}$ is not complete.
Indeed, consider for instance the following sequence.

$$1 \cdot \mathbf{1}_{\{0\}} \;\leq\; 2 \cdot \mathbf{1}_{\{0\}} \;\leq\; 3 \cdot \mathbf{1}_{\{0\}} \;\leq\; \cdots$$

It is $\varphi_{\mathcal{L}}$-convergent in $F_{\mathcal{L}} \cap \mathbb{R}^{\mathbb{R}}$, but it has no supremum in $\mathbb{R}^{\mathbb{R}}$.
On the other hand, it does have a supremum in $F_{\mathcal{L}}$, namely $+\infty \cdot \mathbf{1}_{\{0\}}$.
Because of the above observation, we work with the $[-\infty, +\infty]$-valued Lebesgue integrable functions instead of the $\mathbb{R}$-valued Lebesgue integrable functions.

Example 39. The valuation μ_{S} (see Example 10) is *not* complete.
To see this, we consider the sets $A_1, A_2, \ldots$ given by, for $n \in \mathbb{N}$,

$$A_n \;=\; \{1, \ldots, n\}.$$

Then $A_n \in \mathcal{A}_{\mathrm{S}}$ and $\mu_{\mathrm{S}}(A_n) = 0$ for all $n \in \mathbb{N}$. So we see that

$$A_1 \;\subseteq\; A_2 \;\subseteq\; \cdots$$

is a μ_{S}-convergent sequence. To prove that μ_{S} is not complete, we show that the μ_{S}-convergent sequence $A_1 \subseteq A_2 \subseteq \cdots$ has no supremum in $\mathcal{A}_{\mathrm{S}}$ (see Definition 35).
Suppose (towards a contradiction) that $A_1 \subseteq A_2 \subseteq \cdots$ has a supremum B in $\mathcal{A}_{\mathrm{S}}$. Then in particular $A_n \subseteq B$ for all $n \in \mathbb{N}$. So we have

$$\mathbb{N} \;=\; \textstyle\bigcup_n A_n \;\subseteq\; B.$$

Note that B is the disjoint union of elements from $\mathcal{S}$ (see Example 10). Since all $I \in \mathcal{S}$ are bounded, the set B is bounded. That is, $B \subseteq [a, b]$ for some $a, b \in \mathbb{R}$.
We now see that $\mathbb{N} \subseteq B \subseteq [a, b]$, which is nonsense. So $A_1 \subseteq A_2 \subseteq \cdots$ has no supremum in $\mathcal{A}_{\mathrm{S}}$. Hence μ_{S} is not complete.

Example 40. The valuation φ_{S} (see Example 15) is also *not* complete.
We leave it to the reader to prove this fact.

If $E = \mathbb{R}$, or more generally, if E is σ-Dedekind complete (see Definition 214), then there is a nice description of φ-convergence, see Proposition 42.

Lemma 41. *Let E be an ordered Abelian group.*
Assume that E is σ-Dedekind complete (see Definition 214).
Let L be a lattice, and let $\varphi\colon L \to E$ be a valuation.
Let $a_1 \geq a_2 \geq \cdots$ be a sequence in L.
Then $a_1 \geq a_2 \geq \cdots$ is φ-convergent provided that $a_1 \geq a_2 \geq \cdots$ has a lower bound.

Proof. Let $\ell \in L$ be a lower bound of $a_1 \geq a_2 \geq \cdots$, that is, $\ell \leq a_n$ for all $n \in \mathbb{N}$. We must prove that $a_1 \geq a_2 \geq \cdots$ is φ-convergent, i.e., $\bigwedge_n \varphi(a_n)$ exists.
Note that $\varphi(\ell) \leq \varphi(a_n)$ for all $n \in \mathbb{N}$. So $\varphi(a_1)$, $\varphi(a_2)$, $\ldots$ has a lower bound. But then $\bigwedge_n \varphi(a_n)$ exists, because E is σ-Dedekind complete (see Remark 217). Hence $a_1 \geq a_2 \geq \cdots$ is φ-convergent. □

Proposition 42. *Let E be an ordered Abelian group.*
Assume that E is σ-Dedekind complete (see Definition 214).
Let L be a lattice, and let $\varphi\colon L \to E$ be a complete valuation.
For a sequence $a_1 \geq a_2 \geq \cdots$ in L the following are equivalent.

(i) *$a_1 \geq a_2 \geq \cdots$ is φ-convergent.*
(ii) *$a_1 \geq a_2 \geq \cdots$ has a lower bound in L.*
(iii) *$a_1 \geq a_2 \geq \cdots$ has an infimum, $\bigwedge_n a_n$.*

Proof. The implication "(i) $\Longleftarrow$ (ii)" holds by Lemma 41.
"(ii) $\Longleftarrow$ (iii)" holds, because the infimum $\bigwedge_n a_n$ is a lower bound of $a_1 \geq a_2 \geq \cdots$.
"(iii) $\Longleftarrow$ (i)" holds since φ is complete (see Definition 35). □

The notion of φ-convergence is less trivial in general as the following example shows.

Example 43. We will show that the assumption that E is σ-Dedekind complete in Proposition 42 is necessary for the implication "(iii) $\Longrightarrow$ (i)".
To this end, we extend the Lebesgue integral $\varphi_{\mathcal{L}}$ (see Example 37) to the set

$$F'_{\mathcal{L}} \;:=\; F_{\mathcal{L}} \cup \{\, -\infty \cdot \mathbf{1}\,\}.$$

Note that $F'_{\mathcal{L}}$ is a sublattice of $[-\infty,+\infty]^{\mathbb{R}}$. Let $\varphi'_{\mathcal{L}}\colon F'_{\mathcal{L}} \to \mathbb{L}$ be the map, where $\mathbb{L}$ is the *lexicograpic plane* (see Example 202(iv)), given by, for $f \in F'_{\mathcal{L}}$,

$$\varphi'_{\mathcal{L}}(f) \;=\; \begin{cases} (\;\;0, \varphi_{\mathcal{L}}(f)\,) & \text{if } f \in F_{\mathcal{L}}, \\ (\,-1, 0\,) & \text{if } f = -\infty\cdot\mathbf{1}. \end{cases}$$

Then $\varphi'_{\mathcal{L}}$ is a valuation. In fact, $\varphi'_{\mathcal{L}}$ is a complete valuation as the reader can verify using the following observation. If $a_1 \geq a_2 \geq \cdots$ from $F'_{\mathcal{L}}$ is $\varphi'_{\mathcal{L}}$-convergent, then:

(i) If $a_n \in F_{\mathcal{L}}$ for all $n \in \mathbb{N}$, then $a_1 \geq a_2 \geq \cdots$ is $\varphi_{\mathcal{L}}$-convergent.
(ii) If $a_N = -\infty\cdot\mathbf{1}$ for some $N \in \mathbb{N}$, then $a_n = -\infty\cdot\mathbf{1}$ for all $n \geq N$.

Now, consider the following sequence in $F'_{\mathcal{L}}$.

$$-1\cdot\mathbf{1}_{[0,1]} \;\geq\; -2\cdot\mathbf{1}_{[0,1]} \;\geq\; -3\cdot\mathbf{1}_{[0,1]} \;\geq\; \cdots.$$

This sequence has an infimum in $F'_{\mathcal{L}}$, namely $-\infty\cdot\mathbf{1}$. Nevertheless, the sequence is not $\varphi'_{\mathcal{L}}$-convergent(, because $(0,-1) \geq (0,-2) \geq \cdots$ has no infimum in $\mathbb{L}$).

3.2. R**-completeness.** We now study a notion of completeness for ordered Abelian groups called *R-completeness* that will be useful later on.

Let $\varphi\colon L \to E$ be a valuation. Let $a_1 \leq a_2 \leq \cdots$ and $b_1 \leq b_2 \leq \cdots$ be φ-convergent sequences in L. (see Definition 34).

For the development of the theory, it would be convenient if also

$$a_1 \vee b_1 \;\leq\; a_2 \vee b_2 \;\leq\; \cdots \qquad \text{is } \varphi\text{-convergent.} \tag{17}$$

Unfortunately, this is not always the case (see Example 49). However, if the space E is σ-Dedekind complete (see Appendix A, Definition 214), for instance if $E = \mathbb{R}$, then one can prove that Statement (17) holds.

In fact, if we only assume that E is *R-complete* (see Definition 44) — which is a weaker assumption than that E is Dedekind-complete — then we can still prove that that Statement (17) holds (see Proposition 48).

Definition 44. Let E be an ordered Abelian group. Consider the following.

$$\left[\;\begin{array}{l} \text{Let } x_1 \leq x_2 \leq \cdots \text{ and } y_1 \leq y_2 \leq \cdots \text{ be from } E \text{ such that} \\ \qquad\qquad x_{n+1} - x_n \;\leq\; y_{n+1} - y_n \qquad \text{for all } n. \\ \text{Then } \bigvee x_n \text{ exists whenever } \bigvee y_n \text{ exists.} \end{array}\right.$$

If the above statement holds, we say E is R**-complete**.

Remark 45. The name "R-complete" is due to Willem van Zuijlen [5].

Examples 46. (i) The ordered Abelian group $\mathbb{R}$ is R-complete.

(ii) In fact, any σ-Dedekind complete ordered Abelian group E is R-complete. Indeed, let $x_1 \leq x_2 \leq \cdots$ and $y_1 \leq y_2 \leq \cdots$ be from E such that

$$x_{n+1} - x_n \;\leq\; y_{n+1} - y_n \qquad \text{for all } n, \tag{18}$$

and assume that $\bigvee_n y_n$ exists. We must show that $\bigvee_n x_n$ exists.

Let $n \in \mathbb{N}$ be given. By Statement (18) we see that

$$x_{n+1} - y_{n+1} \;\leq\; x_n - y_n.$$

So with induction on n, we get $x_n - y_n \;\leq\; x_1 - y_1$. Then

$$x_n \;\leq\; (x_1 - y_1) + y_n \;\leq\; (x_1 - y_1) + \textstyle\bigvee_m y_m.$$

So we see that the sequence $x_1, x_2, \dots$ has an upper bound.
So $\bigvee_n x_n$ exists, as E is σ-Dedekind complete. Hence E is R-complete.

(iii) The lexicographic plane $\mathbb{L}$ (see Examples 202(iv)) is R-complete, but $\mathbb{L}$ is not σ-Dedekind complete (see Examples 215(iii)).

(iv) The ordered Abelian group $\mathbb{Q}$ is *not* R-complete.
To see this, and pick $q_1 \le q_2 \le \cdots$ in $\mathbb{Q}$ with

$$q_{n+1} - q_n \;\le\; 2^{-(n+1)} \qquad \text{and} \qquad \textstyle\bigvee_n q_n = \sqrt{2} \text{ in } \mathbb{R}.$$

Note that $q_1 \le q_2 \le \cdots$ has no supremum in $\mathbb{Q}$.
Now, let $y_n := 1 - 2^{-n}$ for all $n \in \mathbb{N}$. Then $y_1 \le y_2 \le \cdots$ has an supremum, namely 1, and we have $y_{n+1} - y_n = 2^{-(n+1)}$. So we see that

$$q_{n+1} - q_n \;\le\; y_{n+1} - y_n \qquad (n \in \mathbb{N}).$$

If $\mathbb{Q}$ were $\mathbb{R}$-complete, then the above implies $q_1 \le q_2 \le \cdots$ would have a supremum in $\mathbb{Q}$, which it does not. Hence $\mathbb{Q}$ is not R-complete.

(v) Let I be a set. For each $i \in I$, let E_i be an R-complete ordered Abelian group. Then the product, $\prod_{i\in I} E_i$, is R-complete.

Remark 47. Let E be an ordered Abelian group. Using the map $x \mapsto -x$, one can easily verify that E is R-complete if and only if the following statement holds.

$$\left[\begin{array}{l} \text{Let } x_1 \ge x_2 \ge \cdots \text{ and } y_1 \ge y_2 \ge \cdots \text{ be from } E \text{ such that} \\ \qquad\qquad x_n - x_{n+1} \;\le\; y_n - y_{n+1} \qquad \text{for all } n. \\ \text{Then } \bigwedge x_n \text{ exists whenever } \bigwedge y_n \text{ exists.} \end{array}\right.$$

Proposition 48. *Let E be an ordered Abelian group which is R-complete. Let L be a lattice, and let $\varphi\colon L \to E$ be a valuation.*

(i) If $a_1 \ge a_2 \ge \cdots$, $b_1 \ge b_2 \ge \cdots$ are φ-convergent sequences from L, then

$$a_1 \wedge b_1 \;\ge\; a_2 \wedge b_2 \;\ge\; \cdots \qquad \textit{and} \qquad a_1 \vee b_1 \;\ge\; a_2 \vee b_2 \;\ge\; \cdots$$

are φ-convergent.

(ii) If $a_1 \le a_2 \le \cdots$, $b_1 \le b_2 \le \cdots$ are φ-convergent sequences from L, then

$$a_1 \wedge b_1 \;\le\; a_2 \wedge b_2 \;\le\; \cdots \qquad \textit{and} \qquad a_1 \vee b_1 \;\le\; a_2 \vee b_2 \;\le\; \cdots$$

are φ-convergent.

Proof. (i) We prove that $a_1 \wedge b_1 \ge a_2 \wedge b_2 \ge \cdots$ is φ-convergent. For this we need to show that $\bigwedge_n \varphi(a_n \wedge b_n)$ exists. Note that since $\bigwedge_n \varphi(a_n)$ and $\bigwedge_n \varphi(b_n)$ exist, we know that $\bigwedge_n\ (\varphi(a_n) + \varphi(b_n))$ exists (by Lemma 208). So by R-completeness, in order to show $\bigwedge_n \varphi(a_n \wedge b_n)$ exists, it suffices to prove that (see Remark 47),

$$\varphi(a_n \wedge b_n) \;-\; \varphi(a_{n+1} \wedge b_{n+1}) \;\le\; (\,\varphi(a_n) + \varphi(b_n)\,) \;-\; (\,\varphi(a_{n+1}) + \varphi(b_{n+1})\,).$$

Phrased differently using "d_φ" (see Definition 18), we need to prove that

$$d_\varphi(a_n \wedge b_n,\, a_{n+1} \wedge b_{n+1}) \;\le\; d_\varphi(a_n, a_{n+1}) + d_\varphi(b_n, b_{n+1}).$$

This follows from Lemma 24.
The proof that $a_1 \vee b_1 \ge a_2 \vee b_2 \ge \cdots$ is φ-convergent is similar.
(ii). Again, similar. □

Example 49. We will prove that the assumption in Proposition 48, that E is R-complete, is necessary.

Let $\mathcal{A}$ be the ring of subsets (see Example 9) of $\mathbb{R}$ generated by the non-empty closed intervals with *rational* endpoints, i.e., subsets of the form $[q, r]$ where $q, r \in \mathbb{Q}$ and $q \le r$. Then there is a unique positive and additive map $\mu\colon \mathcal{A} \to \mathbb{Q}$ such that

$$\mu(\,[q, r]\,) \;=\; r - q \qquad \text{for all } q \le r \text{ from } \mathbb{Q}.$$

Recall that $\mathbb{Q}$ is not R-complete. To prove that the conclusion of Proposition 48 does not hold for $E = \mathbb{Q}$, we will find μ-convergent sequences $A_1 \subseteq A_2 \subseteq \cdots$ and $B_1 \subseteq B_2 \subseteq \cdots$ such that $A_1 \cup B_1 \ \subseteq\ A_2 \cup B_2 \ \subseteq\ \cdots$ is not μ-convergent.

If we have done this, we see that the assumption "E is R-complete" is necessary.

Find rational numbers $\cdots \leq r_2 \leq r_1 < q_1 \leq q_2 \leq \cdots$ such that, in $\mathbb{R}$,

$$\textstyle\bigvee_n q_n \ = \ \sqrt{2} \qquad \text{and} \qquad \bigwedge_n r_n \ = \ \sqrt{2} - 1.$$

Now, let us define $A_1 \subseteq A_2 \subseteq \cdots$ and $B_1 \subseteq B_2 \subseteq \cdots$ in $\mathcal{A}$ by, for $n \in \mathbb{N}$,

$$A_n \ = \ [0, r_1] \qquad \text{and} \qquad B_n \ = \ [r_n, q_n].$$

Then clearly $A_1 \subseteq A_2 \subseteq \cdots$ is φ-convergent. Note that $\mu(B_n) = q_n - r_n$. So, in $\mathbb{R}$,

$$\textstyle\bigvee_n \mu(B_n) \ = \ \bigvee_n q_n - \bigwedge_n r_n \ = \ 1.$$

Hence $\bigvee_n \mu(B_n) = 1$ in $\mathbb{Q}$. So $B_1 \subseteq B_2 \subseteq \cdots$ is a μ-convergent sequence.

However $A_n \cup B_n = [0, q_n]$, and thus $\mu(A_n \cup B_n) = q_n$. So we see that, in $\mathbb{R}$,

$$\textstyle\bigvee_n \mu(\, A_n \cup B_n \,) \ = \ \bigvee_n q_n \ = \ \sqrt{2}.$$

So $\mu(A_1 \cup B_1) \ \leq\ \mu(A_2 \cup B_2) \ \leq\ \cdots$ has no supremum in $\mathbb{Q}$.

Hence $A_1 \cup B_1 \ \subseteq\ A_2 \cup B_2 \ \subseteq\ \cdots$ is not μ-convergent.

3.3. **Convergence Theorems.** The notion of a complete valuation has been based on Levi's Monotone Convergence Theorem (see Example 37). In this subsection, we prove variants of some of the other classical convergence theorems of integration theory. For example, Lebesgue's Dominated Convergence Theorem. It states:

> Let $f_1,\, f_2,\, \ldots$ be a sequence in $F_{\mathcal{L}}$.
> Assume $f_1(x),\, f_2(x),\, \ldots$ converges for almost all $x \in \mathbb{R}$.
> Assume that $f_1,\, f_2,\, \ldots$ is dominated in the sense that $|f_n| \leq D$ for all n for some $D \in F_{\mathcal{L}}$.
> Then there is an $f \in F_{\mathcal{L}}$ with $f_1(x),\, f_2(x),\, \ldots$ converges to $f(x)$ for for almost all $x \in \mathbb{R}$, and $\varphi_{\mathcal{L}}(f) = \lim_n \varphi_{\mathcal{L}}(f_n)$. (19)

The difficulty in the setting of valuation systems is not the proof of the theorem, but its formulation. For instance, it not clear how we should interpret

"$f_1(x),\, f_2(x),\, \cdots$ converges for almost all x"

when the objects f_n are not necessarily functions, but elements of a lattice V.

Let us begin by generalising the notion of convergence in $\mathbb{R}$ to any lattice L. Recall that a sequence $a_1, a_2, \ldots$ in $\mathbb{R}$ is convergent (in the usual sense) if and only if the *limit inferior*, $\lim_N \inf_{n \geq N}\, a_n$, and the *limit superior*, $\lim_N \sup_{n \geq N}\, a_n$, exist and are equal. This leads us to the following definitions.

Definition 50. Let L be a lattice. Let $a_1,\, a_2,\, \ldots$ be a sequence in L.

(i) We say $a_1,\, a_2,\, \ldots$ is **upper convergent** if the following exists.

$$\overline{\lim}_n a_n \ := \ \textstyle\bigwedge_N \bigvee_{n \geq N}\, a_N \vee \cdots \vee a_n.$$

Similarly, we say $a_1,\, a_2,\, \ldots$ is **lower convergent** if the following exists.

$$\underline{\lim}_n a_n \ := \ \textstyle\bigvee_N \bigwedge_{n \geq N}\, a_N \wedge \cdots \wedge a_n.$$

(ii) We say $a_1,\, a_2,\, \ldots$ is **convergent** if it is both upper and lower convergent, and in addition $\overline{\lim}_n a_n = \underline{\lim}_n a_n$. In that case, we write $\lim_n a_n := \overline{\lim}_n a_n$.

(iii) Let $a \in L$ be given. We say $a_1,\, a_2,\, \ldots$ **converges to** a if $a = \lim_n a_n$.

Remark 51. Let L be a lattice. Let $a_1,\, a_2,\, \ldots$ be a sequence in L, which is upper convergent and lower convergent. Then we have the following inequality.

$$\underline{\lim}_n a_n \ \leq\ \overline{\lim}_n a_n$$

Indeed, this follows immediately from the observation that, for every $N \in \mathbb{N}$,

$$\textstyle\bigvee_{n\geq N} a_N \wedge \cdots \wedge a_n \;\leq\; a_N \;\leq\; \bigwedge_{n\geq N} a_N \vee \cdots \vee a_n.$$

Examples 52. (i) In $\mathbb{R}$ we have: A sequence $a_1, a_2, \dots$ is convergent in the sense of Definition 50 if and only if $a_1, a_2, \dots$ is convergent as usual. Moreover, if $a_1, a_2. \dots$ is convergent, then $\lim_n a_n$ from Definition 50 is also the limit of $a_1, a_2, \dots$ in the usual sense.

(ii) Similarly, in $\mathbb{R}^X$, where X is any set, "convergent" from Definition 50 coincides with the usual "pointwise convergent".

Example 53. Let X be a set. Let $A_1, A_2, \dots$ be subsets of X. Then $A_1, A_2, \dots$ is upper and lower convergent in the lattice $\wp X$, and we have, for $x \in X$,

$$\begin{aligned} x \in \overline{\lim}_n A_n \quad &\iff \quad \forall N \;\; \exists n \geq N \quad x \in A_n, \\ x \in \underline{\lim}_n A_n \quad &\iff \quad \exists N \;\; \forall n \geq N \quad x \in A_n. \end{aligned}$$

So we see that $A_1, A_2, \dots$ is *not* convergent iff there is an $\tilde{x} \in X$ such that

$$\forall N \;\; \exists n \geq N \quad \tilde{x} \in A_n \qquad \text{and} \qquad \forall N \;\; \exists n \geq N \quad \tilde{x} \notin A_n.$$

Example 54. For the lattice of Lebesgue integrable functions, $F_{\mathcal{L}}$, the notion of convergence from Def. 50 and the usual pointwise convergence do not coincide.

To see this, consider the following sequence.

$$\mathbf{1}_{[0,1]}, \quad \mathbf{1}_{[1,2]}, \quad \mathbf{1}_{[2,3]}, \quad \dots$$

This sequence converges pointwise to $\mathbf{0}$, but it not convergent in the sense of Def. 50. Indeed, the sequence is not even upper convergent because

$$\mathbf{1}_{[0,1]} \;\leq\; \mathbf{1}_{[0,2]} \;\leq\; \mathbf{1}_{[0,3]} \;\leq\; \cdots$$

has no supremum in $F_{\mathcal{L}}$.

Fortunately, the situation is better for *dominated sequences*.
Let $f_1, f_2, \dots \in F_{\mathcal{L}}$ and $f \in F_{\mathcal{L}}$ be given. Let $D \in F_{\mathcal{L}}$ be given such that $|f_n| \leq D$ for all $n \in \mathbb{N}$. (We say that $f_1, f_2, \dots$ is *dominated* by D.)
The reader can easily verify the following statements (cf. Example 37).

(i) The dominated sequence $f_1, f_2, \dots$ is upper convergent, and for all $x \in \mathbb{R}$,

$$(\overline{\lim}_n f_n)(x) \;=\; \overline{\lim}_n(\, f_n(x)\,).$$

(ii) The dominated sequence $f_1, f_2, \dots$ is lower convergent. and for all $x \in \mathbb{R}$,

$$(\underline{\lim}_n f_n)(x) \;=\; \underline{\lim}_n(\, f_n(x)\,).$$

(iii) The dominated sequence $f_1, f_2, \dots$ converges pointswise to f if and only if $f_1, f_2, \dots$ converges to f in the sense of Definition 50.

Dominated sequences are useful when working with the Lebesgue integrable functions, because $\mathbb{R}$ is σ-Dedekind complete (see Definition 214). However, it turns out that dominated sequences are less useful in general.

Hence we have found a replacement for "$f_1, f_2, \dots$ is dominated", namely,

"$f_1, f_2, \dots$ is *upper and lower φ-convergent*".

Definition 55. Let E be an ordered Abelian group. Let L be a lattice.
Let $\varphi\colon L \to E$ be a valuation. Let $a_1, a_2 \dots \in L$ be given.

(i) We say $a_1, a_2, \dots$ is **upper φ-convergent** if the following exists.

$$\varphi\text{-}\overline{\lim}_n a_n \;:=\; \textstyle\bigwedge_N \bigvee_{n\geq N} \varphi(a_N \vee \cdots \vee a_n)$$

Similarly, we say $a_1, a_2, \dots$ is **lower φ-convergent** if the following exists.

$$\varphi\text{-}\underline{\lim}_n a_n \;:=\; \textstyle\bigvee_N \bigwedge_{n\geq N} \varphi(a_N \wedge \cdots \wedge a_n)$$

(ii) We say $a_1, a_2, \dots$ is **φ-convergent** if it is lower and upper φ-convergent, and in addition $\varphi\text{-}\overline{\lim}_n a_n = \varphi\text{-}\underline{\lim}_n a_n$.

Remark 56. Let E be an ordered Abelian group. Let L be a lattice. Let $\varphi\colon L \to E$ be a valuation. Let $a_1, a_2, \dots$ be a sequence in L, which is upper and lower φ-convergent. We have the following inequality (cf. Remark 51).

$$\varphi\text{-}\underline{\lim}_n a_n \;\leq\; \varphi\text{-}\overline{\lim}_n a_n.$$

Proposition 57. *Let E be a σ-Dedekind complete ordered Abelian group. Let L be a lattice, and let $\varphi\colon L \to E$ be a complete valuation. Then for a sequence $a_1, a_2, \dots$ in L the following are equivalent.*

(i) $a_1, a_2, \dots$ is upper and lower φ-convergent.
(ii) $a_1, a_2, \dots$ has an upper and lower bound.

Proof. "(i) $\Longrightarrow$ (ii)" Assume that $a_1, a_2, \dots$ is upper and lower φ-convergent. We must find $u, \ell \in L$ such that $\ell \leq a_n \leq u$ for all $n \in \mathbb{N}$.

Since $a_1, a_2, \dots$ is upper φ-convergent (see Definition 55), we know that

$$\textstyle\bigvee_n \varphi(a_1 \vee \cdots \vee a_n) \qquad \text{exists.}$$

In other words, we know that

$$a_1 \;\leq\; a_1 \vee a_2 \;\leq\; \cdots \qquad \text{is } \varphi\text{-convergent.}$$

Since φ is complete, $u := \bigvee_n a_n$ exists in L. Note that $a_n \leq u$ for all $n \in \mathbb{N}$.

By a similar reasoning, but using the fact that $a_1, a_2, \dots$ is lower φ-convergent, we can find an $\ell \in L$ such that $\ell \leq a_n$ for all $n \in \mathbb{N}$.

"(i) $\Longleftarrow$ (ii)" Let $\ell, u \in L$ be such that $\ell \leq a_n \leq u$ for all $n \in \mathbb{N}$.

We prove that $a_1, a_2, \dots$ is upper φ-convergent. For this, we must show that the following exists (see Definition 55).

$$\textstyle\bigwedge_N \bigvee_{n \geq N} \varphi(a_N \vee \cdots \vee a_n) \tag{20}$$

Let $N \in \mathbb{N}$ and $n \geq N$ be given. Note that we have

$$\ell \;\leq\; a_N \vee \cdots \vee a_n \;\leq\; u.$$

Since φ is order preserving, this gives us

$$\varphi(\ell) \;\leq\; \varphi(a_N \vee \cdots \vee a_n) \;\leq\; \varphi(u).$$

Since E is σ-Dedekind complete it follows that Expression (20) exists.

We have proven that $a_1, a_2, \dots$ is upper φ-convergent. With a similar reasoning one can prove that $a_1, a_2, \dots$ is lower φ-convergent. □

Example 58. Let $f_1, f_2, \dots$ be Lebesgue integrable functions (see Example 7). Then by Proposition 57 the following statement holds.

$$\left[\begin{array}{c} \text{The sequence } f_1, f_2, \dots \text{ is upper and lower } \varphi_{\mathcal{L}}\text{-convergent.} \\ \Longleftrightarrow \\ \text{There is a Lebesgue integrable } D \text{ with } |f_n| \leq D \text{ for all } n. \end{array}\right.$$

We can now prove a generalisation of the Lemma of Fatou.

Lemma 59 (Fatou)**.** *Let E be an ordered Abelian group. Let L be a lattice, and let $\varphi\colon L \to E$ be a complete valuation. Let $a_1, a_2, \dots$ be an upper φ-convergent sequence in L (see Definition 55) Then $a_1, a_2, \dots$ is upper convergent (see Definition 50), and we have*

$$\varphi(\overline{\lim}_n a_n) \;=\; \varphi\text{-}\overline{\lim}_n a_n.$$

Moreover, if E is a lattice, and if $\overline{\lim}_n \varphi(a_n)$ exists (see Definition 50), then

$$\varphi\text{-}\overline{\lim}_n a_n \;\geq\; \overline{\lim}_n \varphi(a_n).$$

Proof. Let $a_1,\,a_2,\,\ldots$ be an upper φ-convergent sequence. We prove that $a_1, a_2, \ldots$ is upper convergent (see Definition 50), and that $\varphi(\overline{\lim}_n a_n) = \varphi\text{-}\overline{\lim}_n a_n$.

Let $N \in \mathbb{N}$ be given. Note that $\bigvee_{N \geq n} \varphi(a_N \vee \cdots \vee a_n)$ exists because the sequence $a_1,\,a_2,\,\ldots$ is upper φ-convergent (see Definition 55). So the sequence

$$a_N \;\leq\; a_N \vee a_{N+1} \;\leq\; a_N \vee a_{N+1} \vee a_{N+2} \;\leq\; \cdots$$

is φ-convergent (in the sense of Definition 34). For brevity, let us write

$$\overline{a}_N^n \;:=\; a_N \vee \cdots \vee a_{N+n}.$$

Since φ is complete, and $\overline{a}_N^0 \leq \overline{a}_N^1 \leq \cdots$ is φ-convergent, we get $\bigvee_n \overline{a}_N^n$ exists, and

$$\varphi(\overline{a}_N) \;=\; \textstyle\bigvee_n \varphi(\overline{a}_N^n),$$

where $\overline{a}_N := \bigvee_n \overline{a}_N^n$. Note that $\overline{a}_1 \geq \overline{a}_2 \geq \cdots$ is φ-convergent, because

$$\varphi\text{-}\overline{\lim}_n a_n \;=\; \textstyle\bigwedge_N \bigvee_n \varphi(\overline{a}_N^n)$$

exists since $a_1,\,a_2,\,\ldots$ is upper φ-convergent. Since φ is complete, this implies that

$$\textstyle\bigwedge_n \overline{a}_N \text{ exists} \qquad \text{and} \qquad \varphi(\bigwedge_n \overline{a}_N) \;=\; \bigwedge_n \varphi(\overline{a}_N).$$

Now, note that we have the following equality.

$$\textstyle\bigwedge_N \overline{a}_N \;=\; \bigwedge_N \bigvee_{n \geq N} a_n$$

So we see that $a_1, a_2, \ldots$ is upper φ-convergent and that

$$\varphi(\overline{\lim}_n a_n) \;=\; \textstyle\bigwedge_N \varphi(\overline{a}_N) \;=\; \bigwedge_N \bigvee_n \varphi(\overline{a}_N^n) \;=\; \varphi\text{-}\overline{\lim}_n a_n.$$

We have proven the first part of the lemma.

Assume E is a lattice and $\overline{\lim}_n \varphi(a_n)$ exists (see Definition 50). To prove the remainder of the theorem, we need to show that $\varphi\text{-}\overline{\lim}_n a_n \geq \overline{\lim}_n \varphi(a_n)$. That is,

$$\textstyle\bigwedge_N \bigvee_{n \geq N} \varphi(a_N \vee \cdots \vee a_n) \;\geq\; \bigwedge_N \bigvee_{n \geq N} \varphi(a_N) \vee \cdots \vee \varphi(a_n).$$

This is easy. It follows immediately from the fact that

$$\varphi(a_N \vee \cdots \vee a_n) \;\geq\; \varphi(a_N) \vee \cdots \vee \varphi(a_n)$$

for all $N \in \mathbb{N}$ and $n \geq N$. □

Let us now think about "almost everywhere convergent".

Definition 60. Let E be an ordered Abelian group. Let L be a lattice. Let $\varphi\colon L \to E$ be a valuation. Let $a_1, a_2, \ldots \in L$ and $a \in L$ be given.

(i) If $a_1, a_2, \ldots$ is upper and lower convergent, and, with $\approx$ as in Def. 25,

$$\underline{\lim}_n a_n \;\approx\; \overline{\lim}_n a_n,$$

then we say that $a_1, a_2, \ldots$ is **$\approx$-convergent**.

(ii) We say that $a_1, a_2, \ldots$ **$\approx$-converges** to a when $\underline{\lim}_n a_n \approx a \approx \overline{\lim}_n a_n$.

Example 61. Unfortunately, in the lattice of Lebesgue integrable functions, $F_{\mathcal{L}}$, the notion of $\approx$-convergence does not coincide with convergence almost everywhere, as can be seen using a similar argument as before (see Example 54).

Again, the situation is better for dominated sequences.

Let $f_1, f_2, \ldots \in F_{\mathcal{L}}$ and $f \in F_{\mathcal{L}}$. Assume $f_1, f_2, \ldots$ is dominated by some $D \in F_{\mathcal{L}}$, that is, $|f_n| \leq D$ for all $n \in \mathbb{N}$. Then the following statements hold.

(i) The dominated sequence $f_1, f_2, \ldots$ $\approx$-converges to f if and only if $f_1(x),\, f_2(x),\, \ldots$ converges to $f(x)$ for almost all $x \in \mathbb{R}$.

(ii) The dominated sequence $f_1, f_2, \ldots$ is $\approx$-convergent if and only if $f_1(x),\, f_2(x),\, \ldots$ converges for almost all $x \in \mathbb{R}$.

We will prove implication "$\Longleftarrow$" of (i), and leave the rest to the reader.
We must show that $f_1, f_2, \dots$ is upper and lower convergent, and that

$$\underline{\lim}_n f_n \;\approx\; f \;\approx\; \overline{\lim}_n f_n. \tag{21}$$

Since $f_1(x)$, $f_2(x)$, ... converges to $f(x)$ for almost all $x \in \mathbb{R}$, we know that:

$$\underline{\lim}_n(\, f_n(x)\,) \;=\; f(x) \;=\; \overline{\lim}_n(\, f_n(x)\,) \tag{22}$$

for almost all $x \in \mathbb{R}$. So we see that the function given by $x \mapsto \underline{\lim}_n(\, f_n(x)\,)$ is equal almost everywhere to the Lebesgue integrable function f, and hence it is Lebesgue integrable itself. By Example 54(ii) it follows that $f_1, f_2, \dots$ is lower convergent, and that $(\underline{\lim}_n f_n)(x) = \underline{\lim}_n(\, f_n(x)\,)$ for all $x \in \mathbb{R}$. By a similar argument, we see that $f_1, f_2, \dots$ is upper convergent as well, and that $(\overline{\lim}_n f_n)(x) = \overline{\lim}_n(\, f_n(x)\,)$ for all $x \in \mathbb{R}$. Hence we get by Equation (22), for almost all $x \in \mathbb{R}$,

$$(\underline{\lim}_n f_n)(x) \;=\; f(x) \;=\; (\overline{\lim}_n f_n)(x).$$

This proves Equation (21) (see Example 27).

Let us relate φ-convergence and $\approx$-convergence.

Lemma 62. *Let E be an ordered Abelian group.*
Let L be a lattice, and let $\varphi\colon L \to E$ be a complete valuation.
Let $a_1, a_2, \dots$ be an upper and lower φ-convergent sequence in L (see Definition 55). Then the sequence $a_1, a_2, \dots$ is upper and lower convergent (see Definition 50), and the following statements are equivalent.

(i) $a_1, a_2, \dots$ is $\approx$-convergent (see Definition 60).
(ii) $a_1, a_2, \dots$ is φ-convergent (see Definition 55).

Moreover, if either (i) or (ii) holds, we have

$$\varphi(\underline{\lim}_n a_n) \;=\; \varphi(\overline{\lim}_n a_n) \;=\; \varphi\text{-}\lim_n a_n. \tag{23}$$

Proof. Let $a_1, a_2, \dots$ be an upper and lower φ-convergent sequence in L.
By Lemma 59 and its dual, $a_1, a_2, \dots$ is both upper and lower convergent, and

$$\varphi(\underline{\lim}_n a_n) \;=\; \varphi\text{-}\underline{\lim}_n a_n, \qquad \text{and} \qquad \varphi(\overline{\lim}_n a_n) \;=\; \varphi\text{-}\overline{\lim}_n a_n. \tag{24}$$

By Definition 55 and Definition 60 we see that

$$\begin{aligned} a_1, a_2, \dots \text{ is } \varphi\text{-convergent} &\quad\Longleftrightarrow\quad \varphi\text{-}\underline{\lim}_n a_n \;=\; \varphi\text{-}\overline{\lim}_n a_n, \\ a_1, a_2, \dots \text{ is } \approx\text{-convergent} &\quad\Longleftrightarrow\quad \varphi(\,\underline{\lim}_n a_n\,) \;=\; \varphi(\,\overline{\lim}_n a_n\,). \end{aligned}$$

Hence Equation (24) implies that statements (i) and (ii) are equivalent.

Now, assume that (i) (or (ii)) holds. We must show that Statement (23) holds. This follows from Statement (24) since $\varphi\text{-}\lim_n a_n \;=\; \varphi\text{-}\underline{\lim}_n a_n \;=\; \varphi\text{-}\overline{\lim}_n a_n$. □

We now prove a generalisation of Lebesgue's Dominated Convergence Theorem.

Theorem 63 (Lebesgue)**.** *Let E be a* lattice *ordered Abelian group.*
Let L be a lattice, and let $\varphi\colon L \to E$ be a complete valuation.
Let $a_1, a_2, \dots$ be an upper and lower φ-convergent sequence in L (see Def. 55).
Assume that $a_1, a_2, \dots$ is $\approx$-convergent (see Def. 60).
Assume that $\underline{\lim}_n \varphi(a_n)$ and $\overline{\lim}_n \varphi(a_n)$ exist (see Def. 50).
Then the sequence $\varphi(a_1)$, $\varphi(a_2)$, ... converges (see Def. 50), and we have

$$\lim_n \varphi(a_n) \;=\; \varphi(\,\underline{\lim}_n a_n\,) \;=\; \varphi(\,\overline{\lim}_n a_n\,). \tag{25}$$

Proof. Let us first prove that the sequence $\varphi(a_1)$, $\varphi(a_2)$, ... is convergent.
By Lemma (62), we see that $a_1, a_2, \dots$ is φ-convergent (see Definition 55), and that

$$\varphi(\,\underline{\lim}_n a_n\,) \;=\; \varphi(\,\overline{\lim}_n a_n\,) \;=\; \varphi\text{-}\lim_n a_n. \tag{26}$$

By Lemma 59 and its dual, we see that

$$\varphi\text{-}\underline{\lim}_n a_n \;\leq\; \underline{\lim}_n \varphi(a_n) \;\leq\; \overline{\lim}_n \varphi(a_n) \;\leq\; \varphi\text{-}\overline{\lim}_n a_n.$$

But $\varphi\text{-}\underline{\lim}_n a_n = \varphi\text{-}\overline{\lim}_n a_n$, since $a_1, a_2, \dots$ is φ-convergent. So we get

$$\varphi\text{-}\underline{\lim}_n a_n \;=\; \underline{\lim}_n \varphi(a_n) \;=\; \overline{\lim}_n \varphi(a_n) \;=\; \varphi\text{-}\overline{\lim}_n a_n. \tag{27}$$

In particular, $\varphi(a_1)$, $\varphi(a_2)$, $\dots$ is convergent (see Definition 50).

It remains to be shown that Statement (25) holds.
To do this, combine Statement (26) and Statement (27). □

If we assume that E is σ-Dedekind complete we get a more familiar statement.

Theorem 64. *Let E be a* lattice *ordered Abelian group.*
Assume that E is σ-Dedekind complete (see Def. 214).
Let L be a lattice, and let $\varphi\colon L \to E$ be a complete valuation.
Let $a_1, a_2, \dots$ sequence in L which has an upper and lower bound.
Assume that $a_1, a_2, \dots$ is $\approx$-convergent (see Def. 60).
Then the sequence $\varphi(a_1)$, $\varphi(a_2)$, $\dots$ converges (see Def. 50), and we have

$$\lim_n \varphi(a_n) \;=\; \varphi(\,\underline{\lim}_n a_n\,) \;=\; \varphi(\,\overline{\lim}_n a_n\,).$$

Proof. We want to apply Theorem 63. For this, we must prove that $a_1, a_2, \dots$ is upper and lower φ-convergent, and that $\underline{\lim}_n \varphi(a_n)$ and $\overline{\lim}_n \varphi(a_n)$ exist.

Note that $a_1, a_2, \dots$ is upper and lower φ-convergent since $a_1, a_2, \dots$ has an upper and lower bound (see Proposition 57).

Let $u, \ell \in L$ be such that $\ell \leq a_n \leq u$ for all $n \in \mathbb{N}$. Then we have, for all $n \in \mathbb{N}$,

$$\varphi(\ell) \;\leq\; \varphi(a_n) \;\leq\; \varphi(u).$$

Using this, and the fact that E is σ-Dedekind complete, it is not so hard to see that $\underline{\lim}_n \varphi(a_n)$ and $\overline{\lim}_n \varphi(a_n)$ exist (cf. Proposition 57).

Now we can apply Theorem 63, and we are done. □

Example 65. If we apply Theorem 64 to the Lebesgue integral $\varphi_{\mathcal{L}}$, we get the classical form of Lebesgue's Dominated Convergence Theorem (see Statement (19)).

Indeed, let f_1, f_2, $\dots$ be a sequence of Lebesgue integrable functions. Assume there is an Lebesgue integrable function D such that $|f_n| \leq D$ for all $n \in \mathbb{N}$, and assume that $f_1(x)$, $f_2(x)$, $\dots$ converges for almost all $x \in \mathbb{R}$.

We must prove that there is a Lebesgue integrable f such that $f_1(x)$, $f_2(x)$, $\dots$ converges to $f(x)$ for almost all $x \in \mathbb{R}$, and $\varphi_{\mathcal{L}}(f) = \lim_n \varphi_{\mathcal{L}}(f_n)$.

Since $f_1(x)$, $f_2(x)$, $\dots$ converges for almost all $x \in \mathbb{R}$, we know that the sequence f_1, f_2, $\dots$ is $\approx$-convergent (see Example 61(ii)). Now, define

$$f \;:=\; \underline{\lim}_n f_n.$$

It is easy to see that, f_1, f_2, $\dots$ $\approx$-converges to f (see Definition 60).
That is, $f_1(x)$, $f_2(x)$, $\dots$ converges to $f(x)$ for almost all $x \in \mathbb{R}$ (see Example 61(i)).
By Theorem 64 we see that $\varphi(a_1)$, $\varphi(a_2)$, $\dots$ converges, and that

$$\lim_n \varphi_{\mathcal{L}}(f_n) \;=\; \varphi_{\mathcal{L}}(\,\underline{\lim}_n f_n\,) \;=\; \varphi_{\mathcal{L}}(f).$$

So we are done.

There are many variants of the classical convergence theorems of integration. For instance, a variant on Levi's Monotone Convergence Theorem is the following.

> Let $f_1,\, f_2,\, \dots$ be Lebesgue integrable functions.
> Assume that $\bigvee_n \varphi_{\mathcal{L}}(f_n)$ exists.
> Assume that for every $n \in \mathbb{N}$,
> $$f_n(x) \,\leq\, f_{n+1}(x) \qquad \text{for almost all } x \in \mathbb{R}.$$
> Then $\bigvee_n f_n$, the pointwise supremum of $f_1,\, f_2,\, \dots,$ is Lebesgue integrable and
> $$\varphi_{\mathcal{L}}(\textstyle\bigvee_n f_n) \,=\, \bigvee_n \varphi_{\mathcal{L}}(f_n).$$

Note that if we want to prove the above statement it will be useful to know that

$$\varphi_{\mathcal{L}}/{\approx}\colon\; F_{\mathcal{L}}/{\approx} \longrightarrow \mathbb{R}$$

from Proposition 29 is complete. We will prove this in Proposition 66.

Of course, if we apply Lemma 59 and Theorem 63 to $\varphi_{\mathcal{L}}/{\approx}$, we obtain variants of the Lemma of Fatou and the Dominated Convergence Theorem of Lebesgue, respectively. We leave this to the reader.

Proposition 66. *Let L be a lattice. Let E be an ordered Abelian group. Let $\varphi\colon L \to E$ be a complete valuation. Then the valuation*

$$\varphi/{\approx}\colon\; L/{\approx} \longrightarrow E$$

from Proposition 29 is a complete valuation.

Proof. We leave this to the reader. □

There is a small gap that needs to filled before we continue with another topic. Let $\varphi\colon L \to E$ be a valuation. We have defined what it means for a sequence $a_1,\, a_2,\, \dots$ in L to be φ-convergent (see Definition 55), but we have not yet given the meaning of "$a_1,\, a_2,\, \dots$ converges *to* a". We will do this in Definition 67.

Definition 67. Let L be a lattice. Let E be an ordered Abelian group. Let $\varphi\colon L \to E$ be a valuation. Let $a_1,\, a_2,\, \dots$ be a sequence in L. Let $a \in L$ be given. We say $a_1,\, a_2,\, \dots$ **φ-converges** to a provided that

$$a_1,\, a,\, a_2,\, a,\, \dots \qquad \text{is } \varphi\text{-convergent}.$$

Remark 68. Let $\varphi\colon L \to E$ be a valuation. While Definition 67 is certainly reasonable, it is also quite silly, and so one wonders if there is a more direct description of when a sequence φ-converges to an element $a \in L$. If we assume φ is complete, then there is a slightly better description (see Proposition 69).

In Section 9 we will study a notion of convergence (see Definition 188) which was intended to be a more aesthetically pleasing definition of φ-convergence, but which turns out to be strictly weaker than φ-convergence (see Example 190).

Proposition 69. *Let E be an ordered Abelian group. Let L be a lattice, and $\varphi\colon L \to E$ be a complete valuation. Let $a_1,\, a_2,\, \dots$ be a φ-convergent sequence in L, and let $a \in L$. Then $a_1,\, a_2,\, \dots$ φ-converges to a if and only if (see Definition 25)*

$$a \;\approx\; \overline{\lim}_n a_n.$$

(Recall that $a_1,\, a_2,\, \dots$ is upper convergent (see Definition 50) by Lemma 59.)

Proof. We leave this to the reader. □

4. Valuation Systems

Note that the Lebesgue measure $\mu_{\mathcal{L}}$ is a complete valuation (see Example 36), that extends the relatively simple valuation μ_{S} (see Example 10).

We would like to consider $\mu_{\mathcal{L}}$ to be *a completion* of μ_{S}. What should this mean? The following definition seems obvious when one thinks about valuations.

> Let E be an ordered Abelian group.
> Let L and K be lattices.
> Let $\psi\colon K \to E$ and $\varphi\colon L \to E$ be valuations.
> We say ψ is **a completion** of φ provided that
> L is a sublattice of K, ψ extends φ, and ψ is complete.

However, in the more concrete setting of measure theory this broad definition of completion is not that useful. After all, if we are given a completion $\psi\colon K \to \mathbb{R}$ of μ_{S}, then we only know that K is a sublattice of $\mathcal{A}_{\mathrm{S}}$, while we would prefer K to be a sublattice of sets, or resemble it.

To mend this problem we might try to prove that any completion of μ_{S} is essentially a completion on a lattice of subsets. Of course, the meaning of the previous statement is not clear. We suspect that if one gives it an exact meaning, the statement will be either false or trivial. So we will not follow this direction.

Instead, we consider a different notion of completion that involves the the surrounding lattice, $\wp\mathbb{R}$. More precisely, we will see that $\mu_{\mathcal{L}}$ is a completion of μ_{S} *relative to* $\wp\mathbb{R}$, which means that $\mu_{\mathcal{L}}$ extends μ_{S} and that $\mu_{\mathcal{L}}$ is complete *relative to* $\wp\mathbb{R}$ (see Example 80). This naturally leads to the study of the following objects.

$$\wp\mathbb{R} \supseteq \mathcal{A}_{\mathrm{S}} \xrightarrow{\mu_{\mathrm{S}}} \mathbb{R} \qquad\qquad \wp\mathbb{R} \supseteq \mathcal{A}_{\mathcal{L}} \xrightarrow{\mu_{\mathcal{L}}} \mathbb{R}.$$

That is, we are interested in objects of the following shape.

$$V \supseteq L \xrightarrow{\varphi} E,$$

where $\varphi\colon L \to E$ is a valuation, and where V is a lattice such that L is a sublattice of V. We call such objects *valuation systems* (see Definition 72).

The drawback of this approach is that it requires quite a bit of bookkeeping, and so this section is filled with definitions and examples, but there is little theory. We hope the reader will bear with us; we are confident the reader will be rewarded for his/her patience in the next sections.

Since this section is already administrative in nature, we take this chance to put some additional restraints on the notion of valuation system which turns out to be useful later on (see Remark 93). Given a valuation system, $V \supseteq L \xrightarrow{\varphi} E$, we require that E is R-complete (see Def. 44), and that V is σ-distributive (see Def. 70).

Before we give a formal definition of "valuation system" in Subsection 4.2, and define "complete valuation system" in Subsection 4.3, we consider σ-distributive lattices in Subsection 4.1.

We end the section with "convex valuation systems" in Subsection 4.4.

4.1. σ-Distributivity.

Definition 70. Let V be a lattice. We say V is **σ-distributive** provided that

(i) V is **σ-complete**, i.e., for every sequence $c_1, c_2, \ldots$ in V we have

$$\textstyle\bigwedge_n c_n \text{ exists} \qquad \text{and} \qquad \bigvee_n c_n \text{ exists},$$

(ii) and for every $a \in V$ and $c_1, c_2, \ldots \in V$, we have,

$$a \vee \textstyle\bigwedge_n c_n \;=\; \bigwedge_n a \vee c_n \qquad \text{and} \qquad a \wedge \bigvee_n c_n \;=\; \bigvee_n a \wedge c_n.$$

Examples 71. (i) Let X be a set. Then $\wp(X)$ is σ-distributive. Indeed,

$$A \cup \bigcap_n C_n = \bigcap_n A \cup C_n \qquad A \cap \bigcup_n C_n = \bigcup_n A \cap C_n$$

for all A, $C_1, C_2, \ldots \subseteq X$.

(ii) Let C be totally ordered and σ-complete. Then C is σ-distributive.

Indeed, let a, $c_1, c_2, \ldots \in C$ be given. We need to prove that $a \vee \bigwedge_n c_n$ is the supremum of $a \vee c_1$, $a \vee c_2$, ... To this end note that

$$b \leq d_1 \vee d_2 \quad\iff\quad b \leq d_1 \quad\text{or}\quad b \leq d_2 \qquad (b, d_i \in C).$$

(To see this, note that $d_1 \vee d_2 = \max\{d_1, d_2\}$.) Now, for $\ell \in C$, we have

$$\begin{aligned} \forall n[\ \ell \leq a \vee c_n\ ] \quad&\iff\quad \ell \leq a \quad\text{or}\quad \forall n[\ \ell \leq c_n\ ] \\ &\iff\quad \ell \leq a \quad\text{or}\quad \ell \leq \textstyle\bigwedge_n c_n \\ &\iff\quad \ell \leq a \vee \textstyle\bigwedge_n c_n. \end{aligned}$$

So we see that $a \vee \bigwedge_n c_n$ is the greatest lower bound of $a \vee c_1$, $a \vee c_2$ ....

With the same argument, one can prove that $a \wedge \bigvee_n c_n = \bigvee a \wedge c_n$ for all a, $c_1, c_2, \ldots \in C$ such that $\bigvee_n c_n$ exists. Hence C is σ-distributive.

(iii) The lattice of the real numbers $\mathbb{R}$ is a chain and so $\mathbb{R}$ is σ-distributive *if* $\mathbb{R}$ would be σ-complete. However, $\mathbb{R}$ is not σ-complete. Indeed, a sequence $c_1, c_2, \ldots$ in $\mathbb{R}$ has a supremum if and only if it is bounded from above, i.e., there is an $a \in \mathbb{R}$ such that $c_n \leq a$ for all n. Similarly, a sequence $c_1, c_2, \ldots \in \mathbb{R}$ has an infimum if and only if it is bounded from below.

(iv) Let $[-\infty, +\infty]$ be the lattice of the extended real numbers. Then $[-\infty, +\infty]$ is a chain and clearly σ-complete. Hence $[-\infty, +\infty]$ is σ-distributive.

(v) Let I be a set, and for each $i \in I$, let L_i be a σ-distributive lattice. Then the product $L := \prod_{i \in I} L_i$ is σ-distributive.

(vi) Let X be a set. Then lattice $[-\infty, +\infty]^X$ of functions from X to $[-\infty, +\infty]$ is σ-distributive.

4.2. **Valuation Systems.**

Definition 72. We say $V \supseteq L \xrightarrow{\varphi} E$ is a **valuation system** provided that

(i) V is a σ-distributive lattce (see Definition 70);
(ii) L is a sublattice of V;
(iii) E is an ordered Abelian group, which is R-complete (see Definition 44);
(iv) $\varphi\colon L \to E$ is a valuation.

Example 73. Let E be an R-complete ordered Abelian group. Let X be a set, $\mathcal{A}$ a ring of subsets of X, and $\mu\colon \mathcal{A} \to E$ a positive and additive map (see Example 9).

Then we have the following valuation system.

$$\wp X \supseteq \mathcal{A} \xrightarrow{\mu} E$$

Indeed, $\wp X$ is lattice with $\bigwedge_n A_n = \bigcap_n A_n$ and $\bigvee_n A_n = \bigcup_n A_n$ for all $A_i \in \wp X$, the set $\mathcal{A}$ is a sublattice of $\wp X$ by definition, $\wp X$ is σ-distributive (see Examples 71(i)) and we have already seen that $\mu\colon \mathcal{A} \to E$ is a valuation (in Example 9).

In particular, we have the following valuation systems.

$$\wp\mathbb{R} \supseteq \mathcal{A}_{\mathcal{L}} \xrightarrow{\mu_{\mathcal{L}}} \mathbb{R} \qquad\qquad \wp\mathbb{R} \supseteq \mathcal{A}_{\mathrm{S}} \xrightarrow{\mu_{\mathrm{S}}} \mathbb{R}$$

See Example 5 and Example 10.

Example 74. Let E be an R-complete ordered Abelian group. Let F be a Riesz space of functions on a set X (see Example 14), and let $\varphi\colon F \to E$ be a positive and linear map. Then we have the following valuation system.

$$[-\infty, \infty]^X \supseteq F \xrightarrow{\varphi} E$$

Indeed, $[-\infty,\infty]^X$ is a σ-distributive lattice (see Examples 71(vi)). Further, F is a sublattice of $\mathbb{R}^X$ which is in turn a sublattice of $[-\infty,\infty]^X$, and we already know that φ is a valuation (see Example 14).

In particular, since $\mathbb{R}$ is R-complete, we have the following valuation systems

$$[-\infty,+\infty]^X \supseteq (F_{\mathcal{L}} \cap \mathbb{R}^{\mathbb{R}}) \xrightarrow{\varphi_{\mathcal{L}}} \mathbb{R}, \qquad [-\infty,+\infty]^{\mathbb{R}} \supseteq F_{\mathrm{S}} \xrightarrow{\varphi_{\mathrm{S}}} \mathbb{R},$$

see Example 7 and Example 15. Recall that $F_{\mathcal{L}} \cap \mathbb{R}^{\mathbb{R}}$ is a Riesz space of functions on X, while $F_{\mathcal{L}}$ is not. Of course, we also have the following valuation sytem.

$$[-\infty,+\infty]^{\mathbb{R}} \supseteq F_{\mathcal{L}} \xrightarrow{\varphi_{\mathcal{L}}} \mathbb{R}$$

Example 75. Let $I=\{1,2\}$. For each $i \in I$, let $V_i \supseteq L_i \xrightarrow{\varphi_i} E_i$ be a valuation system. Then we have the following valuation system (see Example 16).

$$V_1 \times V_2 \supseteq L_1 \times L_2 \xrightarrow{\varphi_1\times\varphi_2} E_1 \times E_2$$

Indeed, the lattice $V_1 \times V_2$ is σ-distributive (see Examples 71(v)), and the ordered Abelian group $E_1 \times E_2$ is R-complete (see Examples 46(v)). We call this system the *product* of $V_1 \supseteq L_1 \xrightarrow{\varphi_1} E_1$ and $V_2 \supseteq L_2 \xrightarrow{\varphi_2} E_2$. Of course, one can similarly define the product of an I-indexed family of valuation systems where I is any set.

Notation 76. *Let $V \supseteq L \xrightarrow{\varphi} E$ be a valuation system. Let $a_1, a_2, \dots$ be from L. Then $a_1, a_2, \dots$ has a supremum in V and might have a supremum in L. We ignore the latter:* ***with*** $\bigvee_n a_n$ ***we always mean the supremum of*** $a_1, a_2, \dots$ ***in*** V*. Similarly,* ***with*** $\bigwedge_n a_n$ ***we always mean the infimum of*** $a_1, a_2, \dots$ ***in*** V.

4.3. **Complete Valuation Systems.**

Definition 77. Let $V \supseteq L \xrightarrow{\varphi} E$ be a valuation system.

(i) We say $V \supseteq L \xrightarrow{\varphi} E$ is **Π-complete**, or φ is **Π-complete relative to** V, or even φ is **Π-complete** (if no confusion should arise with Definition 35), if for every φ-convergent $a_1 \geq a_2 \geq \cdots$ (see Definition 34), we have,

$$\textstyle\bigwedge_n a_n \in L \quad \text{and} \quad \varphi(\bigwedge_n a_n) \;=\; \bigwedge_n \varphi(a_n).$$

Here, $\bigwedge_n a_n$ is the infimum of $a_1 \geq a_2 \geq \cdots$ in V (see Notation 76).

(ii) Similarly, we say $V \supseteq L \xrightarrow{\varphi} E$ is **Σ-complete**, etc., provided that for every φ-convergent sequence $b_1 \leq b_2 \leq \cdots$ we have

$$\textstyle\bigvee_n b_n \in L \quad \text{and} \quad \varphi(\bigvee_n b_n) \;=\; \bigvee_n \varphi(b_n).$$

(iii) We say $V \supseteq L \xrightarrow{\varphi} E$ is **complete**, etc., provided that $V \supseteq L \xrightarrow{\varphi} E$ is both Π-complete and Σ-complete.

Remark 78. Let $V \supseteq L \xrightarrow{\varphi} E$ be a complete valuation system (see Definition 77). Then the valuation φ is also complete in the sense of Definition 35.
We leave it to the reader to verify this.

Example 79. The Lebesgue integral gives us the valuation system

$$[-\infty,+\infty]^{\mathbb{R}} \supseteq F_{\mathcal{L}} \xrightarrow{\varphi_{\mathcal{L}}} \mathbb{R}$$

(see Example 7 and Example 74); we will prove that this system is complete.

Let $f_1 \leq f_2 \leq \cdots$ be a $\varphi_{\mathcal{L}}$-convergent sequence (in $F_{\mathcal{L}}$). We must prove that

$$\textstyle\bigvee_n f_n \in F_{\mathcal{L}} \qquad \text{and} \qquad \varphi_{\mathcal{L}}(\bigvee_n f_n) \;=\; \bigvee_n \varphi_{\mathcal{L}}(f_n).$$

This follows immediately from Levi's Monotone Convergence Theorem.

Similarly, if $g_1 \geq g_2 \geq \cdots$ is a $\varphi_{\mathcal{L}}$-convergent sequence, then

$$\textstyle\bigwedge_n g_n \in F_{\mathcal{L}} \qquad \text{and} \qquad \varphi_{\mathcal{L}}(\bigwedge_n g_n) \;=\; \bigwedge_n \varphi_{\mathcal{L}}(g_n).$$

So the valuation system $[-\infty,+\infty]^{\mathbb{R}} \supseteq F_{\mathcal{L}} \xrightarrow{\varphi_{\mathcal{L}}} \mathbb{R}$ is complete.

Note that we have given a similar argument earlier (see Example 37) to prove that the valuation $\varphi_{\mathcal{L}}$ is complete in the sense of Definition 35.

Example 80. The Lebesgue measure gives us the valuation system

$$\wp\mathbb{R} \supseteq \mathcal{A}_{\mathcal{L}} \xrightarrow{\mu_{\mathcal{L}}} \mathbb{R}$$

(see Example 5 and Example 73); one can prove that this system is complete.
We leave this to the reader (cf. Example 36).

Convention 81. *The complete valuation systems play a far more important role in the remainder of this thesis than the complete valuations of Definition 35. So:* ***Whenever we later on write "φ is complete" we mean that φ is complete relative to*** *V, where V should be clear from the context.*

4.4. **Convex Valuation Systems.**

We have remarked that the Lebesgue measure is complete (see Example 80). It should be noted that "the Lebesgue measure is complete" has a different meaning in the literature, namely that any subset of a Lebesgue neglegible set is neglegible itself. We call this *convexity* and we will briefly discuss this notion in this subsection.

Definition 82. Let $V \supseteq L \xrightarrow{\varphi} E$ be a valuation system.
We say that $V \supseteq L \xrightarrow{\varphi} E$ is **convex**, if the following statement holds.

$$\left[\begin{array}{l} \text{Let } a \leq b \text{ from } L \text{ with } \varphi(a) = \varphi(b) \text{ be given. Then} \\ \qquad\qquad a \leq z \leq b \qquad \Longrightarrow \qquad z \in L, \\ \text{where } z \in V. \end{array}\right.$$

Examples 83. (i) The Lebesgue measure $\wp\mathbb{R} \supseteq \mathcal{A}_{\mathcal{L}} \xrightarrow{\mu_{\mathcal{L}}} \mathbb{R}$ is convex.
(ii) The Lebesgue integral $[-\infty, +\infty]^{\mathbb{R}} \supseteq F_{\mathcal{L}} \xrightarrow{\varphi_{\mathcal{L}}} \mathbb{R}$ is convex.

Example 84. We give a serious example of a non-convex valuation system.
Let $\mathcal{B}$ be the set of all *Borel* subsets of $\mathbb{R}$, that is, those subsets of $\mathbb{R}$ that can be formed by countable intersection and countable union starting from the open subsets of $\mathbb{R}$. Every Borel subset of $\mathbb{R}$ is Lebesgue measurable.

Let $\mathcal{A}_{\mathcal{B}}$ be the set of Borel subsets of $\mathbb{R}$ with finite Lebesgue measure. Note that we have $\mathcal{A}_{\mathcal{B}} \subseteq \mathcal{A}_{\mathcal{L}}$. Let $\mu_{\mathcal{B}} \colon \mathcal{A}_{\mathcal{B}} \to \mathbb{R}$ be the restriction of $\mu_{\mathcal{L}}$ to $\mathcal{A}_{\mathcal{B}}$.

Recall that $\wp\mathbb{R} \supseteq \mathcal{A}_{\mathcal{L}} \xrightarrow{\mu_{\mathcal{L}}} \mathbb{R}$ is a complete and convex valuation system. It is not hard to see that the valuation system $\wp\mathbb{R} \supseteq \mathcal{A}_{\mathcal{B}} \xrightarrow{\mu_{\mathcal{B}}} \mathbb{R}$ is complete as well. However, we will prove that $\wp\mathbb{R} \supseteq \mathcal{A}_{\mathcal{B}} \xrightarrow{\mu_{\mathcal{B}}} \mathbb{R}$ is *not* convex.

To this end, we will find a negligible Borel set A and a subset Δ of A such that Δ is not a Borel set. This is sufficient to prove that $\wp\mathbb{R} \supseteq \mathcal{A}_{\mathcal{B}} \xrightarrow{\mu_{\mathcal{B}}} \mathbb{R}$ is not convex. Indeed, if $\wp\mathbb{R} \supseteq \mathcal{A}_{\mathcal{B}} \xrightarrow{\mu_{\mathcal{B}}} \mathbb{R}$ were convex, then Δ would be Borel, because

$$\varnothing \subseteq \Delta \subseteq A \qquad \text{and} \qquad \varnothing, A \in \mathcal{A}_{\mathcal{B}} \qquad \text{and} \qquad \mu_{\mathcal{B}}(\varnothing) = 0 = \mu_{\mathcal{B}}(A).$$

To find such A and Δ, we need the following fact (see Corollary 129).

$$\left[\begin{array}{l} \text{There is a set } \mathcal{C}, \text{ a negligible Borel set } A \subset \mathbb{R}, \text{ and maps} \\ \qquad\qquad G \colon \mathcal{C} \longrightarrow \mathcal{B} \qquad \text{and} \qquad p \colon \mathcal{C} \longrightarrow A \\ \text{such that } G \text{ is surjective, and } p \text{ is injective.} \end{array}\right. \tag{28}$$

Now, let Δ be the subset of $p(\mathbb{N}^{\mathbb{N}})$ given by, for all $f \in \mathbb{N}^{\mathbb{N}}$,

$$p(f) \in \Delta \qquad \Longleftrightarrow \qquad p(f) \notin G(f).$$

We prove that Δ is not a Borel set. Suppose that Δ is a Borel set in order to derive a contradiction. Since G is surjective, we have $\Delta \equiv G(f_0)$ for some $f_0 \in \mathbb{N}^{\mathbb{N}}$. Then

$$p(f_0) \in \Delta \qquad \Longleftrightarrow \qquad p(f_0) \notin G(f_0) = \Delta.$$

Which is absurd. Thus Δ is not Borel set.
So we see that Δ is a subset of a negligible Borel set, A, while Δ is not Borel. Hence $\wp\mathbb{R} \supseteq \mathcal{A}_{\mathcal{B}} \xrightarrow{\mu_{\mathcal{B}}} \mathbb{R}$ is not convex.

Proposition 85. *Let $V \supseteq L \xrightarrow{\varphi} E$ be a valuation system.*

(i) *Let $L^\bullet$ be the subset of V given by, for all $z \in V$,*

$$z \in L^\bullet \quad\iff\quad \exists\, a,b \in L\ [\quad a \leq z \leq b \quad\wedge\quad \varphi(a) = \varphi(b) \quad].$$

Then L is a sublattice of $L^\bullet$, which is a sublattice of V.

(ii) *There is a unique order preserving map $\varphi^\bullet : L^\bullet \to E$ which extends φ. Moreover, $\varphi^\bullet$ is a valuation, and $V \supseteq L^\bullet \xrightarrow{\varphi^\bullet} E$ is convex.*

Proof. We leave this to the reader. □

Definition 86. The **convexification** of a valuation system $V \supseteq L \xrightarrow{\varphi} E$ is the valuation system $V \supseteq L^\bullet \xrightarrow{\varphi^\bullet} E$ described in Proposition 85.

Proposition 87. *Let $V \supseteq L \xrightarrow{\varphi} E$ be a Π-complete valuation system. Then the valuation system $V \supseteq L^\bullet \xrightarrow{\varphi^\bullet} E$ is Π-complete as well.*

Proof. Let $a_1 \geq a_2 \geq \cdots$ be a $\varphi^\bullet$-convergent sequence in $L^\bullet$. To prove that the valuation system $V \supseteq L^\bullet \xrightarrow{\varphi^\bullet} E$ is Π-complete, we must show that $\bigwedge_n a_n \in L^\bullet$, and

$$\varphi^\bullet(\textstyle\bigwedge_n a_n) \;=\; \bigwedge_n \varphi^\bullet(a_n).$$

Let $n \in \mathbb{N}$ be given. Note that since $a_n \in L^\bullet$ there are $\ell_n, u_n \in L$ such that

$$\ell_n \;\leq\; a_n \;\leq\; u_n \qquad\text{and}\qquad \varphi(\ell_n) \;=\; \varphi(u_n).$$

Define $\ell := \bigwedge_n \ell_n$ and $u := \bigwedge_n u_n$. Then we have $\ell \leq \bigwedge_n a_n \leq u$. So to prove that $\bigwedge_n a_n \in L^\bullet$, it suffices to show that $\ell, u \in L$, and $\varphi(\ell) = \varphi(u)$.

The trick is to consider $\ell_1' \geq \ell_2' \geq \cdots$ and $u_1' \geq u_2' \geq \cdots$ given by, for $n \in \mathbb{N}$,

$$\ell_n' \;:=\; \ell_1 \wedge \cdots \wedge \ell_n \qquad\text{and}\qquad u_n' \;:=\; u_1 \wedge \cdots \wedge u_n.$$

Note that $\ell = \bigwedge_n \ell_n'$ and $u = \bigwedge_n u_n'$. Let $n \in \mathbb{N}$ be given. We claim that

$$\varphi(\ell_n') \;=\; \varphi^\bullet(a_n) \;=\; \varphi(u_n'). \tag{29}$$

Indeed, since $\varphi(\ell_n) = \varphi(u_n)$ and $\ell_n \leq u_n$, we have $\ell_n \approx u_n$ (see Definition 25). Then by Proposition 28(iii) and induction, we see $\ell_n' \approx u_n'$. Hence $\varphi(\ell_n') = \varphi(u_n')$. Now, as $\ell_n' \leq a_n \leq u_n'$, and $\varphi^\bullet$ is order preserving, we get $\varphi(\ell_n') \leq \varphi^\bullet(a_n) \leq \varphi(u_n')$. Hence we easily see that Statement (29) holds.

Since $a_1 \geq a_2 \geq \cdots$ is $\varphi^\bullet$-convergent, we know that $\bigwedge_n \varphi^\bullet(a_n)$ exists. Further,

$$\textstyle\bigwedge_n \varphi(\ell_n') \;=\; \bigwedge_n \varphi^\bullet(a_n) \;=\; \bigwedge_n \varphi(u_n')$$

by St. (29). So we see that $\ell_1' \geq \ell_2' \geq \cdots$ and $u_1' \geq u_2' \geq \cdots$ are φ-convergent. Because φ is Π-complete relative to V we see that $\bigwedge_n \ell_n' = \ell \in L$ and $u \in L$, and

$$\varphi(\ell) \;=\; \textstyle\bigwedge_n \varphi(\ell_n') \;=\; \bigwedge_n \varphi^\bullet(a_n) \;=\; \bigwedge_n \varphi(u_n') \;=\; \varphi(u). \tag{30}$$

So, since $\ell \leq \bigwedge_n a_n \leq u$, Statement (30) implies that $\bigwedge_n a_n \in L^\bullet$, and

$$\varphi^\bullet(\textstyle\bigwedge_n a_n) \;=\; \bigwedge_n \varphi^\bullet(a_n).$$

Hence $\varphi^\bullet$ is Π-complete relative to V. □

Proposition 88. *Let $V \supseteq L \xrightarrow{\varphi} E$ be a complete valuation system. Then the valuation system $V \supseteq L^\bullet \xrightarrow{\varphi^\bullet} E$ is complete as well.*

Proof. Apply Proposition 87 and its dual. □

5. The Completion

5.1. **Introduction.** Valuation systems were introduced in Section 4 to give meaning to the phrase "the Lebesgue integral $\varphi_{\mathcal{L}}$ is a completion of φ_{S}", namely,

$$[-\infty,+\infty]^{\mathbb{R}} \supseteq F_{\mathcal{L}} \xrightarrow{\varphi_{\mathcal{L}}} \mathbb{R} \text{ is complete} \qquad \text{and} \qquad \varphi_{\mathcal{L}} \text{ extends } \varphi_{\mathrm{S}}.$$

In this section we replace φ_{S} and $[-\infty,+\infty]^{\mathbb{R}}$ by any valuation system $V \supseteq L \xrightarrow{\varphi} E$ and study when, so to say, φ has *a completion with respect to* V, i.e., when there is a sublattice C of V and a valuation $\psi\colon C \to E$ such that

$$V \supseteq C \xrightarrow{\psi} E \text{ is complete} \qquad \text{and} \qquad \psi \text{ extends } \varphi.$$

There is not always a completion (see Example 89). However, if φ has a completion with respect to V, then φ also has a smallest[2] completion with respect to V,

$$V \supseteq \overline{L} \xrightarrow{\overline{\varphi}} E.$$

We describe $\overline{\varphi}$ in Subsection 5.3, and call $\overline{\varphi}$ *the completion* of φ with respect to V.

Π-*Completion.* If we replace "complete" by "Π-complete" in the above discussion the situation is much easier (see Definition 77). Indeed, we will see in Lemma 103 that φ has a Π-completion with respect to V iff

$$\left[\begin{array}{l} \text{for every } b \in L \text{ and } \varphi\text{-convergent } a_1 \geq a_2 \geq \cdots \text{ we have} \\ \qquad \bigwedge_n a_n \leq b \quad \Longrightarrow \quad \bigwedge_n \varphi(a_n) \leq \varphi(b). \end{array}\right.$$

Similar to before, if there is a Π-completion of φ with respect to V, then there is also a smallest Π-completion with respect to V, which will be denoted by

$$V \supseteq \Pi L \xrightarrow{\Pi\varphi} E.$$

We will study $\Pi\varphi$ in Subsection 5.2.

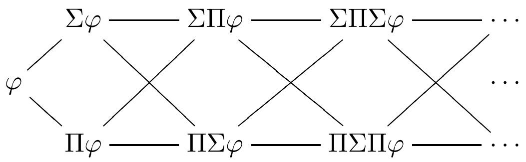
Σ-*Completion.* We can replace "complete" by "Σ-complete" as well.
If it exists, the smallest Σ-completion of φ with respect to V is denoted by

$$V \supseteq \Sigma L \xrightarrow{\Sigma\varphi} E.$$

Since $\Pi\varphi$ and $\Sigma\varphi$ are very similar, we will only study $\Pi\varphi$. All the results that we obtain about $\Pi\varphi$ and all the definitions for $\Pi\varphi$ can be easily translated to results about $\Sigma\varphi$ and definitions for $\Sigma\varphi$, respectively. We leave this to the reader.

Hierarchy of Extensions. Recall that φ is complete with respect to V if and only if φ is both Π-complete and Σ-complete with respect to V (see Definition 77).

So if we want to find a completion of φ with respect to V it is natural to try and see if $\Sigma\Pi\varphi$ is complete (when it exists) with respect to V. Unfortunately, while $\Pi\varphi$ is Π-complete with respect to V, the valuation $\Sigma\Pi\varphi$ need not be Π-complete. Nevertheless, we can continue to apply "Σ" and "Π" whenever possible, and we will see in Subsection 5.5 that this leads to a 'hierarchy' of the following shape.



It will become clear, that if the making of this hierarchy is hindered, e.g., if $\Pi\Sigma\varphi$ has no Σ-completion with respect to V (and so $\Sigma\Pi\Sigma\varphi$ does not exist), then φ cannot have a completion with respect to V. On the other hand, we will see, loosely speaking, that if the making of the hierarchy can proceed unhindered even if we

[2] "Smallest" with respect to the ordering on partial functions given by $f \leq g$ iff g extends f.

go on endlessly using ordinal numbers, that then we eventually obtain the smallest completion $\overline{\varphi}$ of φ with respect to V. This will all become clear in Subsection 5.5.

With some luck, the valuation $\Sigma\Pi\Sigma\varphi$ might already be complete with respect to V. We could then say that "we hit $\overline{\varphi}$ in 3 steps". We will prove in Subsection 5.4 that in general we need to make *uncountably* many steps before we hit $\overline{\varphi}$.

No Completion. Let us end the introducion with an example of a valuation system $V \supseteq L \xrightarrow{\varphi} E$ such that φ has no completion with respect to V.

Example 89. Let L be the sublattice of $\mathbb{R}$ given by

$$L \ := \ \{ \ n^{-1} : \ n \in \mathbb{N} \ \} \cup \{0\}.$$

Let $\varphi\colon L \longrightarrow \mathbb{R}$ be the valuation given by, for all $n \in \mathbb{N}$,

$$\varphi(\, n^{-1} \,) \ = \ 1 \qquad \text{and} \qquad \varphi(\, 0 \,) \ = \ 0.$$

Then we have a valuation system $\mathbb{R} \supseteq L \xrightarrow{\varphi} \mathbb{R}$.

Let C be a sublattice of $\mathbb{R}$, and let $\psi\colon C \longrightarrow \mathbb{R}$ be the valuation given by

$$\mathbb{R} \supseteq C \xrightarrow{\psi} \mathbb{R} \ \text{ is complete} \qquad \text{and} \qquad \psi \text{ extends } \varphi.$$

We will prove that this is not possible.

Consider the sequence $a_1 \geq a_2 \geq \cdots$ in L given by $a_n = n^{-1}$. We have

$$\psi(a_n) \ = \ \varphi(a_n) \ = \ 1.$$

So $\bigwedge_n \psi(a_n) = 1$. In particular, $a_1 \geq a_2 \geq \cdots$ is ψ-convergent (see Definition 34). Since ψ is complete with respect to $\mathbb{R}$ (see Definition 77), we get

$$1 \ = \ \textstyle\bigwedge_n \psi(a_n) \ = \ \psi(\, \bigwedge_n a_n \,) \ = \ \psi(\, \bigwedge_n n^{-1} \,) \ = \ \psi(\, 0 \,) \ = \ 0,$$

which is absurd. Hence φ has no completion with respect to φ.

5.2. **The Π-Extension.**

Definition 90. Let $V \supseteq L \xrightarrow{\varphi} E$ be a valuation system. Define

$$\Pi L \;:=\; \{\ \textstyle\bigwedge_n a_n\colon\ a_1 \geq a_2 \geq \cdots \text{ from } L \text{ is } \varphi\text{-convergent}\ \}.$$

Remark 91. Let $V \supseteq L \xrightarrow{\varphi} E$ be a valuation system.
Note that if φ is Π-complete (see Definition 77), then $\Pi L = L$.

Lemma 92. *Let $V \supseteq L \xrightarrow{\varphi} E$ be a valuation system.*
Then ΠL is a sublattice of V, and L is a sublattice of ΠL.

Proof. We first prove that ΠL is a sublattice of V. Let $a, b \in \Pi L$ be given; we need to prove that $a \wedge b \in \Pi L$ and $a \vee b \in \Pi L$. Choose φ-convergent $a_1 \geq a_2 \geq \cdots$ and $b_1 \geq b_2 \geq \cdots$ with $a = \bigwedge_n a_n$ and $b = \bigwedge_n b_n$. Then $a_1 \wedge b_1 \geq a_2 \wedge b_2 \geq \cdots$ is φ-convergent by Proposition 48, and we have $\bigwedge_n a_n \wedge b_n = a \wedge b$. Hence $a \wedge b \in \Pi L$. Similarly, $a_1 \vee b_1 \geq a_2 \vee b_2 \geq \cdots$ is φ-convergent by Proposition 48 and using σ-distributivity one can prove that $a \vee b = \bigwedge_n a_n \vee b_n$. Hence $a \vee b \in \Pi L$.

To prove that L is a sublattice of ΠL, we first note that L is a subset of ΠL. Now, since both L and ΠL are sublattices of V, and L is a subset of ΠL, we know that L must be a sublattice of ΠL. □

Remark 93. In the proof of Lemma 92, we have used the fact that V is σ-distributive and the fact that E is R-complete (via Proposition 48).

Definition 94. Let $V \supseteq L \xrightarrow{\varphi} E$ be a valuation system.
We say φ is **Π-extendible** if there is a valuation $\psi\colon \Pi L \to E$ with

$$\psi(\textstyle\bigwedge_n a_n) \;=\; \textstyle\bigwedge_n \varphi(a_n) \qquad \text{for all } \varphi\text{-convergent } a_1 \geq a_2 \geq \cdots.$$

Clearly, there can be at most one such map ψ; if it exists, we denote it by

$$\Pi\varphi\colon \Pi L \longrightarrow E.$$

Finally, note that if φ is Π-extendible, then $\Pi\varphi$ extends φ (hence the name).

Remark 95. Let $V \supseteq L \xrightarrow{\varphi} E$ be a valuation system.

(i) Note that if φ is Π-complete with respect to V,
then φ is Π-extendible and $\Pi\varphi = \varphi$.

(ii) On the other hand, if φ is Π-extendible, and $\Pi\varphi = \varphi$,
then φ is Π-complete with respect to V.

Example 96. The following valuation systems are Π-extendible.

$$\wp\mathbb{R} \supseteq \mathcal{A}_{\mathcal{L}} \xrightarrow{\mu_{\mathcal{L}}} \mathbb{R} \qquad \text{and} \qquad [-\infty, +\infty]^{\mathbb{R}} \supseteq F_{\mathcal{L}} \xrightarrow{\varphi_{\mathcal{L}}} \mathbb{R}$$

Indeed, this follows by Remark 95(i) since these valuation systems are Π-complete. More interestingly, the following valuation systems are complete as well.

$$\wp\mathbb{R} \supseteq \mathcal{A}_{\mathrm{S}} \xrightarrow{\mu_{\mathrm{S}}} \mathbb{R} \qquad \text{and} \qquad [-\infty, +\infty]^{\mathbb{R}} \supseteq F_{\mathrm{S}} \xrightarrow{\varphi_{\mathrm{S}}} \mathbb{R}$$

This will follow from Lemma 100.

Example 97. We leave it to the reader to verify that the valuation system

$$\mathbb{R} \supseteq L \xrightarrow{\varphi} \mathbb{R}$$

from Example 89 is *not* Π-extendible.

Lemma 98. *Let $V \supseteq L \xrightarrow{\varphi} E$ be a valuation system.*
If φ is Π-extendible, then $V \supseteq \Pi L \xrightarrow{\Pi\varphi} E$ is Π-complete.

Proof. Let $a^1 \geq a^2 \geq \cdots$ from ΠL be given and suppose that $\bigwedge_n \Pi\varphi(a^n)$ exists; we need to prove that $\bigwedge_n a^n \in \Pi L$ and that $\Pi\varphi(\bigwedge_n a_n) = \bigwedge_n \Pi\varphi(a_n)$ (see Def. 77).

To begin, write $a^n = \bigwedge_n a^n_m$ where $a^n_1 \geq a^n_2 \geq \cdots$ is a φ-convergent sequence in L for each $n \in \mathbb{N}$. Define for each $i \in \mathbb{N}$ an element $b_i \in L$ by

$$b_i \;:=\; \bigwedge\{\, a^n_m \colon n, m \leq i \,\}.$$

Then $b_1 \geq b_2 \geq \cdots$ and $\bigwedge_n b_n = \bigwedge_n a^n$. Recall that $\bigwedge_n \Pi\varphi(a^n)$ exists. *We claim that $\bigwedge_n \Pi\varphi(a^n)$ is the infimum of $\varphi(b_1) \geq \varphi(b_2) \geq \cdots$.* If we can prove this, we are done. Indeed, then $b_1 \geq b_2 \geq \cdots$ is φ-convergent, so $\bigwedge_n a^n = \bigwedge_n b_n \in \Pi L$, and

$$\begin{aligned} \Pi\varphi(\textstyle\bigwedge_n a^n) \;&=\; \Pi\varphi(\textstyle\bigwedge_n b_n) && \text{since } \textstyle\bigwedge_n a^n = \bigwedge_n b_n, \\ &=\; \textstyle\bigwedge_n \varphi(b_n) && \text{since } \varphi \text{ is } \Pi\text{-extendible}, \\ &=\; \textstyle\bigwedge_n \Pi\varphi(a^n) && \text{by the claim.} \end{aligned}$$

Let us prove that $\bigwedge_n \Pi\varphi(a^n)$ is the infimum of $\varphi(b_1) \geq \varphi(b_2) \geq \cdots$.

For each i, we have $b_i \geq \bigwedge_{n \leq i} a^n = a^i$, so $\varphi(b_i) = \Pi\varphi(b_i) \geq \Pi\varphi(a^i)$. Hence we see that $\bigwedge_n \Pi\varphi(a^n)$ is a lower bound of $\varphi(b_1) \geq \varphi(b_2) \geq \cdots$.

On the other hand: Let ℓ be a lower bound of $\varphi b_1 \geq \varphi b_2 \geq \cdots$; we need to prove that $\ell \leq \bigwedge_n \Pi\varphi(a^n)$. For all n and m, we have $a^n_m \geq b_{n \vee m}$ and so $\varphi(a^n_m) \geq \varphi(b_{n\vee m}) \geq \ell$. Hence $\Pi\varphi(a^n) = \bigwedge_m \varphi(a^n_m) \geq \ell$ for all n. So $\bigwedge_n \Pi\varphi(a^n) \geq \ell$.

So $\bigwedge_n \Pi\varphi(a^n)$ is the infimum of $\varphi b_1 \geq \varphi b_2 \geq \cdots$, and we are done. □

Remark 99. Let $V \supseteq L \xrightarrow{\varphi} E$ be a Π-extendible valuation system.
By Lemma 98 we see that $\Pi\varphi$ is Π-complete with respect to V.
So by Remark 95(i) we see that $\Pi\varphi$ is Π-extendible, and that

$$\Pi(\Pi\varphi) \;=\; \Pi\varphi.$$

Lemma 100. *Let $V \supseteq L \xrightarrow{\varphi} E$ be a valuation system.*
Let C be a sublattice of V. Let $\psi\colon C \to E$ be a valuation. Assume

$$\psi \text{ extends } \varphi \qquad \text{and} \qquad V \supseteq C \xrightarrow{\psi} E \;\text{ is } \Pi\text{-complete.}$$

Then φ is Π-extendible and ψ extends $\Pi\varphi$.

Proof. We must prove that φ is Π-extendible and that ψ extends $\Pi\varphi$.
Before we do this, we will prove that for every φ-convergent $a_1 \geq a_2 \geq \cdots$, we have

$$\textstyle\bigwedge_n a_n \in C \qquad \text{and} \qquad \psi(\bigwedge_n a_n) = \bigwedge_n \varphi(a_n). \tag{31}$$

We know that $\bigwedge_n \varphi(a_n)$ exists (since $a_1 \geq a_2 \geq \cdots$ is φ-convergent), and that $\varphi(a_n) = \psi(a_n)$ (since ψ extends φ). So $\bigwedge_n \psi(a_n)$ exists too. Hence $a_1 \geq a_2 \geq \cdots$ is ψ-convergent. Because $V \supseteq C \xrightarrow{\psi} E$ is Π-complete this implies that $\bigwedge_n a_n \in C$ and $\psi(\bigwedge_n a_n) = \bigwedge_n \psi(a_n)$ (see Definition 77). Hence we have proven Statement (31).

Statement (31) implies that $\Pi L \subseteq C$. So in order to prove that φ is Π-extendible, let us consider the valuation $\mu := \psi \,|\, \Pi L$. In order to prove that φ is Π-extendible we must show that $\mu(\bigwedge_n a_n) = \bigwedge_n \varphi(a_n)$ for every φ-convergent sequence $a_1 \geq a_2 \geq \cdots$ (see Def. 94), but this follows immediately from St. (31). Hence φ is Π-extendible.

Finally, since we know that φ is Π-extendible, we can talk about $\Pi\varphi$, and write the second part of St. (31) as $\psi(\bigwedge_n a_n) = \Pi\varphi(\bigwedge_n a_n)$. Hence ψ extends $\Pi\varphi$. □

Corollary 101. *Let $V \supseteq L \xrightarrow{\varphi} E$ be a valuation system.*
Then φ is Π-extendible iff there is a valuation $\psi\colon C \to E$ such that

$$V \supseteq C \xrightarrow{\psi} E \;\text{ is } \Pi\text{-complete} \qquad \text{and} \qquad \psi \text{ extends } \varphi.$$

(So, loosely speaking, φ is Π-extendible iff φ has a Π-complete extension.)

Proof. Combine Lemma 100 and Lemma 98. □

Lemma 102. *Let $V \supseteq L \xrightarrow{\varphi} E$ be a valuation system. Let K be a sublattice of L, and let $\psi\colon K \to E$ be a valuation which extends φ. Suppose that ψ is Π-extendible. Then φ is Π-extendible and $\Pi\psi$ extends $\Pi\varphi$.*

Proof. Note that $\Pi\psi$ extends φ, and $V \supseteq \Pi K \xrightarrow{\Pi\psi} E$ is Π-complete (see Lemma 98). So Lemma 100 implies that φ is Π-extendible and $\Pi\psi$ extends $\Pi\varphi$. $\square$

Lemma 103. *Let $V \supseteq L \xrightarrow{\varphi} E$ be a valuation system. Then φ is Π-extendible if and only if φ has the following property.*

$$\left[\begin{array}{c} \text{For every } b \in L \text{ and } \varphi\text{-convergent } a_1 \geq a_2 \geq \cdots \text{ from } L, \\ \textstyle\bigwedge_n a_n \;\leq\; b \quad\implies\quad \bigwedge_n \varphi(a_n) \;\leq\; \varphi(b). \end{array}\right. \tag{32}$$

Proof. "$\Longrightarrow$" Suppose φ is Π-extendible. Then φ has Property (32), because if $b \in L$ and φ-convergent $a_1 \geq a_2 \geq \cdots$ with $\bigwedge_n a_n \leq b$ are given, then we have

$$\textstyle\bigwedge_n \varphi(a_n) \;=\; \Pi\varphi(\,\bigwedge_n a_n\,) \;\leq\; \Pi\varphi(b) \;=\; \varphi(b).$$

"$\Longleftarrow$" Suppose φ has Property (32); we prove φ is Π-extendible. We claim that

$$\textstyle\bigwedge_n a_n \;\leq\; \bigwedge_n b_n \quad\implies\quad \bigwedge_n \varphi(a_n) \;\leq\; \bigwedge_n \varphi(b_n), \tag{33}$$

where $a_1 \geq a_2 \geq \cdots$ and $b_1 \geq b_2 \geq \cdots$ are φ-convergent sequences in L.

Indeed, if $\bigwedge_n a_n \leq \bigwedge_n b_n$, then $\bigwedge_n a_n \leq b_m$ for all m, so $\bigwedge_n \varphi(a_n) \leq \varphi(b_m)$ for all m (by Property (32)), and hence $\bigwedge_n \varphi(a_n) \leq \bigwedge_n \varphi(b_m)$.

Statement (33) implies that

$$\textstyle\bigwedge_n a_n \;=\; \bigwedge_n b_n \quad\implies\quad \bigwedge_n \varphi(a_n) \;=\; \bigwedge_n \varphi(b_n),$$

so there is a unique map $\psi\colon \Pi L \to E$ such that

$$\textstyle\psi(\,\bigwedge_n a_n\,) \;=\; \bigwedge_n \varphi(a_n) \qquad \text{for all } \varphi\text{-convergent } a_1 \geq a_2 \geq \cdots. \tag{34}$$

In fact, Statement (33) also implies that ψ is order preserving.

To prove that φ is Π-extendible (see Definition 94), it suffices to show that ψ is a valuation. For this, it remains to be shown that ψ is modular (see Definition 3).

Let $a, b \in \Pi L$ be given. To show that ψ is modular, we need to prove that

$$\psi(a \wedge b) \;+\; \psi(a \vee b) \;=\; \psi(a) \;+\; \psi(b).$$

Write $a = \bigwedge_n a_n$ and $b = \bigwedge_n b_n$ where $a_1 \geq a_2 \geq \cdots$ and $b_1 \geq b_2 \geq \cdots$ from L are φ-convergent sequences. Then we have

$$\begin{aligned}
\varphi(a \wedge b) + \varphi(a \vee b) \;&=\; \psi(\textstyle\bigwedge_n a_n \wedge \bigwedge_n b_n) \;+\; \psi(\bigwedge_n a_n \vee \bigwedge_n b_n) \\
&=\; \psi(\textstyle\bigwedge_n a_n \wedge b_n) \;+\; \psi(\bigwedge_n a_n \vee b_n) \\
&=\; \textstyle\bigwedge_n \varphi(a_n \wedge b_n) \;+\; \bigwedge_n \varphi(a_n \vee b_n) \\
&=\; \textstyle\bigwedge_n (\,\varphi(a_n \wedge b_n) \;+\; \varphi(a_n \vee b_n)\,) \\
&=\; \textstyle\bigwedge_n (\,\varphi(a_n) \;+\; \varphi(b_n)\,) \\
&\;\;\vdots \\
&=\; \psi(\textstyle\bigwedge_n a_n) \;+\; \psi(\bigwedge_n b_n).
\end{aligned}$$

Hence ψ is modular, which completes the proof that φ is Π-extendible. $\square$

5.3. **The Smallest Complete Extension.**

In the previous subsection, we described the smallest Π-complete extension of a valuation system (when it exists). In this subsection, we will describe the smallest complete extension of a valuation system (when it exists).

Situation 104. *Let $V \supseteq L \xrightarrow{\varphi} E$ and $V \supseteq C \xrightarrow{\psi} E$ be valuation systems such that*

$$\psi \text{ extends } \varphi \qquad \text{and} \qquad \psi \text{ is complete.}$$

In particular $V \supseteq C \xrightarrow{\psi} E$ must be Π-complete (see Definition 77).
Hence Lemma 100 implies that φ is Π-extendible and that ψ extends $\Pi\varphi$.
Thus, loosely speaking, $\Pi\varphi$ is the smallest extension of φ which is Π-complete.
In this subsection, we identify the smallest extension $\overline{\varphi}$ of φ which is complete.
We tackle this problem in order to familiarise the reader with the notions needed to define "$V \supseteq L \xrightarrow{\varphi} E$ is extendible" (see Definition 141). These notions, which we introduce rather informally in this subsection, will be defined rigorously and in a more general setting later on.

Let us begin. Note that $V \supseteq C \xrightarrow{\psi} E$ is also Σ-complete. Hence φ is Σ-extendible, and ψ extends $\Sigma\varphi$. So we have the following situation.

$$\psi \text{ extends both } \Pi\varphi \text{ and } \Sigma\varphi \qquad \text{and} \qquad V \supseteq C \xrightarrow{\psi} E \text{ is complete.}$$

By a similar reasoning, we see that $\Pi\varphi$ is Σ-extendible, and that $\Sigma\varphi$ is Π-extendible and that ψ extends both $\Sigma\Pi\varphi$ and $\Pi\Sigma\varphi$. (Note that $\Pi(\Pi\varphi) = \Pi\varphi$, see Rem. 99). So we have the following situation.

$$\psi \text{ extends both } \Sigma\Pi\varphi \text{ and } \Pi\Sigma\varphi \qquad \text{and} \qquad V \supseteq C \xrightarrow{\psi} E \text{ is complete.}$$

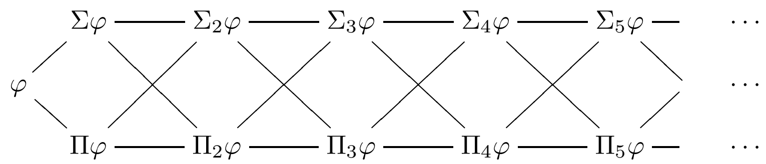
Of course, we can continue this proces. More formally, the clauses

$$\begin{array}{lll} \Pi_{n+1}\varphi \,=\, \Pi(\Sigma_n\varphi) & \Sigma_{n+1}\varphi \,=\, \Sigma(\Pi_n\varphi) & \Pi_0\varphi \,=\, \varphi \,=\, \Sigma_0\varphi \\ \Pi_{n+1}L \,=\, \Pi(\Sigma_n L) & \Sigma_{n+1}L \,=\, \Sigma(\Pi_n L) & \Pi_0 L \,=\, L \,=\, \Sigma_0 L, \end{array}$$

give us valuation systems $V \supseteq \Pi_n L \xrightarrow{\Pi_n\varphi} E$ and $V \supseteq \Sigma_n L \xrightarrow{\Sigma_n\varphi} E$ for all $n \in \omega$.

Note that $\Pi\varphi$ extends φ. Hence $\Sigma_2\varphi$ extends $\Sigma\varphi$ by Lemma 102. Hence $\Pi_3\varphi$ extends $\Pi_2\varphi$. Etcetera. Similarly, $\Sigma\varphi$ extends φ, so $\Pi_2\varphi$ extends $\Pi\varphi$, and so on.

The hierarchy which we have obtained is shown in the following diagram.



We say that the *hierarchy collapsed at* Q, where $Q \in \{\, L,\, \Pi_1 L,\, \Sigma_1 L,\, \Pi_2 L,\, \dots \}$, if

$$\Pi(Q) \,=\, Q \,=\, \Sigma(Q).$$

In that case, let $q\colon Q \to E$ be the associated valuation (either $\Pi_n\varphi$ or $\Sigma_n\varphi$ for some n). Then $V \supseteq Q \xrightarrow{q} E$ is complete, since it is Π-complete and Σ-complete.

Note that the definition of $\Pi_n\varphi$ and $\Sigma_n\varphi$ does not depend on which complete extension ψ of φ is given, only on the fact that such ψ exists. In particular, if $V \supseteq C' \xrightarrow{\psi'} E$ is any complete valuation system such that ψ' extends φ, then ψ' extends $\Pi_n L$ and $\Sigma_n L$. In particular, such ψ' extends q. Hence q is the smallest complete extension of φ we sought.

However, in general the hierarchy need not have collapsed at any $\Pi_n L$ or $\Sigma_n L$, as we will show later on, in Subsection 5.4. So to find the smallest complete extension of φ, we need to push forwards.

To this end, consider the sets $\Pi_\omega L$ and $\Sigma_\omega L$ given by

$$\Pi_\omega L \,:=\, \textstyle\bigcup_n \Pi_n L \qquad \text{and} \qquad \Sigma_\omega L \,:=\, \textstyle\bigcup_n \Sigma_n L.$$

Since $\Pi_n L \subseteq \Sigma_{n+1} L$ and $\Sigma_n L \subseteq \Pi_{n+1} L$ for all n, we see that $\Pi_\omega L = \Sigma_\omega L$.

Now, since $\Pi_n \varphi$ extends $\Pi_m \varphi$ for $n \geq m$, there is a unique map $\Pi_\omega \varphi \colon \Pi_\omega L \to E$ which extends all $\Pi_n \varphi$. One can easily see that $V \supseteq \Pi_\omega L \xrightarrow{\Pi_\omega \varphi} E$ is a valuation system. Similarly, there is a unique map $\Sigma_\omega \varphi \colon \Sigma_\omega L \to E$ which extends all $\Sigma_n \varphi$. Then $V \supseteq \Sigma_\omega L \xrightarrow{\Sigma_\omega \varphi} E$ is a valuation system.

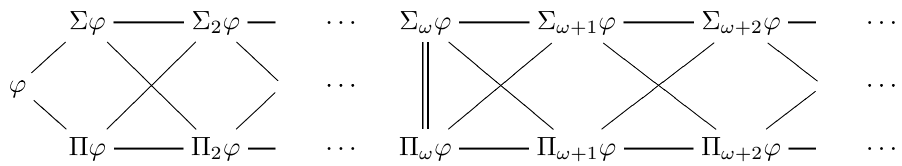
Since $\Pi_{n+1} \varphi$ extends $\Sigma_n \varphi$ for all n, one sees that $\Pi_\omega \varphi = \Sigma_\omega \varphi$.

Again, the hierarchy might have collapsed at $\Pi_\omega L$, i.e.,

$$\Pi(\Pi_\omega L) \;=\; \Pi_\omega L \;=\; \Sigma(\Pi_\omega L).$$

In that case $\Pi_\omega \varphi$ the minimal completion of φ we sought.

However, again the hierarchy might not have collapsed at $\Pi_\omega L$, so we consider the valuations $\Pi_{\omega+n} \varphi := \Pi_n(\Pi_\omega \varphi)$ and $\Sigma_{\omega+n} \varphi := \Sigma_n(\Pi_\omega \varphi)$.



With induction on ordinal numbers, we can continue this process endlessly. However, the collapse of the hierarchy can not be postponed indefinitely.

More formally, let $\overline{L} := \{\, c \in C \colon \exists \alpha [\, c \in \Pi_\alpha L \,]\,\}$. Then we have $\Pi_\alpha L \subseteq \overline{L}$ for every (ordinaln number) α. We want to prove that $\Pi_\alpha L = \overline{L}$ for some α. Define

$$\alpha(c) \;=\; \min\{\; \beta \colon\; c \in \Pi_\beta L \;\} \qquad\qquad (c \in \overline{L}).$$

Then the set of ordinal numbers $\{\, \alpha(c) \colon\; c \in \overline{L} \,\}$ has a supremum, say ξ. We have

$$c \in \Pi_{\alpha(c)} L \;\subseteq\; \Pi_\xi L \qquad\qquad (c \in \overline{L}).$$

So $\overline{L} \subseteq \Pi_\xi L$. But we already had $\Pi_\xi L \subseteq \overline{L}$. Hence $\Pi_\xi L = \overline{L}$.

We claim that the hierarchy has collapsed at $\Pi_\xi L$, i.e.,

$$\Pi(\Pi_\xi L) \;=\; \Pi_\xi L \;=\; \Sigma(\Pi_\xi L).$$

Indeed, we have

$$\Pi_\xi L \;\subseteq\; \Sigma(\Pi_\xi L) \;\subseteq\; \overline{L} \;=\; \Pi_\xi L.$$

So $\Sigma(\Pi_\xi L) = \Pi_\xi L$. Similarly, $\Pi_\xi L = \Pi(\Pi_\xi L)$.

One can easily verify that $\overline{\varphi} := \Pi_\xi \varphi$ is the smallest complete extension of φ.

All in all, we have proven the following.

Proposition 105. *Let* $V \supseteq L \xrightarrow{\varphi} E$ *be valuation system.*
If there is a valuation $\psi \colon C \to E$ *such that*

$$\psi \text{ extends } \varphi \qquad \text{and} \qquad V \supseteq C \xrightarrow{\psi} E \text{ is complete,}$$

then there is a smallest *such valuation,*
that is, there is a valuation $\overline{\varphi} \colon \overline{L} \to E$ *such that*

$$\overline{\varphi} \text{ extends } \varphi \qquad \text{and} \qquad V \supseteq \overline{L} \xrightarrow{\overline{\varphi}} E \text{ is complete,}$$

and such that ψ' *extends* $\overline{\varphi}$ *for every valuation* $\psi' \colon C \to E$ *with*

$$\psi' \text{ extends } \varphi \qquad \text{and} \qquad V \supseteq C \xrightarrow{\psi'} E \text{ is complete.}$$

Moreover, $\overline{\varphi} = \Pi_\xi \varphi$ *for some ordinal number* ξ.

5.4. **The Borel Hierarchy Theorem.**
Before we continue our study of the hierarchy introduced in the Subsection 5.3, let us take a step back and wonder: is this all — the endless hierarchy — neccesary?

Indeed, using the terminology of Subsection 5.3, it is not unthinkable that the hierarchy is always collapsed at, say $\Sigma_{37}L$. In that case the theory would be much simpler; we would only need to use the symbols up to "Σ_{37}". In particular, the involvement of the (infinite) ordinal numbers would not be required.

It turns out that we *do* need a large amount of symbols to desribe the hierarchy. In this subsection we will give an example where the hierarchy can only be collaped at $\Pi_\alpha L$ or at $\Sigma_\alpha L$ if the ordinal number α is *uncountable*, see Proposition 137.

On the bright side, it does not get worse than this: we will see (in Lemma 144) that the hierarchy is always collapsed at $\Pi_{\aleph_1} L$, where $\aleph_1$ is the set of all countable ordinal numbers, i.e., the smallest uncountable ordinal number.

The material in the subsection will not be used later on. So the reader can safely skip this subsection and proceed to Subsection 5.5 on page 56 if so desired.

5.4.1. *Borel Subsets.*
Our example involves Borel sets. Recall that the *Borel subsets* of a topological space X (such as $\mathbb{R}$) are those subsets one can form using countable intersection and countable union starting from the open subsets.

Instead of $\mathbb{R}$, we work with the Borel subsets of the *Baire space*, $\mathbb{N}^\mathbb{N}$. In short, the topology on $\mathbb{N}^\mathbb{N}$ is the product topology when $\mathbb{N}$ is given the discrete toplogy. To understand these words, one might look at [7], but this is not necessary as we will give a more direct description of $\mathcal{T}$ in Subsubsection 5.4.3.

While we could do the following for $\mathbb{R}$ as well, it is much easier for $\mathbb{N}^\mathbb{N}$.

Notation 106. *Let $\mathcal{T}$ denote the set of open subsets of $\mathbb{N}^\mathbb{N}$, and let $\mathcal{B}$ denote the set of Borel subsets of $\mathbb{N}^\mathbb{N}$.*

Note that $\mathcal{B}$ is a sublattice of $\wp(\mathbb{N}^\mathbb{N})$, and $\mathcal{B}$ is a sublattice of $\mathcal{T}$.

Definition 107. Let $\psi\colon \mathcal{B} \to \mathbb{R}$ be the map given by, for all $A \in \mathcal{B}$,

$$\psi(A) \;=\; 0.$$

Then ψ is a valuation, and we have the following valuation system.

$$\wp(\mathbb{N}^\mathbb{N}) \supseteq \mathcal{B} \xrightarrow{\psi} \mathbb{R}.$$

Lemma 108. *The valuation ψ is complete with respect to $\wp(\mathbb{N}^\mathbb{N})$ (see Def. 77).*

Proof. Let $A_1 \supseteq A_2 \supseteq \cdots$ and $B_1 \subseteq B_2 \subseteq \cdots$ be a ψ-convergent sequences in $\mathcal{B}$. To prove that ψ is complete relative to $\wp(\mathbb{N}^\mathbb{N})$ we must show that

$$\textstyle\bigcap_n A_n \in \mathcal{B} \qquad \text{and} \qquad \bigcup_n B_n \in \mathcal{B}.$$

This follows immediately by definition of the Borel subsets. □

Remark 109. Let $\mathcal{A}$ be a sublattice of $\mathcal{B}$ and let

$$\varphi\colon \mathcal{A} \longrightarrow \mathbb{R}$$

be the restriction of ψ to $\mathcal{A}$. Note that we are in Situation 104,

$$\psi \text{ extends } \varphi \qquad \text{and} \qquad \psi \text{ is complete.}$$

Using the notation from Subsection 5.3, let us see what $\Pi\mathcal{A}$ and $\Sigma\mathcal{A}$ are.

Note that every sequence $A_1 \supseteq A_2 \supseteq \cdots$ in $\mathcal{A}$ is φ-convergent, as ψ is constant. Further, given $A_1, A_2, \ldots \in \mathcal{A}$, we have $\bigcap_n A_n = \bigcap_n A'_n$, where $A'_1 \supseteq A'_2 \supseteq \cdots$ are defined by $A'_n = A_1 \cap \cdots \cap A_n$. So we see that

$$\Pi\mathcal{A} \;=\; \{\, \textstyle\bigcap_n A_n \colon\; A_1, A_2, \ldots \in \mathcal{A} \,\}. \tag{35}$$

By a similar reasoning it is not hard to see that

$$\Sigma\mathcal{A} \;=\; \{\; \bigcup_n A_n :\; A_1, A_2, \ldots \,\in\, \mathcal{A}\;\}. \tag{36}$$

5.4.2. *Statement of the Borel Hierarchy Theorem.*
Let us spend words on where we are headed. We will define a sublattice $\mathcal{A}$ of $\wp(\mathbb{N}^{\mathbb{N}})$ in such a way that, using the notation of Remark 109, we have that $\Sigma\mathcal{A}$ is precisely the family of open subsets of $\mathbb{N}^{\mathbb{N}}$, while $\Pi\mathcal{A}$ is the family of closed subsets of $\mathbb{N}^{\mathbb{N}}$. From this information, the reader can deduce with induction and the principle of the excluded middle, that for every ordinal $\alpha > 0$, and all $A \subseteq \mathbb{N}^{\mathbb{N}}$,

$$A \,\in\, \Pi_\alpha\mathcal{A} \qquad\Longleftrightarrow\qquad \mathbb{N}^{\mathbb{N}}\backslash A \,\in\, \Sigma_\alpha\mathcal{A}. \tag{37}$$

The aim of this subsection is to prove the following statement.

$$\left[\;\begin{array}{l}\textbf{Let } \alpha \textbf{ be a } \textbf{\textit{countable}} \textbf{ ordinal number.}\\ \textbf{There is a set } S\in\Sigma(\Pi_\alpha\mathcal{A}) \textbf{ with } S\notin\Pi(\Sigma_\alpha\mathcal{A})\textbf{, and}\\ \textbf{there is a set } P\in\Pi(\Sigma_\alpha\mathcal{A}) \textbf{ with } P\notin\Sigma(\Pi_\alpha\mathcal{A})\textbf{.}\end{array}\right. \tag{38}$$

In particular this means that if the hierarchy has collapsed at $\Pi_\alpha\mathcal{A}$ or at $\Sigma_\alpha\mathcal{A}$ for some ordinal number α then α must be *uncountable*, see Proposition 137.

Statement (38) is known in descriptive set theory as the *Borel Hierarchy Theorem.* We will give a proof of Statement (38) in this subsection that uses the principle of excluded middle and is based on a beautiful paper by Veldman [6, paragraph 5].[3]

5.4.3. *Open Subsets of* $\mathbb{N}^{\mathbb{N}}$*.*
Before we give a definition of $\mathcal{A}$, and start with the proof of Statement (38) let us describe the topology $\mathcal{T}$ on the Baire space $\mathbb{N}^{\mathbb{N}}$ in more detail.

Definition 110. Given $m, n \in \mathbb{N}$, define B^m_n by

$$B^m_n \;:=\; \{\; f\in\mathbb{N}^{\mathbb{N}} :\; f(n) \,=\, m\;\}.$$

Remark 111. For $A \subseteq \mathbb{N}^{\mathbb{N}}$, we have $A \in \mathcal{T}$ if and only if for each $f \in A$ we have,

$$f \,\in\, B^{m_1}_{n_1} \cap \cdots \cap B^{m_K}_{n_K} \;\subseteq\; A,$$

for some $K \in \mathbb{N}$ and $m_1, \ldots, m_K \in \mathbb{N}$ and $n_1, \ldots, n_K \in \mathbb{N}$.

We can formulate Remark 111 more abstractly with some notation.

Definition 112. Let $\mathcal{S}$ and $\mathcal{S}_\cap$ be families of subsets of $\mathbb{N}^{\mathbb{N}}$ given by

$$\begin{aligned}\mathcal{S} \;&:=\; \{\; B^m_n :\; m, n\in\mathbb{N}\;\}\\ \mathcal{S}_\cap \;&:=\; \{\; S_1\cap\cdots\cap S_K :\; K\in\mathbb{N},\; S_k\in\mathcal{S}\;\}.\end{aligned}$$

Remark 113. By Remark 111, we see that $\mathcal{S}$ is a subbasis for the topology $\mathcal{T}$ on $\mathbb{N}^{\mathbb{N}}$, and we see that $\mathcal{S}_\cap$ is a basis for $\mathcal{T}$. In particular, we get

$$\mathcal{T} \;=\; \{\; \bigcup_n A_n :\; A_1, A_2, \ldots \in \mathcal{S}_\cap\;\}. \tag{39}$$

Remark 114. Let $m, n \in \mathbb{N}$. Then $B^m_n \in \mathcal{T}$ by Remark 113. More suprisingly,

$$\mathbb{N}^{\mathbb{N}}\backslash B^m_n \,\in\, \mathcal{T},$$

that is, B^m_n is closed as well. Indeed, this follows by the following equality.

$$\mathbb{N}^{\mathbb{N}}\backslash B^m_n \;=\; \bigcup\{\; B^k_n :\; k\in\mathbb{N},\; k\neq m\;\}.$$

[3] In this paper [6], Veldman (also) gives a proof of a variant of the Borel Hierarchy Theorem using Brouwer's Continuity Principle instead of the principle of excluded middle.

5.4.4. *Definition of the Sublattice $\mathcal{A}$ of $\mathbb{N}^\mathbb{N}$.*
Recall that we want to define a sublattice $\mathcal{A}$ of $\wp(\mathbb{N}^\mathbb{N})$ so that $\Sigma\mathcal{A}$ are the open subsets of $\mathbb{N}^\mathbb{N}$, while $\Pi\mathcal{A}$ are the closed subsets (see Remark 109).

Since the elements of $\mathcal{S}$ are both open and closed, we let $\mathcal{A}$ be the sub-Boolean algebra of $\wp(\mathbb{N}^\mathbb{N})$ generated by $\mathcal{S}$. More concretely:

Definition 115. Let $\mathcal{S}'$, $\mathcal{S}'_\cap$ and $\mathcal{A}$ be the families of subsets of $\mathbb{N}^\mathbb{N}$ given by

$$\begin{aligned} \mathcal{S}' \ &:= \ \{\ B^m_n\colon\ m,n\in\mathbb{N}\ \}\ \cup\ \{\ \mathbb{N}^\mathbb{N}\backslash B^m_n\colon\ m,n\in\mathbb{N}\ \} \\ \mathcal{S}'_\cap \ &:= \ \{\ S_1\cap\cdots\cap S_K\colon\ K\in\mathbb{N},\ S_k\in\mathcal{S}'\ \} \\ \mathcal{A} \ &:= \ \{\ T_1\cup\cdots\cup T_L\colon\ L\in\mathbb{N},\ T_\ell\in\mathcal{S}'_\cap\ \}. \end{aligned}$$

Lemma 116. *The family $\mathcal{A}$ is a sublattice of $\wp(\mathbb{N}^\mathbb{N})$, and*

$$\mathbb{N}^\mathbb{N}\backslash A\ \in\ \mathcal{A} \qquad\iff\qquad A\ \in\ \mathcal{A}$$

for every $A\subseteq\mathbb{N}^\mathbb{N}$, and we have the following equalities.

$$\begin{aligned} \Sigma\mathcal{A}\ &=\ \{\ U\subseteq\mathbb{N}^\mathbb{N}\colon\ U\ \textit{is open}\ \}, \\ \Pi\mathcal{A}\ &=\ \{\ C\subseteq\mathbb{N}^\mathbb{N}\colon\ C\ \textit{is closed}\ \}. \end{aligned}$$

Proof. We leave this to the reader. □

5.4.5. *Encoding the Elements of $\mathcal{A}$.*
Now that we have defined $\mathcal{A}$, we can start the proof of Statement (38). Maybe the most important idea behind the proof presented here is that we can encode the Borel subsets $\mathcal{B}$ of $\mathbb{N}^\mathbb{N}$ as elements of $\mathbb{N}^\mathbb{N}$.

To warm up let us see how we can encode a tuple $a_1\cdots a_n$ of natural numbers as a natural number. We leave it to the reader to find a bijection

$$\langle -,-\rangle :\ \mathbb{N}\times\mathbb{N}\longrightarrow\mathbb{N}\backslash\{1\}.$$

Let $\mathbb{N}^*$ be the set of tuples on $\mathbb{N}$. Given a tuple $a_1\cdots a_n\in\mathbb{N}^*$, define

$$\langle a_1a_2\cdots a_n\rangle\ :=\ \langle a_1,\langle a_2,\ldots\langle a_n,1\rangle\ \cdots\rangle\rangle\,.$$

Then the resulting map $\langle -\rangle:\mathbb{N}^*\to\mathbb{N}$ is a bijection.

Let us now encode the elements of $\mathcal{A}$ (see Def. 115). Given $k\in\mathbb{N}$, let

$$[\![k]\!]^{\mathcal{S}'}\ :=\ \begin{cases} \mathbb{N}^\mathbb{N}\backslash B^m_n & \text{if } k\equiv\langle 1mn\rangle,\\ B^m_n & \text{if } k\equiv\langle 2mn\rangle,\\ \varnothing & \text{otherwise.}\end{cases}$$

Then $[\![-]\!]^{\mathcal{S}'}:\mathbb{N}\to\mathcal{S}'$ is a surjection. Given $k\in\mathbb{N}$ with $k\equiv\langle a_1\cdots a_K\rangle$, let

$$[\![k]\!]^{\mathcal{S}'_\cap}\ :=\ [\![a_1]\!]^{\mathcal{S}'}\ \cap\cdots\cap\ [\![a_K]\!]^{\mathcal{S}'}.$$

Then $[\![-]\!]^{\mathcal{S}'_\cap}:\mathbb{N}\to\mathcal{S}'_\cap$ is a surjection. Given $k\in\mathbb{N}$ with $k\equiv\langle a_1\cdots a_K\rangle$, let

$$[\![k]\!]^{\mathcal{A}}\ :=\ [\![a_1]\!]^{\mathcal{S}'_\cap}\ \cup\cdots\cup\ [\![a_K]\!]^{\mathcal{S}'_\cap}.$$

Then $[\![-]\!]^{\mathcal{A}}:\mathbb{N}\to\mathcal{A}$ is a surjection.

Let $A\in\mathcal{A}$ be given. If $[\![k]\!]^{\mathcal{A}}=A$ for some $k\in\mathbb{N}$ we say that k is a *code* for A. Note that A might have multiple codes. This will not be a problem.

5.4.6. *Encoding Countable Ordinal Numbers.*
Before we can make the step to encode all Borel subsets of $\mathbb{N}^\mathbb{N}$ we need an encoding for the countable ordinal numbers. We need some notation.

Notation 117. *Let $f \in \mathbb{N}^\mathbb{N}$ and $n \in \mathbb{N}$ be given. Define $f^{[n]} \in \mathbb{N}^\mathbb{N}$ by, for $m \in \mathbb{N}$,*

$$f^{[n]}(m) \;=\; f(\,\langle n, m\rangle\,).$$

Since $\langle -, -\rangle$ is a bijection from $\mathbb{N}\times\mathbb{N}$ to $\mathbb{N}\backslash\{1\}$, an element $f \in \mathbb{N}^\mathbb{N}$ is completely determined by $f^{[1]}, f^{[2]}, \ldots$ *and* $f(1)$. More precisely, the assignment

$$f \quad\mapsto\quad f(1),\; f^{[1]},\; f^{[2]},\; \ldots .$$

gives a bijection from $\mathbb{N}^\mathbb{N}$ to $\mathbb{N} \times (\mathbb{N}^\mathbb{N})^\mathbb{N}$.

To encode the countable ordinal numbers, we use special elements of $\mathbb{N}^\mathbb{N}$.

Definition 118. Let **Stp** be the subset of $\mathbb{N}^\mathbb{N}$ inductively given by:
(i) If $f \in \mathbb{N}^\mathbb{N}$ and $f(1) \neq 1$, then $f \in \mathbf{Stp}$.
(ii) If $f \in \mathbb{N}^\mathbb{N}$ and $f(1) = 1$ and $f^{[n]} \in \mathbf{Stp}$ for all $n \in \mathbb{N}$, then $f \in \mathbf{Stp}$.

The elements of **Stp** are called **stumps** and are used in Intuitionistic Mathematics as a replacement for the countable ordinal numbers.

Definition 119. Let $\alpha[-]\colon \mathbf{Stp} \to \aleph_1$ be the map recursively defined by

$$\alpha[f] \;=\; \begin{cases} 0 & \text{if } f(1) \neq 1; \\ \bigvee_{n\in\mathbb{N}} \alpha[f^{[n]}] + 1 & \text{if } f(1) = 1. \end{cases}$$

Recall that $\aleph_1$ is the set of all countable ordinal numbers.

Lemma 120. *The map $\alpha[-]\colon \mathbf{Stp} \longrightarrow \aleph_1$ is surjective.*

Proof. To prove that $\alpha[-]$ is surjective, we must show that for each $\alpha \in \aleph_1$ there is an $f \in \mathbf{Stp}$ with $\alpha[f] = \alpha$. We use induction on $\alpha \in \aleph_1$.

First we must find an $f \in \mathbf{Stp}$ with $\alpha[f] = 0$. Simply take the $f \in \mathbb{N}^\mathbb{N}$ with $f(n) = 37$ for all $n \in \mathbb{N}$. Then $f(1) \neq 0$, so $f \in \mathbf{Stp}$, and $\alpha[f] = 0$.

Let $\alpha \in \aleph_1$ be given, and assume that $\alpha = \alpha[f]$ for some $f \in \mathbf{Stp}$. We need to find a $g \in \mathbf{Stp}$ such that $\alpha[g] = \alpha + 1$. Define $g \in \mathbb{N}^\mathbb{N}$ by

$$g(1) \;=\; 1, \qquad \text{and} \qquad g^{[n]} \;=\; f \quad \text{for all } n \in \mathbb{N}.$$

Then $g \in \mathbf{Stp}$, and $\alpha[g] = \bigvee_{n\in\mathbb{N}} \alpha[f] + 1$. Since $\alpha[f] = \alpha$, we have $\alpha[g] = \alpha + 1$.

Let $\lambda \in \aleph_1$ be a limit ordinal, and assume that $\alpha[-]$ is surjective on λ. We must find an $f \in \mathbf{Stp}$ such that $\alpha[f] = \lambda$. Since $\lambda \in \aleph_1$ there are $\alpha_1, \alpha_2, \ldots \in \lambda$ such that $\lambda = \bigvee_{n\in\mathbb{N}} \alpha_n + 1$. Since $\alpha[-]$ is surjective on λ, we know that there are $f_1, f_2, \ldots \in \mathbf{Stp}$ with $\alpha[f_n] = \alpha_n$. Define $f \in \mathbb{N}^\mathbb{N}$ by

$$f(1) \;=\; 1 \qquad \text{and} \qquad f^{[n]} \;=\; f_n \quad \text{for all } n \in \mathbb{N}.$$

Then $f \in \mathbf{Stp}$, and $\alpha[f] = \bigvee_{n\in\mathbb{N}} \alpha[f^{[n]}] + 1 = \bigvee_{n\in\mathbb{N}} \alpha_n + 1 = \lambda$. □

5.4.7. *Encoding Borel Subsets of $\mathbb{N}^\mathbb{N}$.*
We are now ready to encode the Borel subsets of $\mathbb{N}^\mathbb{N}$.

Definition 121. With recursion on **Stp** define for each $f \in \mathbf{Stp}$ maps

$$[\![-]\!]^\Pi_f \colon \mathbb{N}^\mathbb{N} \longrightarrow \Pi(\Sigma_{\alpha[f]}\mathcal{A}) \qquad \text{and} \qquad [\![-]\!]^\Sigma_f \colon \mathbb{N}^\mathbb{N} \longrightarrow \Sigma(\Pi_{\alpha[f]}\mathcal{A})$$

such that the following clauses hold.
(i) For $f \in \mathbf{Stp}$ with $f(1) \neq 1$ we have

$$[\![g]\!]^\Pi_f \;=\; \bigcap_{n\geq 2} [\![\, g(n)\,]\!]^\mathcal{A} \qquad \text{and} \qquad [\![g]\!]^\Sigma_f \;=\; \bigcup_{n\geq 2} [\![\, g(n)\,]\!]^\mathcal{A}.$$

(ii) For $f \in \mathbf{Stp}$ with $f(1) = 1$ we have

$$[\![g]\!]^{\Pi}_{f} \;=\; \bigcap_{n\in\mathbb{N}} [\![\, g^{[n]} \,]\!]^{\Sigma}_{f^{[n]}} \qquad\text{and}\qquad [\![g]\!]^{\Sigma}_{f} \;=\; \bigcup_{n\in\mathbb{N}} [\![\, g^{[n]} \,]\!]^{\Pi}_{f^{[n]}}.$$

Lemma 122. *For each $f \in \mathbf{Stp}$ the maps $[\![-]\!]^{\Pi}_{f}$ and $[\![-]\!]^{\Sigma}_{f}$ are surjective.*

Proof. We leave this to the reader. □

Remark 123. Let $f \in \mathbf{Stp}$ and $g \in \mathbb{N}^{\mathbb{N}}$ be given. Note that $[\![g]\!]^{\Pi}_{f}$ and $[\![g]\!]^{\Sigma}_{f}$ do not depend on $g(1)$. More precisely, given $g' \in \mathbb{N}^{\mathbb{N}}$ with $g'(n) = g(n)$ for all $n \geq 2$ — so possibly $g(1) \neq g'(1)$. Then we have $[\![g]\!]^{\Pi}_{f} = [\![g']\!]^{\Pi}_{f}$ and $[\![g]\!]^{\Sigma}_{f} = [\![g']\!]^{\Sigma}_{f}$.

We use Remark 123 to combine the maps $[\![-]\!]^{\Pi}_{f}$ and $[\![-]\!]^{\Sigma}_{f}$ into one map $[\![-]\!]^{\mathcal{B}}_{f}$.

Definition 124. Let $[\![-]\!]^{\mathcal{B}}_{-} \colon \mathbb{N}^{\mathbb{N}} \times \mathbf{Stp} \longrightarrow \mathcal{B}$ be given by, for $f \in \mathbf{Stp}$ and $g \in \mathbb{N}^{\mathbb{N}}$,

$$[\![g]\!]^{\mathcal{B}}_{f} \;=\; \begin{cases} [\![g]\!]^{\Pi}_{f} & \text{if } g(1) = 37, \\ [\![g]\!]^{\Sigma}_{f} & \text{if } g(1) \neq 37. \end{cases} \tag{40}$$

We want to prove that $[\![-]\!]^{\mathcal{B}}_{-}$ is surjective. To do this, we need a lemma.

Lemma 125. *We have the following equality.*

$$\mathcal{B} \;=\; \Pi_{\aleph_1}\mathcal{A}. \tag{41}$$

Proof. Note that we have already proven (at the end of in Subsection 5.3) that $\mathcal{B} = \Pi_{\xi}\mathcal{A}$ for *some* ordinal number ξ. Statement (41) is an improvement.

Recall that $\Pi_{\aleph_1}\mathcal{A} \equiv \bigcup_{\alpha\in\aleph_1} \Pi_{\alpha}\mathcal{A}$. Since $\Pi\mathcal{A} \subseteq \Sigma_2\mathcal{A}$, we have

$$\mathcal{T} \;\equiv\; \Sigma\mathcal{A} \;\subseteq\; \Pi_{\aleph_1}\mathcal{A} \;\subseteq\; \mathcal{B}.$$

Recall that $\mathcal{B}$ is the family of all subsets of $\mathbb{N}^{\mathbb{N}}$ that can be formed using countable unions and countable intersections starting from $\mathcal{T}$. So to prove that $\Pi_{\aleph_1}\mathcal{A} = \mathcal{B}$ it suffices to show that $\Pi_{\aleph_1}\mathcal{A}$ is 'closed' under countable unions and intersections, i.e., given $A_1, A_2, \ldots \in \Pi_{\aleph_1}\mathcal{A}$ we must show that $\bigcup_n A_n \in \Pi_{\aleph_1}\mathcal{A}$ and $\bigcap_n A_n \in \Pi_{\aleph_1}\mathcal{A}$. We will only prove that $\bigcup_n A_n \in \Pi_{\aleph_1}\mathcal{A}$; the proof of $\bigcap_n A_n \in \Pi_{\aleph_1}\mathcal{A}$ is similar.

Let $A'_n := A_1 \cup \cdots \cup A_n$ for each $n \in \mathbb{N}$. Then $\bigcup_n A'_n = \bigcup_n A_n$. Since $\Pi_{\aleph_1}\mathcal{A}$ is a sublattice of $\wp(\mathbb{N}^{\mathbb{N}})$, we get that $A'_n \in \Pi_{\aleph_1}\mathcal{A} \equiv \bigcup_{\alpha\in\aleph_1} \Pi_{\alpha}\mathcal{A}$ for all $n \in \mathbb{N}$.

Pick $\alpha_1, \alpha_2, \ldots \in \aleph_1$ such that $A'_n \in \Pi_{\alpha_n}\mathcal{A}$ for all $n \in \mathbb{N}$. Let $\alpha := \bigvee_n \alpha_n$. Then for all $n \in \mathbb{N}$ we have $\alpha_n \leq \alpha$, and $\Pi_{\alpha_n}\mathcal{A} \subseteq \Pi_{\alpha}\mathcal{A}$, and so $A'_n \in \Pi_{\alpha}\mathcal{A}$. By definition of $\Sigma(\Pi_{\alpha}\mathcal{A})$ we have $\bigcup_n A'_n \in \Sigma(\Pi_{\alpha}\mathcal{A}) \equiv \Sigma_{\alpha+1}\mathcal{A}$. Now, note that since $\alpha_1, \alpha_2, \ldots \in \aleph_1$, also $\alpha = \bigvee_n \alpha_n \in \aleph_1$, and so $\alpha + 1 \in \aleph_1$. Hence

$$\textstyle\bigcup_n A_n \;=\; \bigcup_n A'_n \;\in\; \Pi_{\alpha+1}\mathcal{A} \;\subseteq\; \Pi_{\aleph_1}\mathcal{A}.$$

So $\Pi_{\aleph_1}\mathcal{A}$ is closed under countable union. Similarly, $\Pi_{\aleph_1}\mathcal{A}$ is closed under countable intersection. It follows that $\mathcal{B} = \Pi_{\aleph_1}\mathcal{A}$. We have proven Statement (41). □

Proposition 126. *The map $[\![-]\!]^{\mathcal{B}}_{-} \colon \mathbb{N}^{\mathbb{N}} \times \mathbf{Stp} \longrightarrow \mathcal{B}$ is surjective.*

Proof. Let $A \in \mathcal{B}$ be given. We must find $f \in \mathbf{Stp}$ and $g \in \mathbb{N}^{\mathbb{N}}$ such that $A = [\![g]\!]^{\mathcal{B}}_{f}$. By Lemma 125 we know that $A \in \Pi_{\aleph 1}\mathcal{A} \equiv \bigcup_{\alpha\in\aleph_1} \Pi_{\alpha}\mathcal{A}$. Pick an $\alpha \in \aleph_1$ with $A \in \Pi_{\alpha}\mathcal{A}$. Since the map $\alpha[-] \colon \mathbf{Stp} \to \aleph_1$ is a surjection, there is an $f \in \mathbf{Stp}$ such that $\alpha[f] = \alpha$. Since the map $[\![-]\!]^{\Pi}_{f} \colon \mathbb{N}^{\mathbb{N}} \longrightarrow \Pi(\Sigma_{\alpha[f]}\mathcal{A})$ is a surjection and $A \in \Pi_{\alpha}\mathcal{A} \equiv \Pi_{\alpha[f]}\mathcal{A} \subseteq \Pi(\Sigma_{\alpha[f]}\mathcal{A})$, there is an $h \in \mathbb{N}^{\mathbb{N}}$ such that $[\![h]\!]^{\Pi}_{f} = A$.

Now, let $g \in \mathbb{N}^{\mathbb{N}}$ be given by $g^{[n]} = h^{[n]}$ for all $n \in \mathbb{N}$ and $g(1) = 37$. Then

$$[\![g]\!]^{\mathcal{B}}_{f} \;=\; [\![g]\!]^{\Pi}_{f} \;=\; [\![h]\!]^{\Pi}_{f} \;=\; A.$$

So we see that $[\![-]\!]^{\mathcal{B}}_{-} \colon \mathbb{N}^{\mathbb{N}} \times \mathbf{Stp} \longrightarrow \mathcal{B}$ is surjective. □

5.4.8. *Outstanding Debt.*
We take a small detour, because in Example 84 we used a fact, Statement (28), without proof, and we are now in a position to correct this situation.

Notation 127. *Let* $\mathbb{D}$ *denote the* Cantor set*, see* [7, Examples 17.9c].

Lemma 128. $\mathbb{D}$ *is Borel negligble subset of* $\mathbb{R}$*, and there is an injective map*

$$p\colon \mathbb{N}^{\mathbb{N}} \times \mathbf{Stp} \longrightarrow \mathbb{D}.$$

Proof. We leave this to the reader. □

Corollary 129. *We have the following situation.*

$$\mathcal{B} \xleftarrow{\llbracket - \rrbracket^{\mathcal{B}}_{-}} \mathbb{N}^{\mathbb{N}} \times \mathbf{Stp} \xrightarrow{\quad p \quad} \mathbb{D}$$

The map $\llbracket - \rrbracket^{\mathcal{B}}_{-}$ *is surjective, the map* p *is injective, and* $\mathbb{D}$ *is Borel negligible.*

Proof. Combine Lemma 128 and Proposition 126. □

Note that Corollary 129 implies Statement (28). This ends our detour.

5.4.9. *Cataloguing Borel Subsets of* $\mathbb{N}^{\mathbb{N}}$*.*
We have encoded the Borel subsets using elements of $\mathbb{N}^{\mathbb{N}}$. To prove the Borel Hierarchy Theorem (see Statement (38)) we use the encoding to go one step further.

Definition 130. For each $f \in \mathbf{Stp}$, define

$$U^{\Pi}_f \ = \ \{\ h \in \mathbb{N}^{\mathbb{N}} \colon\ h^{[1]} \in \llbracket h^{[2]} \rrbracket^{\Pi}_f \ \}, \qquad U^{\Sigma}_f \ = \ \{\ h \in \mathbb{N}^{\mathbb{N}} \colon\ h^{[1]} \in \llbracket h^{[2]} \rrbracket^{\Sigma}_f \ \}.$$

One can think of the set U^{Π}_f as a *catalogue* of $\Pi(\Sigma_{\alpha[f]}\mathcal{A})$.
Indeed, given $A \in \Pi(\Sigma_{\alpha[f]}\mathcal{A})$ with $A = \llbracket g \rrbracket^{\Pi}_f$ for some $g \in \mathbb{N}^{\mathbb{N}}$, we have

$$A \ = \ \{\ h^{[1]} \colon\ h \in U^{\Pi}_f \ \text{ and } \ h^{[2]} = g \ \}.$$

The following lemma might be the essential part of the Borel Hierarchy Theorem.

Lemma 131. *Let* $f \in \mathbf{Stp}$ *be given. Then we have*

$$U^{\Pi}_f \ \in \ \Pi(\Sigma_{\alpha[f]}\mathcal{A}) \qquad \textit{and} \qquad U^{\Sigma}_f \ \in \ \Sigma(\Pi_{\alpha[f]}\mathcal{A}). \tag{42}$$

To give a proof of Lemma 131 we need some notation and a lemma.

Definition 132. Let $F\colon \mathbb{N}^{\mathbb{N}} \longrightarrow \mathbb{N}^{\mathbb{N}}$ be given.

(i) Given $A \subseteq \mathbb{N}^{\mathbb{N}}$, let $F^*(A) := \{\ g \in \mathbb{N}^{\mathbb{N}} \colon\ F(g) \in A \ \}$.
(ii) We say that F is **continuous** if $F^*(B^m_n) \in \mathcal{T}$ for all $m, n \in \mathbb{N}$.

Lemma 133. *Let* $F\colon \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ *be a continuous function.*
Let α *be an ordinal number with* $\alpha > 0$*. Then, for all* $A \subseteq \mathbb{N}^{\mathbb{N}}$*,*

$$\begin{aligned} F^*(A) \ &\in \ \Pi_\alpha \mathcal{A} \qquad \textit{when} \qquad A \ \in \ \Pi_\alpha\mathcal{A}, \\ F^*(A) \ &\in \ \Sigma_\alpha \mathcal{A} \qquad \textit{when} \qquad A \ \in \ \Sigma_\alpha\mathcal{A}. \end{aligned}$$

Proof. We leave this to the reader. □

Proof of Lemma 131. We prove Statement (42) using induction over $f \in \mathbf{Stp}$.

Let $f \in \mathbf{Stp}$ with $f(1) \neq 1$ be given. We must prove that

$$U^{\Pi}_f \ \in \ \Pi\mathcal{A} \qquad \text{and} \qquad U^{\Sigma}_f \ \in \ \Sigma\mathcal{A}.$$

We will only prove that $U^{\Sigma}_f \in \Sigma\mathcal{A}$ since the proof of $U^{\Pi}_f \in \Pi\mathcal{A}$ is similar.

Let $h \in \mathbb{N}^{\mathbb{N}}$ be given. Note that the following are equivalent.

$$\begin{array}{rcll} h & \in & U_f^{\Sigma}, & \\ h^{[1]} & \in & [\![h^{[2]}]\!]_f^{\Sigma}, & \\ h^{[1]} & \in & \bigcup_{n\geq 2} [\![h^{[2]}(n)]\!]^{\mathcal{A}}, & \\ h^{[1]} & \in & [\![h^{[2]}(n)]\!]^{\mathcal{A}} & \text{for some } n \geq 2, \\ h^{[1]} & \in & [\![k]\!]^{\mathcal{A}} \quad \text{and} \quad h^{[2]}(n) = k & \text{for some } n \geq 2,\ k \in \mathbb{N}. \end{array}$$

Let $F\colon \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ be given by $F(g) = g^{[1]}$. Then $h^{[1]} \in [\![k]\!]^{\mathcal{A}}$ iff $h \in F^*([\![k]\!]^{\mathcal{A}})$. Further, note that $h^{[2]}(n) = k$ iff $h \in B^k_{\langle 2,n\rangle}$. All in all, we get

$$U_f^{\Sigma} \;=\; \bigcup_{k\in\mathbb{N}} \bigcup_{n\geq 2} F^*(\, [\![k]\!]^{\mathcal{A}}\,) \;\cap\; B^k_{\langle 2,n\rangle}. \tag{43}$$

By Equation (43) to prove $U_f^{\Sigma} \in \Sigma\mathcal{A}$ it suffices to show that $F^*(\, [\![k]\!]^{\mathcal{A}}\,) \in \Sigma\mathcal{A}$.

Since it is not hard to see that F is continuous (see Definition 132) and $[\![k]\!]^{\mathcal{A}} \in \Sigma\mathcal{A}$ we get by Lemma 133 that $F^*(\, [\![k]\!]^{\mathcal{A}}\,) \in \Sigma\mathcal{A}$. Hence $U_f^{\Sigma} \in \Sigma\mathcal{A}$.

Recall that we are proving Statement (42) using induction on $f \in \mathbf{Stp}$.
Let $f \in \mathbf{Stp}$ with $f(1) = 1$ be given. Assume that for all $n \in \mathbb{N}$,

$$U^{\Pi}_{f^{[n]}} \in \Pi(\Sigma_{\alpha[f^{[n]}]}\mathcal{A}) \qquad \text{and} \qquad U^{\Sigma}_{f^{[n]}} \in \Sigma(\Pi_{\alpha[f^{[n]}]}\mathcal{A}).$$

We must prove that $U_f^{\Pi} \in \Pi(\Sigma_{\alpha[f]}\mathcal{A})$ and $U_f^{\Sigma} \in \Sigma(\Pi_{\alpha[f]}\mathcal{A})$. We will prove that

$$U_f^{\Sigma} \in \Sigma(\Pi_{\alpha[f]}\mathcal{A}), \tag{44}$$

and we leave the proof of $U_f^{\Pi} \in \Pi(\Sigma_{\alpha[f]}\mathcal{A})$ to the reader.

To proceed, we need some notation. We will define a 'pairing'

$$P\colon \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}} \longrightarrow \mathbb{N}^{\mathbb{N}}.$$

Let $h_1, h_2 \in \mathbb{N}^{\mathbb{N}}$ be given. Define $P(h_1,h_2) \in \mathbb{N}^{\mathbb{N}}$ by: $P(h_1,h_2)(1) = 1$, and

$$(\, P(h_1,h_2)\,)^{[1]} = h_1, \qquad \text{and} \qquad (\, P(h_1,h_2)\,)^{[2]} = h_2,$$

and $(\, P(h_1,h_2)\,)^{[n]}(m) = 1$ for all $n, m \in \mathbb{N}$ with $n > 2$.

Let $h \in \mathbb{N}^{\mathbb{N}}$ be given. Note that the following are equivalent.

$$\begin{array}{rcll} h & \in & U_f^{\Sigma}, & \\ h^{[1]} & \in & [\![h^{[2]}]\!]_f^{\Sigma}, & \\ h^{[1]} & \in & \bigcup_{n\in\mathbb{N}} [\![h^{[2][n]}]\!]^{\Pi}_{f^n}, & \\ h^{[1]} & \in & [\![h^{[2][n]}]\!]^{\Pi}_{f^n} & \text{for some } n \in \mathbb{N}, \\ P(h^{[1]}, h^{[2][n]}) & \in & U^{\Pi}_{f^{[n]}} & \text{for some } n \in \mathbb{N}. \end{array}$$

Now, for each $n \in \mathbb{N}$, let $F_n\colon \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ be given by, for $h \in \mathbb{N}^{\mathbb{N}}$,

$$F_n(h) \;=\; P(h^{[1]}, h^{[2][n]}).$$

Then using the notation of Definition 132 we see that

$$U_f^{\Sigma} \;=\; \bigcup_{n\in\mathbb{N}} F_n^*(\, U^{\Pi}_{f^{[n]}}\,). \tag{45}$$

Recall that we must prove that $U_f^{\Sigma} \in \Sigma(\Pi_{\alpha[f]}\mathcal{A})$. It suffices to prove that

$$F_n^*(\, U^{\Pi}_{f^{[n]}}\,) \;\in\; \Pi(\Sigma_{\alpha[f^{[n]}]}\mathcal{A}), \tag{46}$$

by Statement (45), and because we have

$$\Pi(\Sigma_{\alpha[f^{[n]}]}\mathcal{A}) \;\subseteq\; \Sigma(\Pi_{\alpha[f]}\mathcal{A}).$$

By Lemma 133 to prove that Statement (46) holds it suffices to show that F_n is continuous; we leave this to the reader. Hence we have proven Statement (44).

This completes the proof of Statement (42). $\square$

5.4.10. *Diagonalization.*
With the catalogues U^{Σ}_f and U^{Π}_f at our disposal we use Cantor's diagonal argument to prove the Borel Hierarchy Theorem (see Statement (38)).

Definition 134. Let $f \in \mathbf{Stp}$ be given. Let D^{Π}_f and D^{Σ}_f be give by

$$D^{\Pi}_f \;:=\; \{\; g \in \mathbb{N}^{\mathbb{N}} \colon\; g \notin [\![g]\!]^{\Pi}_f \;\}, \qquad D^{\Sigma}_f \;:=\; \{\; g \in \mathbb{N}^{\mathbb{N}} \colon\; g \notin [\![g]\!]^{\Sigma}_f \;\}.$$

Theorem 135. *Let $f \in \mathbf{Stp}$ be given. Then we have*

$$D^{\Pi}_f \;\in\; \Sigma(\Pi_{\alpha[f]}\mathcal{A}) \qquad \textit{and} \qquad D^{\Pi}_f \;\notin\; \Pi(\Sigma_{\alpha[f]}\mathcal{A}),$$
$$D^{\Sigma}_f \;\in\; \Pi(\Sigma_{\alpha[f]}\mathcal{A}) \qquad \textit{and} \qquad D^{\Sigma}_f \;\notin\; \Sigma(\Pi_{\alpha[f]}\mathcal{A}).$$

Proof. Let $f \in \mathbf{Stp}$ be given. We will only prove that

$$D^{\Pi}_f \;\in\; \Sigma(\Pi_{\alpha[f]}\mathcal{A}) \qquad \text{and} \qquad D^{\Pi}_f \;\notin\; \Pi(\Sigma_{\alpha[f]}\mathcal{A}), \tag{47}$$

because there is a similar proof of $D^{\Sigma}_f \;\in\; \Pi(\Sigma_{\alpha[f]}\mathcal{A})$ and $D^{\Sigma}_f \;\notin\; \Sigma(\Pi_{\alpha[f]}\mathcal{A})$.

Let us first prove that $D^{\Pi}_f \notin \Pi(\Sigma_{\alpha[f]}\mathcal{A})$. So assume $D^{\Pi}_f \in \Pi(\Sigma_{\alpha[f]}\mathcal{A})$ in order to reach a contradiction. Since the map $[\![-]\!]^{\Pi}_f \colon \mathbb{N}^{\mathbb{N}} \longrightarrow \Pi(\Sigma_{\alpha[f]}\mathcal{A})$ is surjective, there is a $g_\delta \in \mathbb{N}^{\mathbb{N}}$ with $[\![g_\delta]\!]^{\Pi}_f = D^{\Pi}_f$. Then by definition of D^{Π}_f,

$$g_\delta \;\in\; D^{\Pi}_f \quad\iff\quad g_\delta \;\notin\; [\![g_\delta]\!]^{\Pi}_f = D^{\Pi}_f. \tag{48}$$

Statement (48) leads to a contradiction. Hence we conclude that $D^{\Pi}_f \notin \Pi(\Sigma_{\alpha[f]}\mathcal{A})$.

Let us prove that $D^{\Pi}_f \in \Sigma(\Pi_{\alpha[f]}\mathcal{A})$. Let $\Delta \colon \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ be given by, for $g \in \mathbb{N}^{\mathbb{N}}$,

$$\Delta(g) \;=\; P(g,g).$$

Then Δ is continuous (see Definition 132) and we have, for $g \in \mathbb{N}^{\mathbb{N}}$,

$$g \;\in\; D^{\Pi}_f \quad\iff\quad \Delta(g) \;\notin\; U^{\Pi}_f \quad\iff\quad g \;\in\; \Delta^*(\,\mathbb{N}^{\mathbb{N}}\backslash U^{\Pi}_f\,).$$

So we see that $D^{\Pi}_f = \Delta^*(\,\mathbb{N}^{\mathbb{N}}\backslash U^{\Pi}_f\,)$. Recall that we must prove that

$$D^{\Pi}_f \;\in\; \Sigma(\Pi_{\alpha[f]}\mathcal{A}).$$

Since Δ is continuous it suffices to show that $\mathbb{N}^{\mathbb{N}}\backslash U^{\Pi}_f \;\in\; \Sigma(\Pi_{\alpha[f]}\mathcal{A})$ by Lemma 133. We already know that $U^{\Pi}_f \in \Pi(\Sigma_{\alpha[f]}\mathcal{A})$, so by Statement (37) we get

$$\mathbb{N}^{\mathbb{N}}\backslash U^{\Pi}_f \;\in\; \Sigma(\Pi_{\alpha[f]}\mathcal{A}).$$

So $D^{\Pi}_f \in \Sigma(\Pi_{\alpha[f]}\mathcal{A})$. Hence we have proven Statement (47). $\square$

Corollary 136. *The Borel Hierarchy Theorem holds, see Statement* (38).

Proposition 137. *Using the terminology from Subsection 5.3,
if α is an ordinal number such that the hierarchy collapsed at $\Pi_\alpha\mathcal{A}$ or at $\Sigma_\alpha\mathcal{A}$, then α must be uncountable.*

Proof. Let α be an ordinal number. We will only prove that α must be uncountable when the hierarchy collapsed at $\Pi_\alpha\mathcal{A}$, because the proof that α must be uncountable when the hierarchy collapsed at $\Sigma_\alpha\mathcal{A}$ is similar.

Assume that the hierarchy hierarchy collapsed at $\Pi_\alpha\mathcal{A}$, that is,

$$\Pi(\Pi_\alpha\mathcal{A}) \;=\; \Pi_\alpha\mathcal{A} \;=\; \Sigma(\Pi_\alpha\mathcal{A}).$$

Assume that α is countable in order to reach a contradiction.
By the Borel Hierarchy Theorem, see Corollary (136), there is an

$$S \in \Sigma(\Pi_\alpha\mathcal{A}) \qquad \text{with} \qquad S \notin \Pi(\Sigma_\alpha\mathcal{A}).$$

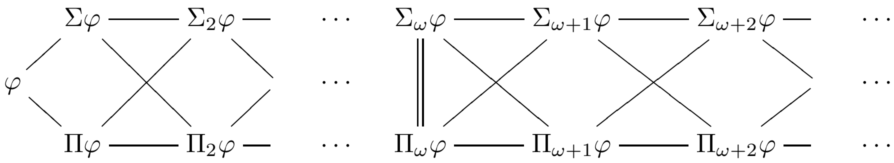
However, we have the following inclusion

$$\Sigma(\Pi_\alpha \mathcal{A}) \;=\; \Pi_\alpha \mathcal{A} \;\subseteq\; \Pi(\Sigma_\alpha \mathcal{A}).$$

So since $S \in \Sigma(\Pi_\alpha \mathcal{A})$, we get $S \in \Pi(\Sigma_\alpha \mathcal{A})$, which contradicts $S \notin \Pi(\Sigma_\alpha \mathcal{A})$.

Hence α is not countable. So α must be uncountable. □

5.5. **The Hierarchy of Extensions.**
To get the smallest complete extension of a valuation φ with respect to some V (when it exists) we can make a hierarchy of extensions of φ, see Subsection 5.3:



We have seen that if φ has a complete extension, then φ also has a smallest complete extension $\overline{\varphi}$, and that $\overline{\varphi} = \Pi_\xi\varphi$ for some ordinal number ξ, see Proposition 105.

Even if we do not know whether φ has a complete extension, we can still try to make the hierarchy, and this is what we are going to do in this subsection.

It is possible that the making of the hierarchy is hindered at some point, e.g., if $\Sigma_2\varphi$ is not Π-extendible, then we can not define $\Pi_3\varphi = \Pi(\Sigma_2\varphi)$.

If we can define the hierarchy up to $\Pi_\alpha\varphi$ unhindered we will say that

$$\varphi \text{ is } \Pi_\alpha\text{-extendible.}$$

We will prove that φ has a complete extension iff φ is $\Pi_{\aleph_1}$-extendible. Moreover, the valuation $\Pi_{\aleph_1}\varphi$ will be the smallest complete extension of φ (see Corollary 145).

Let us begin by giving a formal definition of the hierarchy and "Π_α-extendible".

Definition 138. Let $V \supseteq L \xrightarrow{\varphi} E$ be a valuation system.
We are going to define the following statements and valuation systems.

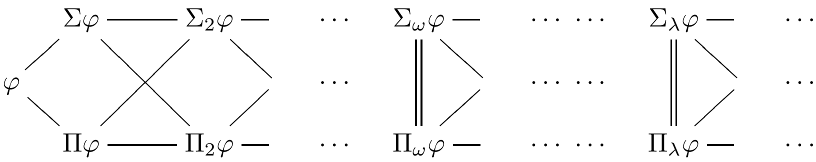
(i) For each ordinal number α, statements

"φ **is** Π_α**-extendible**" and "φ **is** Σ_α**-extendible**".

(ii) For each α such that φ is Π_α-extendible, a valuation system
$$V \supseteq \Pi_\alpha L \xrightarrow{\Pi_\alpha\varphi} E.$$

(iii) For each α such that φ is Σ_α-extendible, a valuation system
$$V \supseteq \Sigma_\alpha L \xrightarrow{\Sigma_\alpha\varphi} E.$$

We will define them in such a way that we get a hierarchy of the following shape.



More precisely, the following statements will be true.

(I) For all $\beta < \gamma$, if φ is Π_γ-extendible, then φ is both Π_β- and Σ_β-extendible, and the map $\Pi_\gamma\varphi$ extends both $\Pi_\beta\varphi$ and $\Sigma_\beta\varphi$.

(II) For all $\beta < \gamma$, if φ is Σ_γ-extendible, then φ is both Π_β- and Σ_β-extendible, and the map $\Sigma_\gamma\varphi$ extends both $\Pi_\beta\varphi$ and $\Sigma_\beta\varphi$.

(III) Let λ be zero or a a limit ordinal. Then φ is Π_λ-extendible if and only if φ is Σ_λ-extendible. Furthermore, if φ is Π_λ-extendible, then $\Pi_\gamma\varphi = \Sigma_\gamma\varphi$.

Now, we define the aforementioned statements and valuation systems by recursion over the ordinal number using the following clauses.

(i) φ is Π_0-extendible and φ is Σ_0-extendible. Moreover,
$$\Pi_0 L = L \qquad \Sigma_0 L = L \qquad \Pi_0\varphi = \varphi \qquad \Sigma_0\varphi = \varphi.$$

(ii) Let α be an ordinal number. Then we have

$$\varphi \text{ is } \Pi_{\alpha+1}\text{-extendible} \iff \left[\begin{array}{l}\varphi \text{ is } \Sigma_\alpha\text{-extendible, and}\\ \Sigma_\alpha\varphi \text{ is } \Pi\text{-extendible.}\end{array}\right.$$

Moreover, if φ is $\Pi_{\alpha+1}$-extendible, then

$$\Pi_{\alpha+1}L \;=\; \Pi(\Sigma_\alpha L) \qquad \text{and} \qquad \Pi_{\alpha+1}\varphi \;=\; \Pi(\Sigma_\alpha\varphi),$$

where $V \supseteq \Pi(\Sigma_\alpha L) \xrightarrow{\Pi(\Sigma_\alpha\varphi)} E$ is the valuation system from Definition 94.

(iii) Let α be an ordinal number. Then we have

$$\varphi \text{ is } \Sigma_{\alpha+1}\text{-extendible} \iff \left[\begin{array}{l}\varphi \text{ is } \Pi_\alpha\text{-extendible, and}\\ \Pi_\alpha\varphi \text{ is } \Sigma\text{-extendible.}\end{array}\right.$$

Moreover, if φ is $\Sigma_{\alpha+1}$-extendible, then

$$\Sigma_{\alpha+1}L \;=\; \Sigma(\Pi_\alpha L) \qquad \text{and} \qquad \Sigma_{\alpha+1}\varphi \;=\; \Sigma(\Pi_\alpha\varphi).$$

(iv) Let λ be a limit ordinal. Then we have

$$\varphi \text{ is } \Pi_\lambda\text{-extendible} \iff \left[\begin{array}{l}\varphi \text{ is } \Pi_\alpha\text{-extendible,}\\ \text{for every } \alpha \in \lambda.\end{array}\right.$$

Moreover, if φ is Π_λ-extendible, then

$$\Pi_\lambda L \;=\; \textstyle\bigcup_{\alpha\in\lambda} \Pi_\alpha L \qquad \text{and} \qquad \Pi_\lambda\varphi(c) \;=\; \Pi_\beta\varphi(c),$$

where $\beta \in \lambda$ and $c \in \Pi_\beta L$.

(v) Let λ be a limit ordinal. Then we have

$$\varphi \text{ is } \Sigma_\lambda\text{-extendible} \iff \left[\begin{array}{l}\varphi \text{ is } \Sigma_\alpha\text{-extendible,}\\ \text{for every } \alpha \in \lambda.\end{array}\right.$$

Moreover, if φ is Σ_λ-extendible, then

$$\Sigma_\lambda L \;=\; \textstyle\bigcup_{\alpha\in\lambda} \Sigma_\alpha L \qquad \text{and} \qquad \Sigma_\lambda\varphi(c) \;=\; \Sigma_\beta\varphi(c),$$

where $\beta \in \lambda$ and $c \in \Sigma_\beta L$.

Definition 139. Let $V \supseteq L \xrightarrow{\varphi} E$ be a valuation system.

(i) We say the **hierarchy has collapsed at** Q if

$$\left[\begin{array}{l}\varphi \text{ is } \Pi_{\alpha+1}\text{-extendible and } \Sigma_{\alpha+1}\text{-extendible, and}\\ \qquad\qquad \Pi(Q) \;=\; Q \;=\; \Sigma(Q),\\ \text{where } Q = \Pi_\alpha\varphi \text{ or } Q = \Sigma_\alpha\varphi.\end{array}\right.$$

(ii) We say the **hierarchy collapses** if the hierarchy has collapsed at some Q.

We will prove that if the hierarchy of a valuation φ collapses then φ has a complete extension (see Lemma 142). After that, we will prove the converse, namely, if φ has a complete extension, then then the hierarchy of φ collapses (see Proposition 148).

Lemma 140. *Let $V \supseteq L \xrightarrow{\varphi} E$ be a valuation system.*

(i) Let $\alpha < \beta$ be ordinal numbers.
Suppose that the hierarchy has collapsed at $\Pi_\alpha\varphi$.
Then φ is Π_β-extendible and Σ_β-extendible, and

$$\Pi_\beta\varphi \;=\; \Pi_\alpha\varphi \;=\; \Sigma_\beta\varphi.$$

(ii) If the hierarchy collapses at $\Pi_\alpha\varphi$ and at $\Sigma_\alpha\varphi$ for some α, then $\Pi_\alpha\varphi = \Sigma_\alpha\varphi$.

(iii) Suppose the hierarchy has collapsed at Q_1 and at Q_2. Then $Q_1 = Q_2$.

(iv) The hierarchy collapses if and only if it has collapsed at $\Pi_\alpha\varphi$ for some α.

Proof. We leave this to the reader. □

Definition 141. Let $V \supseteq L \xrightarrow{\varphi} E$ be a valuation system.

(i) We say that φ (or $V \supseteq L \xrightarrow{\varphi} E$) is **extendible** if the hierarchy collapses.
(ii) Suppose that φ is extendible. Then there is precisely one valuation at which the hierarchy has collapsed (see Lemma 140(iii)); we denote it by

$$V \supseteq \overline{L} \xrightarrow{\overline{\varphi}} E.$$

Lemma 142. *Let $V \supseteq L \xrightarrow{\varphi} E$ be an extendible valuation system. Then*

$$V \supseteq \overline{L} \xrightarrow{\overline{\varphi}} E \text{ is complete.}$$

Proof. To prove that $\overline{\varphi}$ is complete, it suffices to show that $\overline{\varphi}$ is Π-complete and Σ-complete. We know that $\overline{\varphi} = \Pi\overline{\varphi}$ (since the hierarchy has collapsed at $\overline{\varphi}$, see Definition 141(ii) and Definition 139(i)), and that $\Pi\overline{\varphi}$ is Π-complete (see Lemma 98). Hence $\overline{\varphi}$ is Π-complete. Similarly, $\overline{\varphi}$ is Σ-complete. So $\overline{\varphi}$ is complete. □

Remark 143. Let $V \supseteq L \xrightarrow{\varphi} E$ be a valuation system.
(i) Note that if φ is complete with respect to V (see Definition 77), then φ is extendible, and $\overline{\varphi} = \varphi$.
(ii) On the other hand, if φ is extendible and $\overline{\varphi} = \varphi$, then φ is complete with respect to V (see Lemma 142).

Lemma 144. *Let $V \supseteq L \xrightarrow{\varphi} E$ be a valuation system.*
If φ is $\Pi_{\aleph_1}$-extendible, then the hierarchy has collapsed at $\Pi_{\aleph_1}\varphi$.

Proof. Suppose that φ is $\Pi_{\aleph_1}$-extendible. We need to prove that the hierarchy has collapsed at $\Pi_{\aleph_1}\varphi$. For this, we must show that (see Definition 139(i)),

$$\Pi(\Pi_{\aleph_1}\varphi) = \Pi_{\aleph_1}\varphi = \Sigma(\Pi_{\aleph_1}\varphi).$$

Let $a_1 \geq a_2 \geq \cdots$ be a $\Pi_{\aleph_1}\varphi$-convergent sequence. In order to show that $\Pi(\Pi_{\aleph_1}\varphi) = \Pi_{\aleph_1}\varphi$, it suffices to prove that $\bigwedge_n a_n \in \Pi_{\aleph_1}\varphi$.

Since $\aleph_1$ is a limit ordinal, we know that $\Pi_{\aleph_1}L = \bigcup_{\alpha<\aleph_1} \Pi_\alpha L$ (see Defintion 138). Define for each $n \in \mathbb{N}$ an ordinal number $\alpha(n)$ by

$$\alpha(n) \;:=\; \min\{\,\alpha < \aleph_1 \colon\; a_n \in \Pi_\alpha L\,\}.$$

Now, the set $\{\,\alpha(1),\, \alpha(2),\, \ldots\}$ of ordinals has a supremum,

$$\xi \;:=\; \bigvee_n \alpha(n) \;\equiv\; \bigcup_n \alpha(n).$$

Since $\aleph_1$ is the smallest uncountable ordinal, and $\alpha(n) < \aleph_1$, we know that all $\alpha(n)$ are countable. Hence ξ is countable as well, and so $\xi < \aleph_1$.

Now, we have $a_n \in \Pi_\xi L$ for all $n \in \mathbb{N}$. Hence

$$\bigwedge_n a_n \;\in\; \Pi(\Pi_\xi L) \;=\; \Pi_{\xi+1} L \;\subseteq\; \Pi_{\aleph_1} L.$$

So we see that $\Pi(\Pi_{\aleph_1}\varphi) = \Pi_{\aleph_1}\varphi$. Similarly, $\Sigma(\Pi_{\aleph_1}\varphi) = \Pi_{\aleph_1}\varphi$.
Hence the hierarchy has collapsed at $\Pi_{\aleph_1}\varphi$. □

Corollary 145. *Let $V \supseteq L \xrightarrow{\varphi} E$ be a valuation system. Then*

$$\varphi \text{ is extendible} \qquad \Longleftrightarrow \qquad \varphi \text{ is } \Pi_{\aleph_1}\text{-extendible.}$$

Moreover, if φ is extendible, then $\overline{\varphi} = \Pi_{\aleph_1}\varphi$.

Proof. Assume φ is extendible in order to show that φ is $\Pi_{\aleph_1}$-extendible. Then we know that the hierarchy collapses (see Definition 139(ii)). So it collapsed at some $\Pi_\alpha\varphi$ (see Lemma 140(iv)). Pick an ordinal number β with $\beta > \alpha$ and $\beta > \aleph_1$. Then φ is Π_β-extendible by Lemma 140(i). But $\aleph_1 < \beta$, so φ is also $\Pi_{\aleph_1}$-extendible.

Suppose φ is $\Pi_{\aleph_1}$-extendible. Then the hierarchy has collapsed at $\Pi_{\aleph_1}\varphi$ by Lemma 144. Hence φ is extendible and $\overline{\varphi} = \Pi_{\aleph_1}\varphi$ (see Definition 141(ii)). □

Lemma 146. *Let $V \supseteq L \xrightarrow{\varphi} E$ and $V \supseteq C \xrightarrow{\psi} E$ be two valuation systems. Assume that ψ extends φ. Then for every ordinal number α, we have*

$$\begin{aligned} \psi \text{ is } \Pi_\alpha\text{-extendible} &\implies \varphi \text{ is } \Pi_\alpha\text{-extendible} \quad \text{and} \quad \Pi_\alpha\psi \text{ extends } \Pi_\alpha\varphi \\ \psi \text{ is } \Sigma_\alpha\text{-extendible} &\implies \varphi \text{ is } \Sigma_\alpha\text{-extendible} \quad \text{and} \quad \Sigma_\alpha\psi \text{ extends } \Sigma_\alpha\varphi. \end{aligned}$$

Proof. We prove this lemma using induction on α.
(Zero) For $\alpha = 0$, the proposition is trivial.
(Successor) Let α be an ordinal number such that if ψ is Σ_α-extendible, then φ is Σ_α-extendible and $\Sigma_\alpha\psi$ extends $\Sigma_\alpha\varphi$. Suppose ψ is $\Pi_{\alpha+1}$-extendible. We prove

$$\varphi \text{ is } \Pi_{\alpha+1}\text{-extendible} \qquad \text{and} \qquad \Pi_{\alpha+1}\psi \text{ extends } \Pi_{\alpha+1}\varphi. \tag{49}$$

Since ψ is $\Pi_{\alpha+1}$-extendible, we know that (see Definition 138),

$$\psi \text{ is } \Sigma_\alpha\text{-extendible} \qquad \text{and} \qquad \Sigma_\alpha\psi \text{ is } \Pi\text{-extendible}.$$

By assumption, the former implies that φ is Σ_α-extendible and $\Sigma_\alpha\psi$ extends $\Pi_\alpha\psi$; by Lemma 102, the latter implies $\Sigma_\alpha\varphi$ is Π-extendible and $\Pi(\Sigma_\alpha\psi)$ extends $\Pi(\Sigma_\alpha\varphi)$. In other words, we have proven Statement (49).

Let α be an ordinal number such that if ψ is Π_α-extendible, then we have that φ is Π_α-extendible and that $\Pi_\alpha\psi$ extends $\Pi_\alpha\varphi$. Suppose that ψ is $\Sigma_{\alpha+1}$-extendible. By a similar reasoning as before one can prove that

$$\varphi \text{ is } \Sigma_{\alpha+1}\text{-extendible} \qquad \text{and} \qquad \Sigma_{\alpha+1}\psi \text{ extends } \Sigma_{\alpha+1}\varphi.$$

(Limit) Let λ be a limit ordinal such that for all $\alpha < \lambda$, we have

$$\psi \text{ is } \Pi_\alpha\text{-extendible} \implies \varphi \text{ is } \Pi_\alpha\text{-extendible} \quad \text{and} \quad \Pi_\alpha\psi \text{ extends } \Pi_\alpha\varphi.$$

Further, assume ψ is Π_λ-extendible in order to prove that

$$\varphi \text{ is } \Pi_\lambda\text{-extendible} \qquad \text{and} \qquad \Pi_\lambda\psi \text{ extends } \Pi_\lambda\varphi. \tag{50}$$

Let $\alpha < \lambda$ be given. Since ψ is Π_λ-extendible, we know that ψ is Π_α-extendible. So by assumption, φ is Π_α-extendible, and $\Pi_\alpha\psi$ extends $\Pi_\alpha\varphi$.

So we see that φ is Π_λ-extendible. Further (since $\Pi_\lambda\psi$ extends $\Pi_\alpha\psi$), we see that $\Pi_\lambda\psi$ extends all $\Pi_\alpha\varphi$. Hence $\Pi_\lambda\psi$ extends $\Pi_\lambda\varphi$. So we have proven (50). □

Proposition 147. *Let $V \supseteq L \xrightarrow{\varphi} E$ and $V \supseteq C \xrightarrow{\psi} E$ be valuation systems. Assume that ψ extends φ, and that ψ is extendible. Then*

$$\varphi \text{ is extendible} \qquad \text{and} \qquad \overline{\psi} \text{ extends } \overline{\varphi}.$$

Proof. By Corollary 145, we get the conclusion from Lemma 146 with $\alpha = \aleph_1$. □

Proposition 148. *Let $V \supseteq L \xrightarrow{\varphi} E$ and $V \supseteq C \xrightarrow{\psi} E$ be valuation systems. Assume that ψ extends φ and that ψ is complete with respect to V. Then*

$$\varphi \text{ is extendible} \qquad \text{and} \qquad \psi \text{ extends } \overline{\varphi}.$$

(So, loosely speaking, $\overline{\varphi}$ is the smallest complete extension of φ.)

Proof. Since ψ is complete, ψ is clearly extendible and $\overline{\psi} = \psi$ (see Remark 143). Hence φ is extendible and $\psi = \overline{\psi}$ extends $\overline{\varphi}$ by Proposition 147. □

Remark 149. Let $V \supseteq L \xrightarrow{\varphi} E$ be a valuation system.
By Lemma 142 and Proposition 148 we see that

$$\varphi \text{ is extendible} \iff \varphi \text{ has an complete extension}.$$

Hence the name "extendible".

6. Closedness of the Completion under Operations

We have seen how we can obtain the Lebesgue measure and the Lebesgue integral as the (convexification of) the completion of relatively simple valuations systems,

$$\wp\mathbb{R} \supseteq \mathcal{A}_{\mathrm{S}} \xrightarrow{\mu_{\mathrm{S}}} \mathbb{R} \qquad \text{and} \qquad [-\infty,+\infty]^{\mathbb{R}} \supseteq F_{\mathrm{S}} \xrightarrow{\varphi_{\mathrm{S}}} \mathbb{R}.$$

It is now time to derive some simple facts about the completion. In this section we will prove statements of the following form.

(i) If $A, B \in \overline{\mathcal{A}_{\mathrm{S}}}$, then $A \backslash B \in \overline{\mathcal{A}_{\mathrm{S}}}$ (see Example 156).

(ii) If $f, g \in \overline{F_{\mathrm{S}}} \cap \mathbb{R}^{\mathbb{R}}$, then $f + g \in \overline{F_{\mathrm{S}}}$ (see Example 159).

Definition 150. Let P and Q be posets. Let $f\colon S \to Q$ be a map, where $S \subseteq P$. We say f is **σ-preserving with respect to** P provided that

(i) if $\bigwedge_n a_n$ exists (in P) for $a_1 \geq a_2 \geq \cdots$ from S, and if $\bigwedge_n a_n \in S$, then

$$f(\textstyle\bigwedge_n a_n) = \bigwedge_n f(a_n);$$

(ii) if $\bigvee_n b_n$ exists for $b_1 \leq b_2 \leq \cdots$ from S, and if $\bigvee_n b_n \in S$, then

$$f(\textstyle\bigvee_n b_n) = \bigvee_n f(b_n).$$

Let P and Q be posets. Let $f\colon P \to Q$ be a map. We say f is **σ-preserving** provided that f is σ-preserving with respect to P.

Remark 151. If in the setting of Definition 150 f is σ-preserving (with respect some S), then f is order preserving as well.

Theorem 152. *Let $V \supseteq L \xrightarrow{\varphi} E$ and $W \supseteq K \xrightarrow{\psi} F$ be extendible valuation systems. Let $A\colon V \to W$ and $f\colon E \to F$ be σ-preserving maps, such that*

$$A(L) \subseteq K \qquad \text{and} \qquad f \circ \varphi = \psi \circ A|L.$$

$$\begin{array}{ccccc} V & \hookleftarrow & L & \xrightarrow{\varphi} & E \\ {\scriptstyle A}\downarrow & & \downarrow{\scriptstyle A|L} & & \downarrow{\scriptstyle f} \\ W & \hookleftarrow & K & \xrightarrow{\psi} & F \end{array}$$

Then $A(\overline{L}) \subseteq \overline{K}$ and $f \circ \overline{\varphi} = \overline{\psi} \circ A|\overline{L}$.

Proof. We prove with induction that for every ordinal number α we have

$$\begin{aligned} A(\Pi_\alpha L) &\subseteq \overline{K} & f \circ \Pi_\alpha\varphi &= \overline{\psi} \circ A|\Pi_\alpha L \\ A(\Sigma_\alpha L) &\subseteq \overline{K} & f \circ \Sigma_\alpha\varphi &= \overline{\psi} \circ A|\Sigma_\alpha L. \end{aligned} \tag{51}$$

This is sufficient, because $\overline{L} = \Pi_{\aleph_1} L$ and $\overline{\varphi} = \Pi_{\aleph_1}\varphi$ (see Corollary 145).

(i) We prove (51) holds for $\alpha = 0$. Since $\Pi_0\varphi = \Sigma_0\varphi = \varphi$, we need to prove that $A(L) \subseteq \overline{K}$ and $f \circ \varphi = A|L$. But this is valid by assumption.

(ii) Let α be an ordinal number and assume (51) holds for α; we prove (51) holds for $\alpha+1$. We only prove $A(\Pi_{\alpha+1}L) \subseteq \overline{K}$ and $f \circ \Pi_{\alpha+1}\varphi = \overline{\psi} \circ A|\Pi_{\alpha+1}L$; the other part, $A(\Sigma_{\alpha+1}L) \subseteq \overline{K}$ and $f \circ \Sigma_{\alpha+1}\varphi = \overline{\psi} \circ A|\Sigma_{\alpha+1}L$ follows similarly.

Let $a \in \Pi_{\alpha+1}L$ be given. We need to prove that

$$A(a) \in \overline{K} \qquad \text{and} \qquad \overline{\psi}(A(a)) \;=\; f(\Pi_{\alpha+1}\varphi(a)). \tag{52}$$

Recall that $\Pi_{\alpha+1}L = \Pi(\Sigma_\alpha L)$, so write $a = \bigwedge_n a_n$ for some $\Sigma_\alpha\varphi$-convergent $a_1 \geq a_2 \geq \cdots$ and note that $\Pi_{\alpha+1}\varphi(a) = \bigwedge_n \Sigma_\alpha\varphi(a_n)$. We have

$$\begin{aligned} f(\Pi_{\alpha+1}\varphi(a)) &= f(\textstyle\bigwedge_n \Sigma_\alpha\varphi(a_n)) \\ &= \textstyle\bigwedge_n f(\Sigma_\alpha\varphi(a_n)) && \text{since } f \text{ is } \sigma\text{-preserving} \\ &= \textstyle\bigwedge_n \overline{\psi}(A(a_n)) && \text{since (51) holds for } \alpha. \end{aligned}$$

So we see that $A(a_1) \geq A(a_2) \geq \cdots$ is $\overline{\psi}$-convergent. Since $W \supseteq \overline{K} \xrightarrow{\overline{\psi}} F$ is complete, this implies $\bigwedge_n A(a_n) \in \overline{K}$ and $\bigwedge_n \overline{\psi}(A(a_n)) = \overline{\psi}(\bigwedge_n A(a_n))$. Because A is σ-preserving, we have $\bigwedge_n A(a_n) = A(a)$. Hence $A(a) \in \overline{K}$ and

$$\begin{aligned} f(\Pi_{\alpha+1}\varphi(a)) &= \textstyle\bigwedge_n \overline{\psi}(A(a_n)) \\ &= \overline{\psi}(\textstyle\bigwedge_n A(a_n)) \\ &= \overline{\psi}(A(a)). \end{aligned}$$

So we have proven Statement (52).

(iii) Let λ be a limit ordinal, and assume that (51) holds for all $\alpha < \lambda$; we prove that (51) holds for λ. Since $\Pi_\lambda\varphi = \Sigma_\lambda\varphi$, we must prove that

$$A(\Pi_\lambda L) \subseteq \overline{K} \qquad \text{and} \qquad f \circ \Pi_\lambda\varphi \;=\; \overline{\psi} \circ A|\Pi_\lambda L. \tag{53}$$

Let $a \in \Pi_\lambda L$ be given in order to prove $A(a) \in \overline{K}$ and $\overline{\psi}(A(a)) = f(\Pi_\lambda\varphi(a))$. Recall that $\Pi_\lambda L = \bigcup_{\alpha<\lambda} \Pi_\alpha L$, and $\Pi_\lambda\varphi \,|\, \Pi_\alpha L = \Pi_\alpha\varphi$ for all $\alpha < \lambda$. So choose $\alpha < \lambda$ such that $a \in \Pi_\alpha L$. Since (51) holds for α, we know that

$$A(\Pi_\alpha L) \subseteq \overline{K} \qquad \text{and} \qquad f \circ \Pi_\alpha\varphi = \overline{\psi} \circ A|\Pi_\alpha L.$$

Hence $A(a) \in A(\Pi_\alpha L) \subseteq \overline{K}$ and $f(\Pi_\lambda\varphi(a)) = f(\Pi_\alpha\varphi(a)) = \overline{\psi}(A(a))$. □

Example 153. Let $\mathcal{A}$ be a ring of subsets of X. Let $\mu\colon \mathcal{A} \to \mathbb{R}$ be a positive and additive map. Recall that $\wp X \supseteq \mathcal{A} \xrightarrow{\mu} \mathbb{R}$ is a valuation system (see Example 73). Assume that $\wp X \supseteq \mathcal{A} \xrightarrow{\mu} \mathbb{R}$ is extendible.

We would like to prove that $\overline{\mathcal{A}}$ is also a ring of subsets of X (as is $\mathcal{A}$). For the moment, we will prove this under the assumption that $X \in \mathcal{A}$, see Example 156.

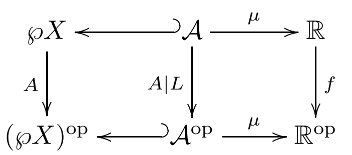
To prove that $\overline{\mathcal{A}}$ is a ring, we need to show that $Z\backslash Y \in \overline{\mathcal{A}}$ for all $Z, Y \in \overline{\mathcal{A}}$. Note that $Z\backslash Y = (X\backslash Y) \cap Z$. So it suffices to show that $X\backslash Y \in \overline{\mathcal{A}}$ for all $Y \in \overline{\mathcal{A}}$.

Consider the order *reversing* maps $A\colon \wp X \to \wp X$ and $f\colon \mathbb{R} \to \mathbb{R}$ given by

$$\begin{aligned} A(Y) &= X\backslash Y & \qquad & (Y \subseteq X) \\ f(x) &= \mu(X) - x & & (x \in \mathbb{R}). \end{aligned}$$

In order to apply Theorem 152 to these maps, let us rebaptise them as order preserving maps $A\colon \wp X \to (\wp X)^{\mathrm{op}}$ and $f\colon \mathbb{R} \to \mathbb{R}^{\mathrm{op}}$ (see Example 17).

We have the following situation.



We leave it to the reader to verify that $(\wp X)^{\mathrm{op}} \supseteq \mathcal{A}^{\mathrm{op}} \xrightarrow{\mu} \mathbb{R}^{\mathrm{op}}$ is again valuation system which is extendibe. We have $A(\mathcal{A}) \subseteq \mathcal{A}$, because $X\backslash Y \in \mathcal{A}$ for all $Y \in \mathcal{A}$ since $\mathcal{A}$ is a ring containing X. Further, since μ is additive, we have

$$\mu(A(Y)) \;=\; \mu(X\backslash Y) \;=\; \mu(X) - \mu(Y) \;=\; f(\mu(Y)).$$

So $\mu \circ A$ and $f \circ \mu$ are identical on $\mathcal{A}$. Note that

$$X\backslash \textstyle\bigcup_n A_n \;=\; \bigcap_n X\backslash A_n \qquad \text{and} \qquad \mu(X) - \bigvee_n x_n \;=\; \bigwedge_n \mu(X) - x_n$$

where $Y_1 \subseteq Y_2 \subseteq \cdots$ are from $\wp X$ and $x_1 \leq x_2 \leq \cdots$ is a bounded sequence in $\mathbb{R}$. So A and f are σ-preserving (see Definition 150).

Hence by Theorem 152, we get $A(\overline{\mathcal{A}}) \subseteq \overline{\mathcal{A}}$ and secondly $f \circ \overline{\mu} = \overline{\mu} \circ A$ on $\overline{\mathcal{A}}$.

From the first fact we get that $X\backslash Y \in \overline{Y}$ for all $Y \in \mathcal{A}$, and hence $\overline{\mathcal{A}}$ is a ring.

From the second fact, we get $\overline{\mu}(X\backslash Y) = \overline{\mu}(X) - \overline{\mu}(Y)$ for all $Y \in \overline{\mathcal{A}}$. From this, one might say, we see that $\overline{\mu}$ is additive. However, we already knew this as $\overline{\mu}$ is modular (and $\mu(\varnothing) = 0$, see Definition 138).

Theorem 154. *Let $V \supseteq L \xrightarrow{\varphi} E$ and $W \supseteq K \xrightarrow{\psi} F$ be extendible valuation systems. Let $A\colon V \to W$ be a σ-preserving map such that $A(L) \subset K$. Let $f\colon E \to F$ be a σ-preserving group-homomorphism such that*

$$d_{\overline{\psi}}(A(c), A(d)) \;\leq\; f(d_{\overline{\varphi}}(c, d)) \tag{54}$$

for all $c, d \in V$ with $A(c), A(d) \in \overline{K}$.

$$\begin{array}{ccccc} V & \hookleftarrow & L & \xrightarrow{\varphi} & E \\ {\scriptstyle A}\downarrow & & \downarrow{\scriptstyle A|L} & & \downarrow{\scriptstyle f} \\ W & \hookleftarrow & K & \xrightarrow{\psi} & F \end{array}$$

Then $A(\overline{L}) \subseteq \overline{K}$.

Proof. With induction, we prove that for every ordinal number α, we have

$$A(\Pi_\alpha L) \subseteq \overline{K} \qquad \text{and} \qquad A(\Sigma_\alpha L) \subseteq \overline{K}.$$

This is sufficient, because $\overline{L} = \Pi_{\aleph_1} L$.

As one can see, such a proof might be quite similar to the proof of Theorem 152. Therefore, we leave the details to the reader and only prove the following statement.

$$A(\Sigma_\alpha L) \subseteq \overline{K} \quad \Longrightarrow \quad A(\Pi_{\alpha+1} L) \subseteq \overline{K}. \tag{55}$$

Assume $A(\Sigma_\alpha L) \subseteq \overline{K}$ and let $a \in \Pi_{\alpha+1} L$ be given; we must prove $A(a) \in \overline{K}$. Write $a = \bigwedge_n a_n$ for some $\Sigma_\alpha\varphi$-convergent sequence $a_1 \geq a_2 \geq \cdots$ in $\Sigma_\alpha L$. Because we have assumed $A(\Sigma_\alpha L) \subseteq \overline{K}$, we know that $A(a_i) \in \overline{K}$. To prove $A(a) \in \overline{K}$, it suffices to show that $A(a_1) \geq A(a_2) \geq \cdots$ is $\overline{\psi}$-convergent. Indeed, then

$$A(a) \;\equiv\; A(\textstyle\bigwedge_n a_n) \;=\; \textstyle\bigwedge_n A(a_n) \;\in\; \overline{K},$$

because A is σ-preserving and $W \supseteq \overline{K} \xrightarrow{\overline{\psi}} F$ is complete.

To prove that the sequence $A(a_1) \geq A(a_2) \geq \cdots$ is $\overline{\psi}$-convergent, we must show that $\bigwedge_n \overline{\psi}(A(a_n))$ exists. Note that by Inequality (54), we have

$$\begin{aligned} \overline{\psi}(A(a_{n+1})) \;-\; \overline{\psi}(A(a_n)) \;&=\; d_{\overline{\psi}}(A(a_{n+1}), A(a_n)) \\ &\leq\; f(d_{\overline{\varphi}}(a_{n+1}, a_n)) \;=\; f(\overline{\varphi}(a_{n+1}, a_n)) \;-\; f(\overline{\varphi}(a_n)). \end{aligned}$$

So since F is R-complete (see Definition 44), in order to show that $\bigwedge_n \overline{\psi}(A(a_n))$ exists, it suffices to prove that $\bigwedge_n f(\overline{\varphi}(a_n))$ exists. For this we need to prove that $\bigwedge_n \overline{\varphi}(a_n)$ exists (as f is σ-preserving). That is, we must show that $a_1 \geq a_2 \geq \cdots$ is $\overline{\varphi}$-convergent. Of course, this follows quickly from the fact that $a_1 \geq a_2 \geq \cdots$ is $\Sigma_\alpha\varphi$-convergent. We have proven Statement (55). ☐

Proposition 155. *Let $V \supseteq L \xrightarrow{\varphi} E$ be an extendible valuation system. Note that its completion is denoted by $V \supseteq \overline{L} \xrightarrow{\overline{\varphi}} E$. Given $\ell \leq u$ from L, consider*

$$[\ell, u] \supseteq L \cap [\ell, u] \xrightarrow{\varphi|[\ell,u]} E;$$

it is an extendible valuation system. Note that its completion is denoted by

$$[\ell, u] \supseteq \overline{L \cap [\ell, u]} \xrightarrow{\overline{\varphi|[\ell,u]}} E.$$

We have $\overline{L \cap [\ell, u]} = \overline{L} \cap [\ell, u]$. Moreover, $\overline{\varphi}$ and $\overline{\varphi|[\ell, u]}$ are identical on $\overline{L \cap [\ell, u]}$.

Proof. One can easily see that $\overline{\varphi}|[\ell, u]$ extends $\varphi|[\ell, u]$ and that the valuation system

$$[\ell, u] \supseteq \overline{L} \cap [\ell, u] \xrightarrow{\overline{\varphi}|[\ell,u]} E$$

is complete. Hence $\varphi|[\ell, u]$ is extendible, and $\overline{\varphi}|[\ell, u]$ extends $\overline{\varphi|[\ell, u]}$ (see Proposition 147). In particular, $\overline{L \cap [\ell, u]} \subseteq \overline{L} \cap [\ell, u]$ and $\overline{\varphi}$ and $\overline{\varphi|[\ell, u]}$ are identical on $\overline{L \cap [\ell, u]}$. It remains to be shown that

$$\overline{L} \cap [\ell, u] \;\subseteq\; \overline{L \cap [\ell, u]}. \tag{56}$$

To this end, consider the map $\varrho\colon V \to [\ell, u]$ given by $\varrho(x) = \ell \vee (x \wedge u)$. Note that $\varrho(x) = x$ for all $x \in [\ell, u]$, and thus $\varrho(\overline{L}) = \overline{L} \cap [\ell, u]$. So in order to prove (56), we must show that $\varrho(\overline{L}) \subseteq \overline{L \cap [\ell, u]}$. To do this, we apply Theorem 154.

$$\begin{array}{ccccc} V & \hookleftarrow & L & \xrightarrow{\ \varphi\ } & E \\ \downarrow{\scriptstyle \varrho} & & \downarrow{\scriptstyle \varrho|L} & & \downarrow{\scriptstyle 1_E} \\ [\ell, u] & \hookleftarrow & L \cap [\ell, u] & \xrightarrow{\ \varphi|[\ell,u]\ } & E \end{array}$$

We must verify that ϱ is σ-preserving and that

$$d_{\overline{\varphi|[\ell,u]}}(\varrho(c), \varrho(d)) \ \leq\ d_{\overline{\varphi}}(c, d) \tag{57}$$

for all $c, d \in V$ with $\varrho(c), \varrho(d) \in \overline{L \cap [\ell, u]}$. One can easily see that ϱ is σ-preserving, because V is σ-distributive (see Definition 70). Concerning Inequality (57), note that for $c, d \in V$ with $\varrho(c), \varrho(d) \in \overline{L \cap [\ell, u]}$ we have

$$\begin{array}{rcll} d_{\overline{\varphi|[\ell,u]}}(\varrho(c), \varrho(d)) & = & d_{\overline{\varphi}}(\varrho(c), \varrho(d)) & \text{since } \overline{\varphi}|\,\overline{L \cap [\ell, u]} = \overline{\varphi|[\ell, u]} \\ & = & d_{\overline{\varphi}}(\,\ell \vee (c \wedge u),\ \ell \vee (d \wedge u)\,) & \text{by definition of } \varrho \\ & \leq & d_{\overline{\varphi}}(\,c \wedge u,\ d \wedge u\,) & \text{by Lemma 24} \\ & \leq & d_{\overline{\varphi}}(c, d) & \text{by Lemma 24.} \end{array}$$

Hence Theorem 154 is applicable, and we obtain Inequality (56). □

Example 156. Let $\mathcal{A}$ be a ring of subsets of X. Let $\mu\colon \mathcal{A} \to \mathbb{R}$ be a positive and additive map. Recall that $\wp X \supseteq \mathcal{A} \xrightarrow{\mu} \mathbb{R}$ is a valuation system (see Example 73). Assume that $\wp X \supseteq \mathcal{A} \xrightarrow{\mu} \mathbb{R}$ is extendible (see Definition 141).

We prove that $\overline{\mathcal{A}}$ is a ring. (In Example 153, we saw that this is the case if $X \in \mathcal{A}$.)

Let $Y, Z \in \overline{\mathcal{A}}$ be given. To prove that $\overline{\mathcal{A}}$ is a ring, we must show that

$$Y \backslash Z \in \overline{\mathcal{A}}.$$

We restrict our attention to the interval $I := [\,\varnothing,\, Y \cup Z\,]$. Note that $\mathcal{A} \cap I$ is a ring of subset of $Y \cup Z$ with $Y \cup Z \in \mathcal{A} \cap I$. So by Example 153, we know that $\overline{\mathcal{A} \cap I}$ is a ring. Note that $Y, Z \in \overline{\mathcal{A} \cap I}$ because $\overline{\mathcal{A}} \cap I = \overline{\mathcal{A} \cap I}$ by Proposition 155. So we get $Y \backslash Z \in \overline{\mathcal{A} \cap I}$, and thus $Y \backslash Z \in \overline{\mathcal{A}} \cap I$ by Proposition 155.

Hence $\overline{\mathcal{A}}$ is a ring of subsets of X.

Theorem 157. *Let $V \supseteq L \xrightarrow{\varphi} E$ and $W \supseteq K \xrightarrow{\psi} F$ be extendible valuation systems. Let R be a sublattice of V with $L \subseteq R$. Let $f\colon E \to F$ be a σ-preserving map, and let $A\colon R \to W$ be σ-preserving with respect to V. Assume that $A(L) \subseteq K$ and $f \circ \varphi = \psi \circ A|L$.*

$$\begin{array}{ccccccc} V & \hookleftarrow & R & \hookleftarrow & L & \xrightarrow{\ \varphi\ } & E \\ & & \downarrow{\scriptstyle A} & & \downarrow{\scriptstyle A|L} & & \downarrow{\scriptstyle f} \\ & & W & \hookleftarrow & K & \xrightarrow{\ \psi\ } & F \end{array}$$

Assume that R is convex in V, and that for every $c \in R$, there are φ-convergent sequences $a_1 \leq a_2 \leq \cdots$ and $b_1 \geq b_2 \geq \cdots$ such that $\bigwedge_n a_n \leq c \leq \bigvee_n b_n$.

Then $A(\overline{L} \cap R) \subseteq \overline{K}$ and $f \circ \overline{\varphi} = \overline{\psi} \circ A$ on $\overline{L} \cap R$.

Proof. Let us first prove the following special case.

$$\left[\ \begin{array}{l} \text{Let } c \in \overline{L} \cap R \text{ with } \ell \leq c \leq u \text{ for some } \ell, u \in L. \text{ Then} \\ \qquad A(c) \in \overline{K} \qquad \text{and} \qquad f(\overline{\varphi}(c)) = \overline{\psi}(A(c)). \end{array}\right. \tag{58}$$

Let $c \in \overline{L} \cap R$ with $\ell \leq c \leq u$ for some $\ell, u \in L$ be given. Then clearly $c \in [\ell, u]$. Further, $[\ell, u] \subseteq R$ since R is convex and $\ell, u \in R$ (as $L \subseteq R$). So we have:

$$\begin{array}{ccccc} [\ell,u] & \hookleftarrow & L \cap [\ell,u] & \xrightarrow{\varphi|[\ell,u]} & E \\ {\scriptstyle A|[\ell,u]}\downarrow & & \downarrow{\scriptstyle A \,|\, L\cap[\ell,u]} & & \downarrow{\scriptstyle f} \\ W & \hookleftarrow & K & \xrightarrow{\psi} & F \end{array}$$

Moreover, by Proposition 155, we know that $c \in \overline{L \cap [\ell, u]}$ and $\overline{\varphi}(c) = \overline{\varphi|[\ell, u]}(c)$. Hence Theorem 152 yields $A(c) \in \overline{K}$ and $\overline{\psi}(A(c)) = f(\overline{\varphi|[\ell, u]}(c))$. But then $\overline{\psi}(A(c)) = f(\overline{\varphi}(c))$. This proves Statement (58).

We proceed by proving another special case.

$$\left[\begin{array}{c} \text{Let } c \in \overline{L} \cap R \text{ and suppose } c \geq \ell \text{ for some } \ell \in L. \text{ Then} \\ A(c) \in \overline{K} \qquad \text{and} \qquad f(\overline{\varphi}(c)) = \overline{\psi}(A(c)). \end{array}\right. \tag{59}$$

Let $c \in \overline{L} \cap R$ with $c \geq \ell$ for some ℓ be given. Pick φ-convergent $u_1 \leq u_2 \leq \cdots$ such that $c \leq \bigvee_n u_n$. Since $u_1 \geq u_2 \geq \cdots$ is φ-convergent and $c \in \overline{L}$, we know that $c \wedge u_1 \leq c \wedge u_2 \leq \cdots$ is $\overline{\varphi}$-convergent (see Proposition 48). Since $V \supseteq \overline{L} \overset{\overline{\varphi}}{\to} E$ is complete, this implies $\overline{\varphi}(c) = \overline{\varphi}(\bigvee_n c \wedge u_n) = \bigvee_n \overline{\varphi}(c \wedge u_n)$. We get:

$$\begin{aligned} f(\overline{\varphi}(c)) &= f(\textstyle\bigvee_n \overline{\varphi}(c \wedge u_n)) \\ &= \textstyle\bigvee_n f(\overline{\varphi}(c \wedge u_n)) \qquad \text{since } f \text{ is } \sigma\text{-preserving} \end{aligned}$$

Note that $\ell \leq c \wedge u_n \leq u_n$. So by (58), we get $A(c \wedge u_n) \in \overline{K}$ and:

$$f(\overline{\varphi}(c)) = \textstyle\bigvee_n \overline{\psi}(A(c \wedge u_n))$$

From this we see $A(c \wedge u_1) \leq A(c \wedge u_2) \leq \cdots$ is $\overline{\psi}$-convergent. Since $W \supseteq \overline{K} \overset{\overline{\psi}}{\to} F$ is complete, we get $f(\overline{\varphi}(c) = \bigvee_n A(c \wedge u_n) \in \overline{K}$ and

$$\begin{aligned} f(\overline{\varphi}(c)) &= \overline{\psi}(\textstyle\bigvee_n A(c \wedge u_n)) \\ &= \overline{\psi}(A(\textstyle\bigvee_n c \wedge u_n)) \qquad \text{since } A \text{ is } \sigma\text{-preserving} \\ &= \overline{\psi}(A(c)). \end{aligned}$$

This completes the proof of Statement (59).

We are now ready to give the proof of the general case. Let $c \in R \cap \overline{L}$ be given. We need to prove that $A(c) \in \overline{K}$ and $f(\overline{\varphi}(c)) = \overline{\psi}(A(c))$. Pick φ-convergent $\ell_1 \geq \ell_2 \geq \cdots$ such that $\bigwedge_n \ell_n \leq c$. Since $\ell_1 \geq \ell_2 \geq \cdots$ is φ-convergent and $c \in \overline{L}$, we know that $\ell_1 \vee c \geq \ell_2 \vee c \geq \cdots$ is $\overline{\varphi}$-convergent. Since $V \supseteq \overline{L} \overset{\overline{\varphi}}{\to} E$ is complete, this implies that $\overline{\varphi}(c) = \overline{\varphi}(\bigwedge_n \ell_n \vee c) = \bigwedge_n \overline{\varphi}(\ell_n \vee c)$. We get:

$$\begin{aligned} f(\overline{\varphi}(c)) &= f(\textstyle\bigwedge_n \overline{\varphi}(\ell_n \vee c)) \\ &= \textstyle\bigwedge_n f(\overline{\varphi}(\ell_n \vee c)) \qquad \text{since } f \text{ is } \sigma\text{-preserving} \end{aligned}$$

Note that $\ell_n \leq \ell_n \vee c$. Further, since R is a sublattice of V, and $c \in R$, $\ell_n \in L \subseteq R$, we get $\ell_n \vee c \in R$. So by (59), we have $A(\ell_n \vee c) \in \overline{K}$ and:

$$f(\overline{\varphi}(c)) = \textstyle\bigwedge_n \overline{\psi}(A(\ell_n \vee c))$$

From this we see $A(\ell_1 \vee c) \geq A(\ell_2 \vee c) \geq \cdots$ is $\overline{\psi}$-convergent. Since $W \supseteq \overline{K} \overset{\overline{\psi}}{\to} F$ is complete, we get $f(\overline{\varphi}(c) = \bigwedge_n A(\ell_n \vee c) \in \overline{K}$ and

$$\begin{aligned} f(\overline{\varphi}(c)) &= \overline{\psi}(\textstyle\bigwedge_n A(\ell_n \vee c)) \\ &= \overline{\psi}(A(\textstyle\bigwedge_n \ell_n \vee c)) \qquad \text{since } A \text{ is } \sigma\text{-preserving} \\ &= \overline{\psi}(A(c)). \end{aligned}$$

We are done. □

Proposition 158. *Let $V \supseteq L \xrightarrow{\varphi} E$ be an extendible valuation system. Let R be a sublattice of V endowed with a group structure. Assume L is a subgroup of R and that φ is a group homomorphism (recall that E is an ordered Abelian* group*).*

Further, assume that R is convex and that for every $c \in R$, there are φ-convergent sequences $a_1 \geq a_2 \geq \cdots$ and $b_1 \leq b_2 \leq \cdots$ such that $\bigwedge_n a_n \leq c \leq \bigvee_n b_n$.

Then $\overline{L} \cap R$ is a subgroup of R, and $\overline{\varphi}|R$ is a group homomorphism.

Proof. In order to show that $\overline{L} \cap R$ is a subgroup of R, we must prove the following.

(i) If $a, b \in \overline{L} \cap R$, then $a + b \in \overline{L}$.
(ii) If $a \in \overline{L} \cap R$, then $-a \in \overline{L}$.

We only give a proof for (i). It will then be clear how to prove (ii).

We aim to apply Theorem 157. To this end, the reader can easily verify that $V \times V \supseteq L \times L \xrightarrow{\varphi \times \varphi} E \times E$ is an extendible valuation system; that its completion is $V \times V \supseteq \overline{L} \times \overline{L} \xrightarrow{\overline{\varphi} \times \overline{\varphi}} E \times E$; that $R \times R$ is a convex sublattice of $V \times V$; that the assignment $(c, d) \mapsto c + d$ yields a σ-preserving map $A \colon R \times R \to V$ with respect to $V \times V$; that the map $f \colon E \times E \to E$ given by $f(x, y) = x + y$ is σ-preseving.

Further, note that $A(L \times L) \subseteq L$ because L is a subgroup of R. Note that for all $c_1, c_2 \in R \times R$ there are φ-convergent $\ell_1^i \geq \ell_2^i \geq \cdots$ and $u_1^i \leq u_2^i \leq \cdots$ such that $\bigwedge_n \ell_n^i \leq c_i \leq \bigvee_n u_n^i$, and thus $\bigwedge_n (\ell_n^1, \ell_n^2) \leq (c_1, c_2) \leq \bigvee_n (u_n^1, u_n^2)$, where $(\ell_1^1, \ell_1^2) \geq (\ell_2^1, \ell_2^2) \geq \cdots$ and $(u_1^1, u_1^2) \leq (u_2^1, u_2^2) \leq \cdots$ are $\varphi \times \varphi$-convergent. Finally, note that $f \circ (\varphi \times \varphi) = \varphi \circ A|(L \times L)$, because φ is a group homomorphism.

$$\begin{array}{ccccccc}
V \times V & \hookleftarrow & R \times R & \hookleftarrow & L \times L & \xrightarrow{\varphi \times \varphi} & E \times E \\
 & & {\scriptstyle +}\downarrow & & {\scriptstyle +}\downarrow & & \downarrow{\scriptstyle +} \\
 & & V & \hookleftarrow & L & \xrightarrow{\quad\varphi\quad} & E
\end{array}$$

So we are in a position to apply Theorem 157. It gives us that $A(\overline{L} \times \overline{L} \cap R \times R) \subseteq L$ and $f \circ (\overline{\varphi} \times \overline{\varphi}) = \overline{\varphi} \circ A$ on $\overline{L} \times \overline{L} \cap R \times R$. In other words, if $c, d \in \overline{L} \cap R$, then $c + d \in \overline{L}$, and $\overline{\varphi}(c + d) = \overline{\varphi}(c) + \overline{\varphi}(d)$. Hence we have proven statement (i), and at the same time we have shown that $\overline{\varphi}$ is a group homomorphism. □

Example 159. Let X be a set. Let F be a Riesz space of functions on X. Let $\varphi \colon F \to R$ be a positive linear map. Recall that $[-\infty, \infty]^X \supseteq F \xrightarrow{\varphi} \mathbb{R}$ is a valuation system. Assume that φ is extendible.

We would like to prove that $\overline{F}$ is a Riesz space of functions and $\overline{\varphi}$ is linear. However, since addition is only defined on $R := \mathbb{R}^X$, we will instead show that $\overline{F} \cap R$ is a Riesz space of functions and that $\overline{\varphi}|R$ is linear. Moreover, we assume that for every $f \in \overline{F} \cap R$ there are φ-convergent sequences $\ell_1 \geq \ell_2 \geq \cdots$ and $u_1 \leq u_2 \leq \cdots$ such that $\bigwedge_n \ell_n \leq f \leq \bigvee_n u_n$.

To prove that $\overline{F} \cap R$ is a Riesz space, we must show that

(i) $f + g \in \overline{F}$ for all $f, g \in \overline{F} \cap R$, and
(ii) $\lambda \cdot f \in \overline{F}$ for all $\lambda \in \mathbb{R}$ and $f \in \overline{F} \cap R$.

We only prove the first statement; we leave it to the reader to prove the second.

Of course, it suffices to establish that $\overline{F} \cap R$ is a subgroup of R. To do this, we apply Proposition 158. Indeed, one can easily see that all the prerequisites are met. To name a few: one sees that R is a sublattice of V, that F is a subgroup of R (since F is a Riesz space of functions), that φ is a group homomorphism (since φ is linear), and that R is convex (since $\mathbb{R}$ is convex in $[-\infty, \infty]$).

Proposition 158 not only gives us that $\overline{F} \cap R$ is a subgroup of R, but also that $\varphi|R$ is a group homomorphism. We leave it to the reader to prove that $\overline{\varphi}|R$ is homogeneous, i.e., $\overline{\varphi}(\lambda \cdot f) = \lambda \cdot \overline{\varphi}(f)$ for all $f \in \overline{F} \cap R$ and $\lambda \in \mathbb{R}$.

Hence $\overline{F} \cap R$ is a Riesz space of functions, and $\overline{\varphi}|R$ is linear.

7. Extendibility

Let $V \supseteq L \xrightarrow{\varphi} E$ be a valuation space, and suppose we want to prove that φ can be extended to a complete valuation. We have seen that it suffices to prove that φ is $\Pi_{\aleph_1}$-extendible (see Corollary 145). However, to prove φ is $\Pi_{\aleph_1}$-extendible already seems like a monumental task when one has only barely started to unfold the definition of "φ is $\Pi_{\aleph_1}$-extendible" (see Definition 138):

$$\begin{array}{rcr}
\varphi \text{ is } \Pi\text{-extendible,} & \text{and} & \varphi \text{ is } \Sigma\text{-extendible;} \\
\Pi\varphi \text{ is } \Pi\text{-extendible,} & \text{and} & \Sigma\varphi \text{ is } \Sigma\text{-extendible;} \\
\Pi_2\varphi \text{ is } \Pi\text{-extendible,} & \text{and} & \Sigma_2\varphi \text{ is } \Sigma\text{-extendible;} \\
\vdots & \vdots & \vdots \\
\Pi_\omega\varphi \text{ is } \Pi\text{-extendible,} & \text{and} & \Sigma_\omega\varphi \text{ is } \Sigma\text{-extendible;} \\
\Pi_{\omega+1}\varphi \text{ is } \Pi\text{-extendible,} & \text{and} & \Sigma_{\omega+1}\varphi \text{ is } \Sigma\text{-extendible;} \\
\vdots & \vdots & \vdots
\end{array}$$

It turns out that for some E the situation is more tractable. For instance, we will see that if $E = \mathbb{R}$, then to prove that φ is extendible it suffices to show that φ is Π_2-extendible or Σ_2-extendible. Actually, we have a sharper result: it suffices to show that φ is *continuous* (see Definition 160). Those E for which we have

$$\varphi \text{ is continuous} \qquad \Longrightarrow \qquad \varphi \text{ is extendible.}$$

will be called *benign* (see Definition 165).

7.1. Continuous Valuations.

Below we define what it means for a valuation system to be continuous. We will see that we have the following implications

$$\begin{array}{ccccc}
\Sigma_2\text{-extendible} & & & & \Sigma\text{-extendible} \\
& \searrow & & \nearrow & \\
& & \text{continuous} & & \\
& \nearrow & & \searrow & \\
\Pi_2\text{-extendible} & & & & \Pi\text{-extendible.}
\end{array}$$

In fact, we prove that φ is continuous if and only if it can be extended to $\Pi L \cup \Sigma L$ in some sense (see Lemma 162), so that we might have dubbed it "$\Pi \cup \Sigma$-extendible".

Definition 160. Let $V \supseteq L \xrightarrow{\varphi} E$ be a valuation system.
We say φ (or more precisely $V \supseteq L \xrightarrow{\varphi} E$) is **continuous** provided that

$$\textstyle\bigwedge_n a_n \leq \bigvee_n b_n \qquad \Longrightarrow \qquad \bigwedge_n \varphi(a_n) \leq \bigvee_n \varphi(b_n)$$

for all φ-convergent $a_1 \geq a_2 \geq \cdots$ and φ-convergent $b_1 \leq b_2 \leq \cdots$.

Example 161. We leave it to the reader to verify that the valaution systems

$$\wp\mathbb{R} \supseteq \mathcal{A}_\mathrm{S} \xrightarrow{\mu_\mathrm{S}} \mathbb{R} \qquad \text{and} \qquad [-\infty, +\infty]^\mathbb{R} \supseteq F_\mathrm{S} \xrightarrow{\varphi_\mathrm{S}} \mathbb{R}$$

are continuous.

Lemma 162. *Let $V \supseteq L \xrightarrow{\varphi} E$ be a valuation system. The following are equivalent.*

(i) φ is continuous.

(ii) φ is Π-extendible and Σ-extendible, and there is an order preserving map

$$f \colon \Pi L \cup \Sigma L \to E$$

that extends both $\Pi\varphi$ and $\Sigma\varphi$.

Proof. (i) $\Longrightarrow$ (ii) Suppose that φ is continuous. By Lemma 103, we see that φ is Π-extendible. Similarly, φ must be Σ-extendible. We need to find an order preserving map $f\colon \Pi L \cup \Sigma L \to E$ that extends both $\Pi\varphi$ and $\Sigma\varphi$. We have little choice,

$$f(c) \ := \ \begin{cases} \Pi\varphi(c) & \text{if } c \in \Pi L; \\ \Sigma\varphi(c) & \text{if } c \in \Sigma L. \end{cases} \tag{60}$$

To see that Equation (60) is a valid definition of a map $f\colon \Pi L \cup \Sigma L \to E$, we need to verify that $\Pi\varphi$ and $\Sigma\varphi$ are identical on $\Pi L \cap \Sigma L$. Let $c \in \Pi L \cap \Sigma L$ be given. We must prove $\Pi\varphi(c) = \Sigma\varphi(c)$. Choose φ-convergent $a_1 \geq a_2 \geq \cdots$ and φ-convergent $b_1 \leq b_2 \leq \cdots$ such that $\bigwedge_n a_n = c = \bigvee_n b_n$.

Then $b_n \leq a_n$ for all n, so $\varphi(b_n) \leq \varphi(a_n)$ for all n. Hence

$$\Sigma\varphi(c) = \textstyle\bigvee_n \varphi(b_n) \ \leq \ \bigwedge_n \varphi(a_n) = \Pi\varphi(c).$$

Conversely, we have $\bigwedge_n a_n \leq \bigvee_n b_n$, so since φ is continuous we get

$$\Pi\varphi(c) = \textstyle\bigwedge_n \varphi(a_n) \ \leq \ \bigvee_n \varphi(b_n) = \Sigma\varphi(c).$$

Hence $\Pi\varphi(c) = \Sigma\varphi(c)$. So Equation (60) is a valid definition of f.

Since by defintion, f extends both $\Pi\varphi$ and $\Sigma\varphi$, it only remains to be shown that f is order preserving. Let $c, d \in \Pi L \cup \Sigma L$ with $c \leq d$ be given. We prove

$$f(c) \ \leq \ f(d).$$

Of course, if c, d are both in ΠL, then we done, because $\Pi\varphi$ is order preserving and f extends $\Pi\varphi$. Similarly, if $c, d \in \Sigma L$, we also immediately get $f(c) \leq f(d)$.

Suppose $c \in \Pi L$ and $d \in \Sigma L$. Choose φ-convergent sequences $b_1 \leq b_2 \leq \cdots$ and $a_1 \geq a_2 \geq \cdots$ such that $\bigvee_n b_n = c$ and $\bigwedge_n a_n = d$. Then $b_m \leq \bigvee_n b_n \leq \bigwedge_n a_n \leq a_m$ for all m, so $\varphi(b_m) \leq \varphi(a_m)$ for all m, and hence

$$f(c) = \Sigma\varphi(c) = \textstyle\bigvee_n \varphi(b_n) \ \leq \ \bigwedge_n \varphi(a_n) = \Pi\varphi(d) = f(d).$$

Suppose $c \in \Pi L$ and $d \in \Sigma L$. Choose φ-convergent sequences $a_1 \geq a_2 \geq \cdots$ and $b_1 \leq b_2 \leq \cdots$ such that $\bigwedge_n a_n = c$ and $\bigvee_n b_n = d$. Then $\bigwedge_n a_n \leq \bigvee_n b_n$. So since f is continuous, we get $f(c) = \Pi\varphi(c) = \bigwedge_n \varphi(a_n) \leq \bigvee_n \varphi(b_n) = \Sigma\varphi(d) = f(d)$.

(ii) $\Longrightarrow$ (i) Let $f\colon \Pi L \cup \Sigma L \to E$ be an order preserving map that extends both $\Pi\varphi$ and $\Sigma\varphi$. We prove that φ is continuous. Let φ-convergent sequences $a_1 \geq a_2 \geq \cdots$ and $b_1 \leq b_2 \leq \cdots$ with $\bigwedge_n a_n \leq \bigvee_n b_n$ be given. We need to prove that

$$\textstyle\bigwedge_n \varphi(a_n) \ \leq \ \bigvee_n \varphi(b_n).$$

This is easy; since f is order preserving and extends $\Pi\varphi$ and $\Sigma\varphi$, we get

$$\textstyle\bigwedge_n \varphi(a_n) \ = \ \Pi\varphi(\bigwedge_n a_n) \ = \ f(\bigwedge_n a_n) \ \leq \ f(\bigvee_n b) \ = \ \Sigma\varphi(\bigvee_n b_n) \ = \ \bigvee_n \varphi(b_n). \quad \square$$

Corollary 163. *Let $V \supseteq L \xrightarrow{\varphi} E$ be a valuation system.*

(i) If φ is continuous, then φ is Π-extendible and Σ-extendible.
(ii) φ is continuous provided that φ is either Π_2-extendible or Σ_2-extendible.

Proof. Point (i) follows immediately from Lemma 162. Point (ii) is also a consequence of Lemma 162. Indeed, assume that φ is Π_2-extendible. We prove that φ is continuous. Note that $\Pi_2\varphi$ is order preserving and extends both $\Pi\varphi$ and $\Sigma\varphi$. Hence φ satisfies condition (i) of Lemma 162. Thus φ is continuous. $\square$

Lemma 164. *Let $V \supseteq L \xrightarrow{\varphi} E$ be a valuation system.*
Let K be a sublattice of L such that $\psi := \varphi|K$ is continuous.
Then φ is continuous under the following assumptions.

(i) Given a φ-convergent sequence $a_1 \geq a_2 \geq \cdots$ in L, we have

$$\textstyle\bigwedge_n \varphi(a_n) \;=\; \bigvee \{\; \Pi\psi(\ell)\colon\; \ell \in S \;\},$$

for some $S \subseteq \Pi K$ with $\ell \leq \bigwedge_n a_n$ for all $\ell \in S$.

(ii) Given a φ-convergent sequence $b_1 \leq b_2 \leq \cdots$ in L, we have

$$\textstyle\bigvee_n \varphi(b_n) \;=\; \bigvee \{\; \Sigma\psi(u)\colon\; u \in T \;\},$$

for some $T \subseteq \Sigma K$ with $\bigvee_n b_n \leq u$ for all $u \in T$.

Proof. Let φ-convergent sequences $a_1 \geq a_2 \geq \cdots$ and $b_1 \leq b_2 \leq \cdots$ from L with $\bigwedge_n a_n \leq \bigvee_n b_n$ be given. To prove that φ is continuous (see Definition 160), we must show that $\bigwedge_n \varphi(a_n) \leq \bigvee_n \varphi(b_n)$. Let $\ell \in S$ and $u \in T$ be given. Note that $\ell \leq \bigwedge_n a_n \leq \bigvee_n b_n \leq u$, so $\Pi\psi(\ell) \leq \Sigma\psi(u)$ since ψ is continuous.

Hence $\bigwedge_n \varphi(a_n) \leq \bigvee_n \varphi(b_n)$ by Assumptions (i) and (ii). □

7.2. **Benign E.**

Definition 165. Let E be an ordered Abelian group. We say E is **benign** provided that for every valuation system $V \supseteq L \xrightarrow{\varphi} E$, we have

$$\varphi \text{ is continuous} \qquad \Longrightarrow \qquad \varphi \text{ is extendible.}$$

Example 166. We will prove that $\mathbb{R}$ is benign (see Corollary 187).

Example 167. Let I be a set and let X_i be a benign ordered Abelian group for every $i \in I$. We leave it to the reader to verify that $\prod_{i\in I} X_i$ is benign.

8. Uniformity on E

To prove that $\mathbb{R}$ is benign (see Definition 165), we study ordered Abelian groups E which are endowed with a certain uniformity (such as $\mathbb{R}$) in Subsection 8.1. We prove that all such E are benign (see Theorem 186), in the following way.

Let $V \supseteq L \xrightarrow{\varphi} E$ be a valuation system. Recall that in order to prove that E is benign we must show that if φ is continuous, then φ is extendible (see Definition 165). We will first prove that if φ is continuous, then both $\Pi\varphi$ and $\Sigma\varphi$ are continuous (see Lemma 183). Then, by induction, we see that φ is both Π_n-extendible and Σ_n-extendible, and both $\Pi_n\varphi$ and $\Sigma_n\varphi$ are continuous, for every $n \in \mathbb{N}$. Hence φ is Π_ω-extendible. However, it is not clear a priori that $\Pi_\omega\varphi$ is continuous.

Secondly, we prove that if φ is Π_λ-extendible for some ordinal number λ, then $\Pi_\lambda\varphi$ is continuous. So by induction we see that φ is both Π_α-extendible and Σ_α-extendible, and both $\Pi_\alpha\varphi$ and $\Sigma_\alpha\varphi$ are continuous, for every ordinal number α (see Lemma 185). Hence φ is extendible (see Corollary 145).

To prove the second statement we use the fact that elements of $\Pi_\alpha L$ (or $\Sigma_\alpha L$) can be approximated from below by elements of ΠL, in some sense (see Lemma 181). We will express this by ΣL is *lower* $\Pi_\alpha\varphi$*-dense* in $\Pi_\alpha L$. We will formally introduce this notion, and study it, in Subsection 8.2.

8.1. Fitting Uniformity.

Definition 168. Let E be an ordered Abelian group. A **fitting uniformity** on E is a *countable* set Φ of binary relations on E with the following properties.

(i) We have $s\,\varepsilon\,s$ for all $\varepsilon \in \Phi$ and $s \in E$.

(ii) There is a map $\wedge\colon \Phi \times \Phi \to \Phi$ such that
$$s \;\; \varepsilon\wedge\delta \;\; t \quad\Longrightarrow\quad s\,\varepsilon\,t \text{ and } s\,\delta\,t \qquad (\varepsilon,\delta \in \Phi,\ s,t \in E).$$

(iii) There is a map ${}^{-}\!/2\colon \Phi \to \Phi$ such that
$$r \;\; {}^{\varepsilon}\!/2 \;\; s \;\; {}^{\varepsilon}\!/2 \;\; t \quad\Longrightarrow\quad r\,\varepsilon\,t \qquad (\varepsilon \in \Phi,\ r,s,t \in E).$$

(iv) Given $\varepsilon \in \Phi$ and $r,s,t \in E$ with $r \leq s \leq t$, we have
$$r\,\varepsilon\,t \quad\Longrightarrow\quad r\,\varepsilon\,s \text{ and } s\,\varepsilon\,t.$$

(v) Let $s,t \in E$ with $s \leq t$. Then $s = t$ provided that $s\,\varepsilon\,t$ for all $\varepsilon \in \Phi$.

(vi) If a sequence $s_1 \geq s_2 \geq \cdots$ from E has an infimum $s \in E$, then
$$\forall\varepsilon \in \Phi \;\; \exists N \in \mathbb{N} \;\; s\,\varepsilon\,s_N.$$

(vii) Let $s_1 \geq s_2 \geq \cdots$ be a sequence in E, and assume that for every $\varepsilon \in \Phi$ there is an $N \in \mathbb{N}$ such that $s_n \;\varepsilon\; s_{N_\varepsilon} \quad (n \geq N)$.
Then $s_1 \geq s_2 \geq \cdots$ has an infimum $\bigwedge_n s_n$.

(viii) Let $r,s,t \in E$ and $\varepsilon \in \Phi$ be given. Then $s\,\varepsilon\,t$ implies $r+s \;\varepsilon\; r+t$.

Example 169. We define a fitting uniformity on $\mathbb{R}$. For each natural number n, let ε_n be the binary relation on $\mathbb{R}$ given by
$$s \;\varepsilon_n\; t \quad\iff\quad s \leq t \quad\text{and}\quad t - s \leq 2^{-n}.$$
Then $\Phi := \{\varepsilon_n \colon n \in \mathbb{N}\}$ is a fitting uniformity on $\mathbb{R}$.
(Take $\varepsilon_n \wedge \varepsilon_m := \varepsilon_{n\vee m}$ and ${}^{\varepsilon_n}\!/2 := \varepsilon_{n+1}$ for all $n,m \in \mathbb{N}$.)

Remark 170. The fitting uniformities defined here are related to the *uniform spaces* (or more precisely, *quasi uniform spaces*) studied in topology, see [7].

However we do not involve uniform spaces, because the usual way of reasoning about them does not seem to fit well with property (iv). Moreover, we do not wish to assume that the reader is familiar with uniform spaces.

To the list of properties that a fitting uniformity must have (see Definition 168), we add some easy observations in Lemma 171. When we speak of "property (q)", where q is some Roman numeral, we refer to this list.

Lemma 171. *Let E be an ordered Abelian group with a fitting uniformity Φ.*

(ix) Let $s, t \in E$ and $\varepsilon \in \Phi$ be given. Then $s \,\varepsilon\, t$ implies $-t \;\varepsilon\; -s$.

(x) If a sequence $s_1 \leq s_2 \leq \cdots$ from E has an supremum $s \in E$, then

$$\forall \varepsilon \in \Phi \quad \exists N \in \mathbb{N} \quad s_N \,\varepsilon\, s.$$

(xi) Let $s_1 \leq s_2 \leq \cdots$ be a sequence in E, and assume that for every $\varepsilon \in \Phi$ there is an $N_\varepsilon \in \mathbb{N}$ such that $s_{N_\varepsilon} \;\varepsilon\; s_n \quad (n \geq N_\varepsilon)$.
Then $s_1 \leq s_2 \leq \cdots$ has a supremum $\bigvee_n s_n$.

Proof. (ix) Let $s, t \in E$ and $\varepsilon \in \Phi$ be given, and assume $s \,\varepsilon\, t$. We prove $-t \;\varepsilon\; -s$. Indeed, by property (viii), we have

$$-t \;=\; -(t+s) \,+\, s \quad \varepsilon \quad -(t+s) \,+\, t \;=\; -s.$$

(x) Let $s_1 \leq s_2 \leq \cdots$ be a sequence in E which has a supremum s in E. Let $\varepsilon \in \Phi$ be given. We need to find an $N \in \mathbb{N}$ such that $s_N \;\varepsilon\; s$.

Let us consider the sequence $-s_1 \geq -s_2 \geq \cdots$. By Lemma 206 the sequence $-s_1 \geq -s_2 \geq \cdots$ has an infimum, $-s$. By property (vi) we have $-s \;\varepsilon\; -s_N$ for some N. Then by property (ix), we get $s_N \;\varepsilon\; s$, and we are done.

(xi) Similar: apply property (vii) to the sequence $-s_1 \geq -s_2 \geq \cdots$. □

Notation 172. *Let E be an ordered Abelian group with a fitting uniformity Φ.*

(i) Given binary relations ε and δ on E (for instance, $\varepsilon, \delta \in \Phi$), we write

$$\varepsilon \;\leq\; \delta \qquad \iff \qquad \forall s,t \in E \;[\; s \;\varepsilon\; t \implies s \;\delta\; t \;].$$

(ii) Given binary relations ε and δ on E, let $\varepsilon + \delta$ be the relation on E given by

$$s \quad \varepsilon + \delta \quad t \qquad \iff \qquad \exists q \in E \;[\; s \;\varepsilon\; q \;\delta\; t \;].$$

Remark 173. The operation "+" defined in Notation 172(ii) is associative, but not in general commutative (contrary to the expectation the symbol "+" evokes).

The chosen notation does have advantages: property (iii) can be written as

$$\varepsilon/2 \,+\, \varepsilon/2 \;\leq\; \varepsilon \qquad (\varepsilon \in \Phi).$$

Lemma 174. *Let E be an ordered Abelian group with fitting uniformity Φ. Then E is R-complete (see Definition 44).*

Proof. Let $x_1 \leq x_2 \leq \cdots$ and $y_1 \leq y_2 \leq \cdots$ from E be given such that

$$x_{N+1} - x_N \;\leq\; y_{N+1} - y_N \qquad (N \in \mathbb{N}). \tag{61}$$

Assume $\bigvee_n y_n$ exists. To that E is R-complete, we must show that $\bigvee_n x_n$ exists.

Let $\varepsilon \in \Phi$ be given. By property (xi), we know that to prove $\bigvee_n x_n$ exists, it suffices to find $N \in \mathbb{N}$ such that $x_N \,\varepsilon\, x_n$ for all $n \geq N$.

By property (x), we know there is an N such that $y_N \,\varepsilon \bigvee_m y_m$. Let $n \geq N$ be given. We will prove that $x_N \,\varepsilon\, x_n$. We already know $y_N \,\varepsilon\, y_n$ by property (iv) because $y_N \leq y_n \leq \bigvee_m y_m$ and $y_N \,\varepsilon \bigvee_n y_m$. So $0 \;\varepsilon\; (y_n - y_N)$ by property (viii).

From Inequality (61) one can easily derive that

$$0 \;\leq\; x_n - x_N \;\leq\; y_n - y_N.$$

Since $0 \;\varepsilon\; (y_n - y_N)$ we get $0 \;\varepsilon\; (x_n - x_N)$ by property (iv).

Hence $x_N \,\varepsilon\, x_n$ by property (viii). So we are done. □

The following lemma will be useful.

Lemma 175. *Let E be an ordered Abelian group with fitting uniformity Φ.*

Let $S \subseteq E$ be non-empty and downwards directed*, i.e., for all $s_1, s_2 \in S$, there is an $s \in S$ such that $s \leq s_1$ and $s \leq s_2$.*

Let $t \in E$ be a lower bound of S which is close to S in the sense that

$$\forall \varepsilon \in \Phi \ \exists s \in S \quad t \, \varepsilon \, s. \tag{62}$$

Then t is the infimum of S.

Proof. To show that t is the infimum of S, we need to prove that $\ell \leq t$ for every lower bound ℓ of S. To do this, we take a detour.

Let $\varepsilon_1, \varepsilon_2, \cdots$ be an enumeration of Φ. Using Equation (62), and the fact that S is non-empty and directed, choose $s_1 \geq s_2 \geq \cdots$ in S such that

$$t \ \varepsilon_n \ s_n \qquad (n \in \mathbb{N}). \tag{63}$$

We will prove that $s_1 \geq s_2 \geq \cdots$ has an infimum s and that $s = t$.

This is sufficient to prove that t is the infimum of S. Indeed, if ℓ is a lower bound of S, then ℓ is a lower bound of $s_1 \geq s_2 \geq \cdots$, and so $\ell \leq \bigwedge_n s_n = t$.

We use property (vi) to show that $s_1 \geq s_2 \geq \cdots$ has an infimum. Given $\varepsilon \in \Phi$, we need to find an N such that $s_n \ \varepsilon \ s_N$ for all $n \geq N$. Pick k such that $\varepsilon = \varepsilon_k$ and take $N = k$. Let $n \geq N$ be given. Note that $t \leq s_n \leq s_N = s_k$ and $t \ \varepsilon_k \ s_k$ by Equation (63). So we have $s_n \ \varepsilon_k \ s_N$ by property (iv).

Hence property (vi) implies that $s_1 \geq s_2 \geq \cdots$ has an infimum, s. It remains to be shown that $s = t$. For this we use property (v).

Note that $t \leq s$ because $t \leq s_n$ for all n. Let $\varepsilon \in \Phi$ be given. We need to prove that $t \, \varepsilon \, s$. Choose k such that $\varepsilon = \varepsilon_k$. Then $t \leq s \leq s_k$ and $t \ \varepsilon_k \ s_k$ by Equation (63). So $t \ \varepsilon_k \ s$ by property (iv). Hence $s = t$ by property (v). □

8.2. **Denseness.** Throughout this subsection, E will be an ordered Abelian group endowed with a fitting uniformity Φ (see Definition 168).

Definition 176. Let $V \supseteq L \xrightarrow{\varphi} E$ be a valuation system. Let $S \subseteq T$ be subsets of L. We say S is **lower φ-dense** in T provided that the following condition holds.

For every $a \in T$ and $\varepsilon \in \Phi$ there is an $\ell \in S$ such that

$$\ell \leq a \quad \text{and} \quad \varphi(\ell) \, \varepsilon \, \varphi(a).$$

The notion of **upper φ-denseness** is defined similarly.

Example 177. Let $V \supseteq L \xrightarrow{\varphi} E$ be a Σ-extendible valuation system (see Def. 94). *Then L is lower $\Sigma\varphi$-dense in ΣL.*

Indeed, given $a \in \Sigma L$ and $\varepsilon \in \Phi$, we need to find an $\ell \in L$ such that $\varphi(\ell) \ \varepsilon \ \Sigma\varphi(a)$. Write $a = \bigvee_n a_n$ for some φ-convergent sequence $a_1 \leq a_2 \leq \cdots$. Then we have

$$\Sigma\varphi(a) \ = \ \textstyle\bigvee_n \varphi(a_n).$$

By property (x), there is an N such that $\varphi(a_N) \ \varepsilon \ \Sigma\varphi(a)$. So take $\ell = a_N$.

Lemma 178. *Let $V \supseteq L \xrightarrow{\varphi} E$ be a valuation system.*

(i) *Let $R \subseteq S \subseteq T$ be subsets of L. Suppose R is lower φ-dense in S, and suppose that S is lower φ-dense in T. Then R is lower φ-dense in T.*

(ii) *Let R be a subset of L, and let $\mathcal{S}$ be a family of subsets of L. If R is lower φ-dense in each $S \in \mathcal{S}$, then R is lower φ-dense in $\bigcup \mathcal{S}$.*

Proof. (i) Let $t \in T$ and $\varepsilon \in \Phi$ be given. To prove R is lower φ-dense in T, we need to find an $r \in R$ with $r \leq t$ and $\varphi(r) \ \varepsilon \ \varphi(t)$. This is easy. Choose an $s \in S$ such that $s \leq t$ and $\varphi(s) \ {}^{\varepsilon}\!/_2 \ \varphi(t)$ (see Definition 168(iii) for the meaning of "${}^{\varepsilon}\!/_2$"). Choose an $r \in R$ such that $r \leq s$ and $\varphi(r) \ {}^{\varepsilon}\!/_2 \ \varphi(s)$. Then $r \leq s$ and $\varphi(r) \ \varepsilon \ \varphi(t)$.

(ii) We leave this to the reader. □

The proof that E is benign hinges on the following lemma.

Lemma 179. *Let $V \supseteq L \xrightarrow{\varphi} E$ be a valuation system. Let K be a lower φ-dense sublattice of L. Then for every φ-convergent sequence $a_1 \geq a_2 \geq \cdots$ from L and $\varepsilon \in \Phi$ there is a φ-convergent sequence $\tilde{a}_1 \geq \tilde{a}_2 \geq \cdots$ from K with*

$$\tilde{a}_n \ \leq \ a_n \qquad \textit{and} \qquad \textstyle\bigwedge_n \varphi(\tilde{a}_n) \ \ \varepsilon \ \ \bigwedge_n \varphi(a_n). \tag{64}$$

Proof. Let $a_1 \geq a_2 \geq \cdots$ be a φ-convergent sequence in L, and let $\varepsilon \in \Phi$ be given. We need to find a φ-convergent sequence $\tilde{a}_1 \geq \tilde{a}_2 \geq \cdots$ in K which satisfies Condition (64). To this end, we seek a sequence $\tilde{a}_1 \geq \tilde{a}_2 \geq \cdots$ in K such that

$$\varphi(\tilde{a}_n)\ \eta\ \varphi(a_n) \qquad \text{and} \qquad \forall i \in \mathbb{N}\ \ \exists N \in \mathbb{N}\ \ \forall n \geq N\ [\ \ \varphi\tilde{a}_n\ \varepsilon_i\ \varphi\tilde{a}_N\ \ ], \tag{65}$$

where $\varepsilon_1, \varepsilon_2, \ldots$ is an enumeration of Φ, and $\eta \in \Phi$ with $2\eta \leq \varepsilon$ (see Notation 172).

Such a sequence $\tilde{a}_1 \geq \tilde{a}_2 \geq \cdots$ is φ-convergent (by property (vii)). We prove that $\tilde{a}_1 \geq \tilde{a}_2 \geq \cdots$ satisfies Condition (64). Indeed: We know $\bigwedge_n \varphi(\tilde{a}_n)$ exists. Hence, there is an $N \in \mathbb{N}$ with $\bigwedge_n \varphi(\tilde{a}_n)\ \eta\ \varphi(\tilde{a}_N)$ by property (vi). Then

$$\textstyle\bigwedge_n \varphi(\tilde{a}_n) \quad \eta \quad \varphi(\tilde{a}_N) \quad \eta \quad \varphi(a_N).$$

So we have $\bigwedge_n \varphi(\tilde{a}_n)\ \varepsilon\ \varphi(a_N)$. But $\bigwedge_n \varphi(\tilde{a}_n) \ \leq\ \bigwedge_n \varphi(a_n) \ \leq\ \varphi(a_N)$. Thus $\bigwedge_n \varphi(\tilde{a}_n)\ \varepsilon\ \bigwedge_n \varphi(a_n)$ by property (iv). Hence $\tilde{a}_1 \geq \tilde{a}_2 \geq \cdots$ satisfies Condition (64).

Finding a sequence $\tilde{a}_1 \geq \tilde{a}_2 \geq \cdots$ which satisfies Condition (65) is a subtle affair. Pick $\eta_1, \eta_2, \ldots$ and $\zeta_1, \zeta_2, \ldots$ from Φ (using properties (iii) and (ii)) such that

$$2\eta_i \leq \varepsilon_i, \qquad \eta_i \leq \eta, \qquad 2\zeta_i \leq \eta_i, \qquad 2\zeta_{i+1} \leq \zeta_i.$$

Then we have

$$\zeta_i + \cdots + \zeta_j \leq \eta_i \qquad (i, j \in \mathbb{N},\ i \leq j). \tag{66}$$

Pick $\ell_1, \ell_2, \ldots$ from K such that $\ell_n \leq a_n$ and $\varphi(\ell_n)\ \ \zeta_n\ \ \varphi(a_n)$ and define

$$\tilde{a}_{ij} \ = \ \ell_i \wedge \cdots \wedge \ell_j, \qquad\qquad \tilde{a}_n \ = \ \tilde{a}_{1n} = \ell_1 \wedge \cdots \wedge \ell_n,$$

where $i, j, n \in N$ with $i \leq j$. Then $\tilde{a}_{ij} \in K$ and $\tilde{a}_n \leq \ell_n \leq a_n$. We will prove that the sequence $\tilde{a}_1 \geq \tilde{a}_2 \geq \cdots$ satisfies Condition (65).

Note that for all $i, j \in \mathbb{N}$ with $i \leq j$, we have, by Lemma 24,

$$d_\varphi(\tilde{a}_{ij}, a_j) \ = \ d_\varphi(\ell_i \wedge \cdots \wedge \ell_j,\ a_i \wedge \cdots \wedge a_j) \ \leq \ d_\varphi(\ell_i, a_i) + \cdots + d_\varphi(\ell_j, a_j).$$

Since $\varphi(\ell_k)\ \zeta_k\ \varphi(a_k)$ for all k, the inequality above yields, using property (viii),

$$\varphi(\tilde{a}_{ij}) \quad \zeta_i + \cdots + \zeta_j \quad \varphi(a_j).$$

So because $\zeta_i + \cdots + \zeta_j \leq \eta_i$ (see Inequality (66)), we have

$$\varphi(\tilde{a}_{ij}) \quad \eta_i \quad \varphi(a_j). \tag{67}$$

In particular, we get $\varphi(\tilde{a}_n) \equiv \varphi(\tilde{a}_{1n})\ \eta_1\ \varphi(a_n)$. Hence $\varphi(\tilde{a}_n)\ \eta\ \varphi(a_n)$ as $\eta_1 \leq \eta$.

Let $i \in \mathbb{N}$ be given. To prove that $\tilde{a}_1 \geq \tilde{a}_2 \geq \cdots$ satisfies Condition (65), it remains to be shown that there is an $N \in \mathbb{N}$ such that

$$\varphi(\tilde{a}_n) \ \ \varepsilon_i \ \ \varphi(\tilde{a}_N) \qquad (n \geq N) \tag{68}$$

Using property (vi), determine $N \geq i$ such that $\bigwedge_n \varphi(a_n)\ \eta_i\ \varphi(a_N)$. We will show that Statement (68) holds. Let $n \geq N$ be given. Note that by Lemma 23,

$$d_\varphi(\tilde{a}_n, \tilde{a}_N) \ = \ d_\varphi(\,\tilde{a}_{i-1} \wedge \tilde{a}_{in},\ \tilde{a}_{i-1} \wedge \tilde{a}_{iN}\,) \ \leq \ d_\varphi(\tilde{a}_{in}, \tilde{a}_{iN}).$$

So to prove Statement (68), it suffices to show that $\varphi(\tilde{a}_{in})\ \varepsilon_i\ \varphi(\tilde{a}_{iN})$.

Recall that $\bigwedge_m \varphi(a_m)\ \ \eta_i\ \ \varphi(a_N)$ by choice of N. Then in particular, we get $\varphi(a_n)\ \eta_i\ \varphi(a_N)$ by property (iv). Further, $\varphi(\tilde{a}_{in})\ \eta_i\ \varphi(a_n)$ by Inequality (67). So

$$\varphi(\tilde{a}_{in})\ \eta_i\ \varphi(a_n)\ \eta_i\ \varphi(a_N).$$

Hence $\varphi(\tilde{a}_n)\ \varepsilon_i\ \varphi(a_N)$, because $2\eta_i \leq \varepsilon_i$. Note that $\varphi(\tilde{a}_{in}) \leq \varphi(\tilde{a}_{iN}) \leq \varphi(a_N)$.

So by property (iv), we get $\varphi(\tilde{a}_{in})\ \varepsilon_i\ \varphi(\tilde{a}_{iN})$. □

Corollary 180. *Let $V \supseteq L \xrightarrow{\varphi} E$ be a Π-extendible valuation system. Let K be a sublattice of L. Then*

$$K \text{ is lower dense in } L \quad\Longrightarrow\quad \Pi K \text{ is lower dense in } \Pi L.$$

Proof. Follows immediately from Lemma 179. □

Lemma 181. *Let $V \supseteq L \xrightarrow{\varphi} E$ be a valuation system which is both Σ-extendible and Π-extendible. Then for every ordinal number α:*

(i) If φ is Π_α-extendible, then

$$\Pi L \text{ is upper dense in } \Pi_\alpha L, \quad \text{and} \qquad \Sigma L \text{ is lower dense in } \Pi_\alpha L.$$

(ii) If φ is Σ_α-extendible, then

$$\Pi L \text{ is upper dense in } \Sigma_\alpha L, \quad \text{and} \qquad \Sigma L \text{ is lower dense in } \Sigma_\alpha L.$$

Proof. We use induction on α.

For $\alpha = 0$, Statements (i) and (ii) are trivial.

Let α be an ordinal number such that Statement (i) holds for α in order to prove that Statement (ii) holds for $\alpha+1$. Suppose φ is $\Sigma_{\alpha+1}$-extendible. We need to prove that ΠL is upper dense in $\Sigma_{\alpha+1}L$ and that ΣL is lower dense in $\Sigma_{\alpha+1}L$.

Note that φ is Π_α-extendible, because φ is $\Sigma_{\alpha+1}$-extendible.

By Statement (i) for α, we know that ΠL is lower dense in $\Pi_\alpha L$. Further, $\Pi_\alpha L$ is lower dense in $\Sigma(\Pi_\alpha L) = \Sigma_{\alpha+1}L$ by Example 177. So we see that ΠL is lower dense in $\Sigma_{\alpha+1}L$ by Lemma 178(i).

By Statement (i) for α, we know that ΣL is upper dense in $\Pi_\alpha L$. So by the dual of Corollary 180, we have $\Sigma L = \Sigma(\Sigma L)$ is upper dense in $\Sigma(\Pi_\alpha L) = \Sigma_{\alpha+1}L$.

Hence, Statement (ii) holds for $\alpha+1$ (if Statement (i) holds for α).

Similarly, if Statement (ii) holds for α, then Statement (i) holds for $\alpha+1$.

Let λ be a limit ordinal such that Statement (i) holds for all $\alpha < \lambda$. We prove that Statement (i) holds for λ. Suppose that φ is Π_λ-extendible. We need to prove that ΠL is upper dense in $\Pi_\lambda L$ and ΣL is lower dense in $\Pi_\lambda L$.

We know that φ is Π_α-extendible for all $\alpha < \lambda$.

As Statement (i) holds for all $\alpha < \lambda$, we see that ΠL is upper dense in all $\Pi_\alpha L$. So by Lemma 178(ii), ΠL is upper dense in $\Pi_\lambda L = \bigcup_{\alpha<\lambda} \Pi_\alpha L$.

Similarly, ΣL is lower dense in $\Sigma_\lambda L = \bigcup_{\alpha<\lambda} \Sigma_\alpha L$. □

Corollary 182. *Let $V \supseteq L \xrightarrow{\varphi} E$ be a valuation system. Let K be a lower φ-dense sublattice of L and assume that $\psi := \varphi|K$ is Π-extendible. Let $a_1 \geq a_2 \geq \cdots$ be a φ-convergent sequence in L. Then*

$$\textstyle\bigwedge_n \varphi(a_n) \;=\; \bigvee \left\{\, \Pi\psi(\ell)\colon\; \ell \in S \,\right\}, \tag{69}$$

where $S \;:=\; \left\{\, \bigwedge_n \tilde{a}_n\colon\; \psi\text{-convergent } \tilde{a}_1 \geq \tilde{a}_2 \geq \cdots \text{ with } \tilde{a}_n \leq a_n \text{ for all } n \,\right\}.$

Proof. To prove Statement (69), we apply the dual of Lemma 175. We need to verify that $\Pi\psi(S) := \{\, \Pi\psi(\ell)\colon \ell \in S \,\}$ is upwards directed, that $\bigwedge_n \varphi(a_n)$ is a lower bound of $\Pi\psi(S)$, and that

$$\forall \varepsilon \in \Phi \;\; \exists \ell \in S \quad \Pi\psi(\ell) \;\; \varepsilon \;\; \textstyle\bigwedge_n \varphi(a_n). \tag{70}$$

To begin, note that Statement (70) follows immediately from Lemma 179.

Let ψ-convergent $\tilde{a}_1 \geq \tilde{a}_2 \geq \cdots$ with $\tilde{a}_n \leq a_n$ for all n be given. Then we have $\psi(\tilde{a}_n) = \varphi(\tilde{a}_n) \leq \varphi(a_n)$ for all n, so $\Pi\psi(\bigwedge_n \tilde{a}_n) = \bigwedge_n \psi(\tilde{a}_n) \leq \bigwedge_n \varphi(a_n)$. Hence $\bigwedge_n \varphi(a_n)$ is a lower bound of $\Pi\psi(S)$.

To prove that $\Pi\psi(S)$ is upwards directed, it suffices to show that S is upwards directed (as $\Pi\psi$ is order preserving). Let ψ-convergent sequences $\tilde{a}_1 \geq \tilde{a}_2 \geq \cdots$ and $\tilde{a}'_1 \geq \tilde{a}'_2 \geq \cdots$ with $\tilde{a}_n \leq a_n$ and $\tilde{a}'_n \leq a_n$ be given. Then

$$\tilde{a}_1 \vee \tilde{a}'_1 \;\leq\; \tilde{a}_1 \vee \tilde{a}'_2 \;\leq\; \cdots$$

is again a ψ-convergent sequence by Proposition 48. Further $\tilde{a}_n \vee \tilde{a}'_n \leq a_n$ for all n. Hence $\bigwedge_n \tilde{a}_n \vee \tilde{a}'_n \in S$. But also $\bigwedge_n \tilde{a}_n \leq \bigwedge_n \tilde{a}_n \vee \tilde{a}'_n$ and $\bigwedge_n \tilde{a}'_n \leq \bigwedge_n \tilde{a}_n \vee \tilde{a}'_n$. So we see that S is upwards directed. □

Lemma 183. *Let $V \supseteq L \xrightarrow{\varphi} E$ be a valuation system.*
Assume φ is continuous. Then $\Pi\varphi$ is continuous.

Proof. Note that L is an upper $\Pi\varphi$-dense sublattice of ΠL (see Example 177). We apply Lemma 164 to prove that φ is continuous. We must verify that Conditions (i) and (ii) of Lemma 164 hold.

(i) Let $a_1 \geq a_2 \geq \cdots$ be a $\Pi\varphi$-convergent sequence in ΠL. We need to find $S \subseteq \Pi L$ such that $\bigwedge_n \varphi(a_n) = \bigvee S$ and $\ell \leq \bigwedge_n a_n$ for all $\ell \in S$. By Lemma 98, we know that $\Pi\varphi$ is Π-complete. Hence $\bigwedge_n a_n \in \Pi L$. So simply take $S = \{\bigwedge_n a_n\}$.

(ii) Follows immediately from Corollary 182. □

Lemma 184. *Let $V \supseteq L \xrightarrow{\varphi} E$ be a valuation system.*
Let K be a sublattice of L. Then φ is continuous provided that:

(i) The restriction $\varphi|K$ of φ to K is continuous.
(ii) K is lower and upper φ-dense in L.

Proof. This follows from Lemma 164. Indeed, condition (i) holds by Corollary 182, and condition (ii) holds by the dual of Corollary 182. □

Lemma 185. *Let $V \supseteq L \xrightarrow{\varphi} E$ be a continuous valuation system, and α an ordinal. Then φ is both Π_α-extendible and Σ_α-extendible, and $\Pi_\alpha\varphi$ and $\Sigma_\alpha\varphi$ are continuous.*

Proof. With induction on α.

For $\alpha = 0$, the statement is trivial.

Let α be an ordinal number and assume that φ is Π_α-extendible and $\Pi_\alpha\varphi$ is continuous. We prove that φ is $\Sigma_{\alpha+1}$-extendible and $\Sigma_{\alpha+1}\varphi$ is continuous. Indeed, since $\Pi_\alpha\varphi$ is continuous, $\Pi_\alpha\varphi$ is Σ-extendible and so φ is $\Sigma_{\alpha+1}$-extendible. Finally, $\Sigma(\Pi_\alpha\varphi) = \Sigma_{\alpha+1}\varphi$ is continuous by the dual of Lemma 183.

Similarly, if φ is Σ_α-extendible and $\Sigma_\alpha\varphi$ is continuous, then φ is $\Pi_{\alpha+1}$-extendible and $\Pi_{\alpha+1}\varphi$ is continuous.

Let λ be a limit ordinal such that for each $\alpha < \lambda$, we have that φ is Π_α-extendible and $\Pi_\alpha\varphi$ is continuous. Note that φ is Π_λ-extendible. We prove that $\Pi_\lambda\varphi$ is continuous. For this, we use Lemma 184. Consider $\psi := \Pi_2\varphi$. By assumption, ψ is continuous. We know that $\Pi_\lambda\varphi$ extends ψ, and that ψ extends both $\Pi\varphi$ and $\Sigma\varphi$. Since ΠL is lower dense in $\Pi_\alpha L$, and ΣL is upper dense in $\Pi_\alpha L$ (by Lemma 181), we get that $K := \Pi_2 L$ is both upper and lower dense in $\Pi_\alpha L$. So by Lemma 184, we see that $\Pi_\lambda\varphi$ is continuous. (Of course, the argument is also valid for other choices for ψ, such as $\Sigma_3\varphi$ and $\Pi_{42}\varphi$.) □

Theorem 186. *Let E be an ordered Abelian group.*
If E has a fitting uniformity, then E is benign.

Proof. Let $V \supseteq L \xrightarrow{\varphi} E$ be a continuous valuation system. To prove that E is benign, we must show that φ is extendible (see Definition 165). It suffices to prove that φ is $\Pi_{\aleph_1}$-extendible by Corollary 145. Now apply Lemma 185. □

Corollary 187. *The ordered Abelian group $\mathbb{R}$ is benign.*

9. Fubini's Theorem

In this section we study Fubini's Theorem. We have not found a satisfactory generalisation of this theorem to the setting of valuations. However, we will see that it is possible to split the proof of Fubini's Theorem into two parts, so that the first part (Subsection 9.1) is algebraic in nature and specific to the setting of step functions, and the second part (Subsection 9.2) is more analytic in nature and a consequence of a general extension theorem for valuations (see Theorem 199).

9.1. **Algebraic Part.** Let us first formulate Fubini's Theorem. This takes time.

Let X be a set, let $\mathcal{A}_X$ be a ring of subsets of X, and let

$$\mu_X\colon \mathcal{A}_X \to \mathbb{R}$$

be a positive and additve map (see Example 9).

Similarly, let Y be a set, let $\mathcal{A}_Y$ be a ring of subsets of Y, and let

$$\mu_Y\colon \mathcal{A}_Y \to \mathbb{R}$$

be a positive and additve map.

Now, let $\mathcal{A}_{X\times Y}$ be the ring of subsets of $X \times Y$ generated by the subsets

$$\{\ A\times B\colon\quad A\in\mathcal{A}_X,\quad B\in\mathcal{A}_Y\ \ \}$$

Let $\mu_{X\times Y}\colon \mathcal{A}_{X\times Y}\to\mathbb{R}$ be the unique positive and additive map such that

$$\mu_{X\times Y}(\,A\times B\,)\ =\ \mu_X(A)\,\cdot\,\mu_Y(B)$$

for all $A\in\mathcal{A}_X$ and $B\in\mathcal{A}_Y$. Such $\mu_{X\times Y}$ exists, as the reader can verify.

Let F_X be the set of all $\mathcal{A}_X$-stepfunctions, i.e., functions of the form

$$\textstyle\sum_{n=1}^N\ \lambda_n\cdot\mathbf{1}_{A_n},$$

where $A_1,\dots,A_N\in\mathcal{A}_X$ and $\lambda_i\in\mathbb{R}$. As the reader may verify, the expression

$$\varphi_X(\textstyle\sum_{n=1}^N\ \lambda_n\cdot\mathbf{1}_{A_n}\,)\ =\ \textstyle\sum_{n=1}^N\ \lambda_n\cdot\mu_X(A_n)$$

determines a unique positive and linear map $\varphi_X\colon F_X\to\mathbb{R}$.

Similarly, we get a map $\varphi_Y\colon F_Y\to\mathbb{R}$, and a map $\varphi_{X\times Y}\colon F_{X\times Y}\to\mathbb{R}$.

One can verify that any $f\in F_{X\times Y}$ is of the form

$$\textstyle\sum_{n=1}^N\ \lambda_n\cdot\mathbf{1}_{A_n\times B_n},$$

where $A_1,\dots,A_N\in\mathcal{A}_X$, and $B_1,\dots,B_N\in\mathcal{A}_Y$, and $\lambda_n\in\mathbb{R}$.

So it is not hard to verify that the equality

$$\mathcal{F}_X(\ \textstyle\sum_{n=1}^N\lambda_n\cdot\mathbf{1}_{A_n\times B_n}\ )\quad=\quad\textstyle\sum_{n=1}^N\lambda_n\cdot\mu_X(A_n)\cdot\mathbf{1}_{B_n}$$

gives us positive and linear map $\mathcal{F}_X\colon F_{X\times Y}\to F_Y$.

Let $f\in F_{X\times Y}$ be given. For each $y\in Y$, define $f^y\in F_X$ by, for all $x\in X$,

$$f^y(x)\ =\ f(x,y)$$

One can easily verify that we have, for all $y\in Y$,

$$\mathcal{F}_X(f)(y)\ =\ \varphi_X(f^y).$$

So, informally, $\mathcal{F}_X(f)=\int f(x,y)\,dx$.

Since φ_X is linear one quickly sees that $\varphi_Y\circ\mathcal{F}_X=\varphi_{X\times Y}$. Informally,

$$\int\!\!\int f(x,y)\ dx\,dy\ =\ \int f\qquad(f\in F_{X\times Y}).$$

This is Fubini's Theorem for stepfunctions, $F_{X\times Y}$.

Of course, we want to prove Fubini's Theorem for the extension $\overline{F}_{X\times Y}$.

So let us assume φ_X, φ_Y, and $\varphi_{X\times Y}$ are extendible (see Definition 141).

Alternatively, we can assume that φ_Y and φ_Y are continuous (see Definition 160); we leave it to the reader to verify that then $\varphi_{X\times Y}$ is continuous (which is not too easy), and so $\varphi_{X\times Y}$ is extendible, since $\mathbb{R}$ is benign (see Definition 165).

Note that it is not possible to find an $\overline{\mathcal{F}}_Y : \overline{F}_{X\times Y} \to \overline{F}_Y$ such that, for all $y \in Y$,

$$\overline{\mathcal{F}}_Y(f)(y) \;=\; \overline{\varphi}_X(f^y).$$

So to formulate Fubini's Theorem for $\overline{F}_{X\times Y}$ we need a slightly different approach than the one we used for the stepfunctions.

Consider the space $E_Y := \overline{F_Y}/{\approx}$ (see Proposition 29). We leave it to the reader to verify that E_Y can be endowed with the structure of an ordered Abelian group, and a fitting uniformity (see Definition 168) such that the map $F_{X\times Y} \to E_Y$ given by $f \mapsto \mathcal{F}_X(f)/\approx$ is a group homomorphism.

We can now formulate Fubini's Theorem as follows.

The valuation

$$\mathcal{F}_X : F_{X\times Y} \to \overline{F_Y}/{\approx}$$

is extendible, and $\mathrm{dom}(\overline{\varphi}_{X\times Y}) \subseteq \mathrm{dom}(\overline{\mathcal{F}}_X)$, and (71)

$$\overline{\varphi}_{X\times Y}(f) \;=\; (\overline{\varphi}/{\approx} \,\circ\, \overline{\mathcal{F}_X}\,)(f)$$

for all $f \in \overline{F}_{X\times Y}$.

Of course, to be true to the usual formulation of Fubini's Theorem we would need to prove that that $\overline{\mathcal{F}}_X(f)(y) = \overline{\varphi}_X(f^y)$ for *almost* all $y \in Y$. We will not do this.

9.2. **Extension of Operations.** Let $V \supseteq L \xrightarrow{\varphi} E$ be a valuation system. In Section 6 we saw that the completion $\overline{L}$ of φ is *closed* under various operations. It is also possible to *extend* operations to $\overline{L}$, which are (initially only) defined on L. The aim of this subsection is to prove Theorem 199 which is an example of this principle in case that E has a fitting uniformity Φ (see Definition 168).

It should be noted that from the methods found in the proof of Theorem 199 one can easily obtain a stronger version of this theorem. More interestingly, the patterns in the proof strongly suggest that we should make a study of the uniform structure on L given by the relations $\overline{\varepsilon}$ (where $\varepsilon \in \Phi$) defined by

$$a \;\overline{\varepsilon}\; b \qquad\Longleftrightarrow\qquad 0 \;\varepsilon\; d_\varphi(a,b) \qquad\qquad (a,b \in L).$$

However, we have refrained from proving a stronger version of the theorem and introducing yet another notion of uniform structure. Indeed, we have not found a clear favorite among the several approaches to the strengthening of the theorem and the axiomatisation of the uniform structure on L. Accordingly, we introduce few new notions, and the proofs in this subsection are sometimes ad hoc.

One new notion we do present is that of *weak φ-convergence* (see Definition 188). As the name suggests, φ-convergence (see Definition 55) implies weak φ-convergence (see Lemma 189), but the reverse implication does not hold (see Example 190). Nevertheless, any weakly φ-convergent sequence has a φ-convergent subsequence (see Proposition 192).

Due to this all the notions of φ-convergent and weakly φ-convergent can be used somewhat interchangeably. The main merit of "weak φ-convergent" is that some useful statements concerning it (see Lemma 193 and Lemma 194) can be easily proven, while it is not clear if the same statement (or a variant) holds for "φ-convergent".

The main application of Theorem 199 is the proof of Fubini's Theorem 200. Let us start the work towards a proof.

Definition 188. Let E be an ordered Abelian group.
Let Φ be a fitting uniformity on E.

Let L be a lattice, and let $\varphi\colon L \to E$ be a valuation.
Let $a \in L$ and let $a_1, a_2, \ldots$ be a sequence in L.
We say $a_1, a_2, \ldots$ **weakly φ-converges** to a if

$$\forall \varepsilon \in \Phi \quad \exists N \quad \forall n \geq N \quad [\ \ 0 \quad \varepsilon \quad d_\varphi(a_n, a)\ \ ].$$

Lemma 189. *Let E be an ordered Abelian group.
Let Φ be a fitting uniformity on E.
Let L be a lattice, and let $\varphi\colon L \to E$ be a complete valuation.
Let $a \in L$ and let $a_1, a_2, \ldots$ be a sequence in L. We have:*

$$a_1, a_2, \ldots \ \varphi\text{-converges to } a \quad \Longrightarrow \quad a_1, a_2, \ldots \ \text{weakly } \varphi\text{-converges to } a.$$

Proof. Let $\varepsilon \in \Phi$ be given. To prove that $a_1, a_2, \ldots$ weakly φ-converges to a we must find an $N \in \mathbb{N}$ such that $0\ \varepsilon\ d_\varphi(a_n, a)$ for all $n \geq N$.

To find such N takes some preparation, so bear with us.

Since $a_1, a_2, \ldots$ φ-converges to a, i.e., $a_1, a, a_2, a, \ldots$ is φ-convergent, we know that $a_1, a, a_2, a, \ldots$ is upper φ-convergent. That is, we know the following exists.

$$u \ := \ \textstyle\bigwedge_N \bigvee_{n \geq N} \ \varphi(a \vee a_N \vee \cdots \vee a_n) \tag{72}$$

In particular, we see that for each $N \in \mathbb{N}$, the sequence

$$a \vee a_N \ \leq \ a \vee a_N \vee a_{N+1} \ \leq \ \cdots$$

is φ-convergent (in the sense of Definition 34). Since φ is complete, we see that $\overline{a}_N := \bigvee_{n \geq N} a \vee a_n$ exists in L and that $\varphi(\overline{a}_N) = \bigvee_{n \geq N} \varphi(a \vee a_N \vee \cdots \vee a_n)$. Now, note that we can phrase Statement (72) as $u = \bigwedge_N \varphi(\overline{a}_N)$.

Since $a_1, a_2, \ldots$ φ-converges to a, we know that the sequence $a_1, a, a_2, a, \ldots$ is lower φ-convergent. That is, the following exists.

$$\ell \ := \ \textstyle\bigvee_N \bigwedge_{n \geq N} \ \varphi(a \wedge a_N \wedge \cdots \wedge a_n) \tag{73}$$

In particular we see that for each $N \in \mathbb{N}$ the sequence

$$a \wedge a_N \ \geq \ a \wedge a_N \wedge a_{N+1} \ \geq \ \cdots$$

is φ-convergent. As before, $\underline{a}_N := \bigwedge_{n \geq N} a \wedge a_n$ exists, and $\ell = \bigvee_N \varphi(\underline{a}_N)$.

Now, note that for each $N \in \mathbb{N}$ and $n \geq N$ we have

$$\underline{a}_N \ \leq \ a \wedge a_n \ \leq \ a \vee a_n \ \leq \ \overline{a}_N.$$

In particular, we have the following inequalities.

$$\varphi(\underline{a}_N) \ \leq \ \varphi(a \wedge a_n) \ \leq \ \varphi(a \vee a_n) \ \leq \ \varphi(\overline{a}_N). \tag{74}$$

Recall that we want to prove (for some N) that $0\ \varepsilon\ d_\varphi(a, a_n)$. That is, we must show that $\varphi(a \wedge a_n)\ \varepsilon\ \varphi(a \vee a_n)$ (see Definition 168(viii)). To prove this it suffices to show that $\varphi(\underline{a}_N)\ \varepsilon\ \varphi(\overline{a}_N)$ as we can see from Statement (74) (see Definition 168(iv)).

So to complete the proof of this lemma, we need to find an $N \in \mathbb{N}$ with

$$\varphi(\underline{a}_N) \ \ \varepsilon \ \ \varphi(\overline{a}_N).$$

Since $a_1, a, a_2, a, \ldots$ is φ-convergent, we know that $u = \ell$. Now, recall that we have $u = \bigwedge_N \varphi(\overline{a}_N)$ and $\ell = \bigwedge_N \varphi(\overline{a}_N)$. Determine an N with

$$\varphi(\underline{a}_N) \ \ {}^{\varepsilon}\!/_{2} \ \ \ell \qquad \text{and} \qquad u \ \ {}^{\varepsilon}\!/_{2} \ \ \varphi(\overline{a}_N)$$

using Definition 168(vi) and Lemma 171(x). Hence we see that $\varphi(\underline{a}_N)\ \varepsilon\ \varphi(\overline{a}_N)$. $\square$

Example 190. Let φ be a complete valuation. We know that φ-convergence implies weak φ-convergence (see Lemma 188). The reverse implication does not hold. Indeed, consider the Lebesgue integral $\varphi_{\mathcal{L}}\colon F_{\mathcal{L}} \to \mathbb{R}$ and the sequence $f_1, f_2, \ldots$ of functions on $\mathbb{R}$ given by $f_n = \frac{1}{n} \cdot \mathbf{1}_{[n,n+1]}$. Note that

$$d_{\varphi_{\mathcal{L}}}(f_n, \mathbf{0}) \ = \ \varphi_{\mathcal{L}}(|f_n|) \ = \ \tfrac{1}{n}.$$

So we see that $f_1, f_2, \ldots$ weakly $\varphi_\mathcal{L}$-converges to $\mathbf{0}$.

However, we prove that the sequence $f_1, f_2, \ldots$ does not $\varphi_\mathcal{L}$-converge to $\mathbf{0}$. Indeed, assume (towards a contradiction) that $f_1, f_2, \ldots$ does $\varphi_\mathcal{L}$-converge to $\mathbf{0}$. Then $f_1, f_2, \ldots$ is $\varphi_\mathcal{L}$-convergent. So in particular $f_1, f_2, \ldots$ is upper $\varphi_\mathcal{L}$-convergent (see Definition 55). So we know that the following exists.

$$\varphi_\mathcal{L}\text{-}\overline{\lim}_n f_n \;=\; \textstyle\bigwedge_N \bigvee_{n\geq N} \varphi_\mathcal{L}(f_N \vee \cdots \vee f_n)$$

Now, note that any $N \in \mathbb{N}$ and $n \geq N$ we have $f_N \vee \cdots \vee f_n = f_N + \cdots + f_n$, so

$$\varphi_\mathcal{L}(f_N \vee \cdots \vee f_n) \;=\; \varphi_\mathcal{L}(f_N) + \cdots + \varphi_\mathcal{L}(f_n) \;=\; \tfrac{1}{N} + \cdots + \tfrac{1}{n}.$$

So we see that $\sum_n \frac{1}{n} = \bigvee_n \varphi_\mathcal{L}(f_1 \vee \cdots \vee f_n)$, which is absurd. Hence $f_1, f_2, \ldots$ does not $\varphi_\mathcal{L}$-converge to $\mathbf{0}$.

Let φ be a complete valuation. If $a_1, a_2, \ldots$ weakly φ-converges to a, then $a_1, a_2, \ldots$ might not φ-converge to a (as we saw in Example 190). However, there is always a subsequence of $a_1, a_2, \ldots$ which does φ-converge to a (see Proposition 192). To prove this, we need a lemma.

Lemma 191. *Let E be an ordered Abelian group.*
Let Φ be a fitting uniformity on E.
Let L be a lattice, and let $\varphi\colon L \to E$ be a valuation.
Let $a \in L$ and let $a_1, a_2, \ldots$ be a sequence in L that weakly φ-converges to a.
Assume that $\sum_n d_\varphi(a, a_n) \;:=\; \bigvee_N \sum_{n=1}^N d_\varphi(a, a_n)$ exists.
Then $a_1, a_2, \ldots$ φ-converges to a.

Proof. To prove that $a_1, a_2, \ldots$ φ-converges to a, we must show that $a_1, a, a_2, a, \ldots$ is φ-convergent (see Definition 67). For this, we must show that the following exist,

$$\begin{aligned} u \;&:=\; \textstyle\bigwedge_N \bigvee_{n\geq N} \varphi(a \vee a_N \vee \cdots \vee a_n) \\ \ell \;&:=\; \textstyle\bigvee_N \bigwedge_{n\geq N} \varphi(a \wedge a_N \wedge \cdots \wedge a_n), \end{aligned}$$

and we must prove that $\ell = u$.

Let $N \in \mathbb{N}$ be given. We prove that $\bigvee_{n\geq N} \varphi(a \vee a_N \vee \cdots \vee a_n)$ exists. Let us write $a'_n := a \vee a_N \vee \cdots \vee a_n$ for brevity. To prove that $\bigvee_{n\geq N} \varphi(a'_n)$ exists, we want to use the fact that E is R-complete (see Proposition 174). So the task at hand is to study given $n \geq N$ the value $\varphi(a'_{n+1}) - \varphi(a'_n)$. Note that

$$\varphi(a'_{n+1}) - \varphi(a'_n) \;=\; d_\varphi(\,a'_{n+1},\, a'_n\,) \;=\; d_\varphi(\,a'_n \vee a_{n+1},\; a'_n \vee a\,), \tag{75}$$

since $a'_{n+1} = a'_n \vee a_{n+1}$ and $a'_n = a'_n \vee a$ (as $a \leq a'_n$). By Lemma 23 we have

$$d_\varphi(\,a'_n \vee a_{n+1},\; a'_n \vee a\,) \;\leq\; d_\varphi(\,a_{n+1},\, a\,). \tag{76}$$

So if we combine Statement (75) and Statement (76) we get

$$\varphi(a'_{n+1}) - \varphi(a'_n) \;\leq\; d_\varphi(a_{n+1}, a). \tag{77}$$

Recall that we have assumed that $\sum_n d_\varphi(a_n, a)$ exists. From this, Statement (77), and the fact that E is R-complete it follows that $\bigvee_{n\geq N} \varphi(a'_n)$ exists.

We prove that $u := \bigwedge_N \bigvee_{n\geq N} \varphi(a \vee a_N \vee \cdots \vee a_n)$ exists. Again we use the fact that E is R-complete: it is sufficient to prove that $\xi_N - \xi_{N+1} \leq d_\varphi(a, a_N)$ where

$$\xi_N \;:=\; \textstyle\bigvee_{n\geq N} \varphi(a \vee a_N \vee \cdots \vee a_n).$$

Let $n \in \mathbb{N}$ be given. It is useful to begin by considering the value $\varphi(a''_N) - \varphi(a''_{N+1})$ where $a''_N := a \vee a_N \vee \cdots \vee a_n$ for all $N \leq n$. We obtain

$$\varphi(a''_N) - \varphi(a''_{N+1}) \;\leq\; d_\varphi(a, a_N)$$

using a similar reasoning as before. Written differently, we have

$$\varphi(a \vee a_N \vee \cdots \vee a_n) \;\leq\; d_\varphi(a, a_N) \;+\; \varphi(a \vee a_{N+1} \vee \cdots \vee a_n)$$

for all $N \in \mathbb{N}$ and $n \geq N$. This implies

$$\begin{aligned} \bigvee_{n \geq N} \varphi(a \vee a_N \vee \cdots \vee a_n) &\leq d_\varphi(a, a_N) + \varphi(a \vee a_{N+1} \vee \cdots \vee a_n) \\ &\leq d_\varphi(a, a_N) + \bigvee_{n \geq N+1} \varphi(a \vee a_{N+1} \vee \cdots \vee a_n). \end{aligned}$$

Or in other words, $\xi_N \leq d_\varphi(a, a_N) + \xi_{N+1}$. Hence we have proven:

$$u := \bigwedge_N \bigvee_{n \geq N} \varphi(a \vee a_N \vee \cdots \vee a_n) \quad \text{exists.}$$

Of course, the above argument can be adapted to yield:

$$\ell := \bigvee_N \bigwedge_{n \geq N} \varphi(a \wedge a_N \wedge \cdots \wedge a_n) \quad \text{exists.}$$

It remains to be shown that $\ell = u$. Let $\varepsilon \in \Phi$ be given. Since reader can easily verify that $\ell \leq u$, to prove that $\ell = u$, it suffices to show that $\ell \, \varepsilon \, u$ (see Definition 168(v)). Let $N \in \mathbb{N}$ be given. Note that we have the following inequalities.

$$\bigwedge_{n \geq N} \varphi(a \wedge a_N \wedge \cdots \wedge a_n) \leq \ell \leq u \leq \bigvee_{n \geq N} \varphi(a \vee a_N \vee \cdots \vee a_n).$$

So to prove $\ell \, \varepsilon \, u$, it suffices to show that for some N (see Definition 168(iv)),

$$\bigwedge_{n \geq N} \varphi(a \wedge a_N \wedge \cdots \wedge a_n) \quad \varepsilon \quad \bigvee_{n \geq N} \varphi(a \vee a_N \vee \cdots \vee a_n). \tag{78}$$

Since $\sum_n d_\varphi(a, a_n)$ exists, we can find an $N \in \mathbb{N}$ such that (see Lemma 171(x))

$$0 \quad \varepsilon/4 \quad d_\varphi(a, a_N) + \cdots + d_\varphi(a, a_n) \qquad (n \geq N). \tag{79}$$

We will prove that Statement (78) holds for this N. Since $\bigwedge_{n \geq N} \varphi(a \wedge a_N \wedge \cdots \wedge a_n)$ and $\bigvee_{n \geq N} \varphi(a \vee a_N \vee \cdots \vee a_n)$ exist, we can find $n \geq N$ such that

$$\begin{aligned} \bigwedge_{n \geq N} \varphi(a \wedge a_N \wedge \cdots \wedge a_n) \quad &\varepsilon/4 \quad \varphi(a \wedge a_N \wedge \cdots \wedge a_n) \\ \varphi(a \vee a_N \vee \cdots \vee a_n) \quad &\varepsilon/4 \quad \bigvee_{n \geq N} \varphi(a \vee a_N \vee \cdots \vee a_n). \end{aligned}$$

So to prove that Statement (78) it suffices to show that

$$\varphi(a \wedge a_N \wedge \cdots \wedge a_n) \quad \varepsilon/2 \quad \varphi(a \vee a_N \vee \cdots \vee a_n). \tag{80}$$

Note that we have the following inequalities.

$$\varphi(a \wedge a_N \wedge \cdots \wedge a_n) \leq \varphi(a) \leq \varphi(a \vee a_N \vee \cdots \vee a_n).$$

So to prove that Statement (80) holds, we will show that

$$\varphi(a \wedge a_N \wedge \cdots \wedge a_n) \quad \varepsilon/4 \quad \varphi(a) \quad \varepsilon/4 \quad \varphi(a \vee a_N \vee \cdots \vee a_n). \tag{81}$$

Now, note that vteration of Lemma 24 yields

$$\begin{aligned} \varphi(a \wedge a_N \wedge \cdots \wedge a_n) - \varphi(a) &= d_\varphi(\, a \wedge a_N \wedge \cdots \wedge a_n, \ a \,) \\ &= d_\varphi(\, a \wedge a_N \wedge \cdots \wedge a_n, \ a \wedge a \wedge \cdots \wedge a \,) \\ &\leq d_\varphi(a_N, a) + \cdots + d_\varphi(a_n, a). \end{aligned}$$

So by Statement (79) and Definition 168(viii) we get

$$\varphi(a \wedge a_N \wedge \cdots \wedge a_n) \quad \varepsilon/4 \quad \varphi(a).$$

Using a similar argument, we obtain

$$\varphi(a) \quad \varepsilon/4 \quad \varphi(a \vee a_N \vee \cdots \vee a_n).$$

So we have proven Statement (81) and thereby completed the proof. □

Proposition 192. *Let E be an ordered Abelian group.*
Let Φ be a fitting uniformity on E.
Let L be a lattice, and $\varphi \colon L \to E$ a valuation.
Let $a_1, a_2, \ldots$ be a sequence in L that weakly φ-converges to some $a \in L$.
Then: there are $j_1 < j_2 < \cdots$ in $\mathbb{N}$ such that $a_{j_1}, a_{j_2}, \ldots$ φ-converges to a.

Proof. It suffices to find $j_1 < j_2 < \cdots$ in $\mathbb{N}$ such that $\sum_k \ d_\varphi(a, a_{j_k})$ exists. Indeed, then we have $a_{j_1}, a_{j_2}, \ldots$ weakly φ-converging to a since $a_1, a_2, \ldots$ φ-converges to a. So by Lemma 191 we $a_{j_1}, a_{j_2}, \ldots$ φ-converges to a, as we must prove.

Let $\varepsilon'_1, \varepsilon'_2, \ldots$ be an enumeration of Φ. Pick $\varepsilon_1, \varepsilon_2, \ldots$ in Φ such that for all n,

$$\varepsilon_n \ \leq \ \varepsilon'_n \qquad \text{and} \qquad \varepsilon_{n+1} \ \leq \ \varepsilon_n/2.$$

Note that for all $N \in \mathbb{N}$ and $n \geq N+1$ we have (see Notation 172),

$$\varepsilon_{N+1} + \cdots + \varepsilon_n \ \leq \ \varepsilon_N. \tag{82}$$

Pick $j_1 < j_2 < \cdots$ in $\mathbb{N}$ such that for all $k \in \mathbb{N}$,

$$0 \quad \varepsilon_{k+1} \quad d_\varphi(a, a_{j_k}).$$

Then by Statement (82) and Definition 168(viii) for all $N \in \mathbb{N}$ and $n \geq N$,

$$0 \quad \varepsilon_N \quad d_\varphi(a, a_{j_N}) \ + \ \cdots \ + \ d_\varphi(a, a_{j_n}). \tag{83}$$

Recall that we need to prove that $\sum_k \ d_\varphi(a, a_{j_k})$ exists. Let $\varepsilon \in \Phi$ be given. By Lemma 171(xi) it suffices to find an $N \in \mathbb{N}$ such that for all $n \geq N$,

$$0 \quad \varepsilon \quad d_\varphi(a, a_{j_N}) \ + \ \cdots \ + \ d_\varphi(a, a_{j_n}). \tag{84}$$

Since $\varepsilon'_1, \varepsilon'_2, \ldots$ enumerates Φ we can find an $N \in \mathbb{N}$ such that $\varepsilon'_N = \varepsilon$. Recall that $\varepsilon \leq \varepsilon'_N \leq \varepsilon_N$. So Statement (84) follows directly from Statement (83). □

Lemma 193. *Let E be an ordered Abelian group.*
Let Φ be a fitting uniformity on E.
Let L be a lattice, and $\varphi\colon L \to E$ a valuation.
Let $a_1, a_2, \ldots$ be a sequence in L which weakly φ-converges to some $a \in L$
For each $N \in \mathbb{N}$, let $b^N_1, b^N_2, \ldots$ be a sequence in L which weakly φ-converges to a_N.
Then there are $j_1 < j_2 < \cdots$ in $\mathbb{N}$ such that $b^1_{j_1},\ b^2_{j_2},\ \ldots$ weakly φ-converges to a.

Proof. To find a suitable sequence $j_1 < j_2 < \cdots$ we need some preparation.

We know that Φ is countable (see Definition 168). Let $\varepsilon'_1,\ \varepsilon'_2,\ \ldots$ be an enumeration of Φ. Define a sequence $\varepsilon_1 \geq \varepsilon_2 \geq \cdots$ in Φ (see Notation (ii)) by

$$\varepsilon_1 \ = \ \varepsilon'_1 \qquad \text{and} \qquad \varepsilon_{n+1} \ = \ \varepsilon_n \wedge \varepsilon'_{n+1}.$$

Note that we have $\varepsilon_n \leq \varepsilon'_n$ for all n.

Let $N \in \mathbb{N}$ be given. Since $b^N_1, b^N_2, \ldots$ weakly φ-converges to a_N, we know by Definition 188 that there is an $M \in \mathbb{N}$ such that $d_\varphi(b^N_n, a_N)\ \varepsilon_N\ 0$ for all $n \geq M$.

Now, choose $j_1 < j_2 < \cdots$ such that $d_\varphi(b^N_n, a_N)\ \varepsilon_N\ 0$ for all $n \geq j_N$.

We will prove that $b^1_{j_1},\ b^2_{j_2}, \ldots$ weakly φ-converges to a. Let $\varepsilon \in \Phi$ be given. We must find an $\mathfrak{n} \in \mathbb{N}$ such that $d_\varphi(b^N_{j_N}, a_N)\ \varepsilon\ 0$ for all $N \geq \mathfrak{n}$ (see Definition 188).

Find an $k \in \mathbb{N}$ such that $\varepsilon/2 = \varepsilon'_k$. (Recall that $\varepsilon'_1, \varepsilon'_2, \ldots$ enumerates Φ.)

Pick $\mathfrak{n} \geq k$ such that $d_\varphi(a_N, a)\ \varepsilon/2\ 0$. We prove that $d_\varphi(b^N_{j_N}, a)\ \varepsilon\ 0$ for all $N \geq \mathfrak{n}$. Let $N \geq \mathfrak{n}$ be given. We have $d_\varphi(b^N_{j_N}, a_N)\ \varepsilon_N\ 0$ by choice of j_N. So in particular $d_\varphi(b^N_{j_N}, a_N)\ \varepsilon/2\ 0$ since $\varepsilon/2 = \varepsilon'_k \geq \varepsilon_k \geq \varepsilon_{\mathfrak{n}} \geq \varepsilon_N$.

Further, we have $d_\varphi(a_N, a)\ \varepsilon/2\ 0$ since $N \geq \mathfrak{n}$.

So by property (viii) of a fitting uniformity (see Definition 168) we have

$$d_\varphi(b^N_{j_N}, a_N) \ + \ d_\varphi(a_N, a) \quad \varepsilon/2 \quad d_\varphi(a_N, a) \quad \varepsilon/2 \quad 0.$$

So by property (iii) of a fitting uniformity we have

$$d_\varphi(b^N_{j_N}, a_N) \ + \ d_\varphi(a_N, a) \quad \varepsilon \quad 0.$$

Now, by points (i) and (iv) of Lemma 21 we get

$$0 \ \leq \ d_\varphi(b^N_{j_N}, a) \ \leq \ d_\varphi(b^N_{j_N}, a_N) \ + \ d_\varphi(a_N, a).$$

So by property (iv) we get $d_\varphi(b^N_{j_N}, a)\ \varepsilon\ 0$. □

Lemma 194. *Let E be an ordered Abelian group.*
Let Φ be a fitting uniformity on E.
Let L be a lattice, and $\varphi\colon L \to E$ a valuation. Let $a, b \in L$ be given.
Let $a_1, a_2, \dots$ be a sequence in L which weakly φ-converges to a.
Let $b_1, b_2, \dots$ be a sequence in L which weakly φ-converges to b.
Then $a_1 \wedge b_1,\ a_2 \wedge b_2,\ \dots$ weakly φ-converges to $a \wedge b$,
and $a_1 \vee b_1,\ a_2 \vee b_2,\ \dots$ weakly φ-converges to $a \vee b$.

Proof. We will only prove that $a_1 \wedge b_1,\ a_2 \wedge b_2,\ \dots$ weakly φ-converges to $a \wedge b$.

Let $\varepsilon \in \Phi$ be given. To prove $a_1 \wedge b_1,\ a_2 \wedge b_2,\ \dots$ weakly φ-converges to $a \wedge b$, we must find an $N \in \mathbb{N}$ such that

$$0 \quad \varepsilon \quad d_\varphi(\, a_n \wedge b_n,\ a \wedge b\,) \qquad\qquad (n \geq N). \tag{85}$$

Since $a_1, a_2, \dots$ weakly φ-converges to a and $b_1, b_2, \dots$ weakly φ-converges to b we know that there is an $N \in \mathbb{N}$ such that

$$0 \;\; {}^{\varepsilon}\!/_2 \;\; d_\varphi(a_n, a) \qquad \text{and} \qquad 0 \;\; {}^{\varepsilon}\!/_2 \;\; d_\varphi(b_n, b) \qquad\qquad (n \geq N). \tag{86}$$

We will prove that Statement (85) holds for this N.

To this end, note by Lemma 21(i) and Lemma 24 we have, for all $n \in \mathbb{N}$,

$$0 \quad \leq \quad d_\varphi(\, a_n \wedge b_n,\ a \wedge b\,) \quad \leq \quad d_\varphi(a_n, a) \,+\, d_\varphi(b_n, b)$$

So by property (iv) of Φ (see Def. 168) to prove (85) it suffices to show that

$$0 \quad \varepsilon \quad d_\varphi(a_n, a) \,+\, d_\varphi(b_n, b) \tag{87}$$

for any $n \geq N$. By Statement (86) and property (viii) of Φ, we get

$$0 \;\; {}^{\varepsilon}\!/_2 \;\; d_\varphi(a_n, a) \quad {}^{\varepsilon}\!/_2 \quad d_\varphi(a_n, a) \,+\, d_\varphi(b_n, b)$$

for all $n \geq N$. So we see that Statement (87) holds by property (viii) of Φ. □

Example 195. Given Lemma 48, one might surmise that Lemma 194 holds if one replaces "weakly φ-converges" by "φ-converges". This is not the case, as we will show.

Recall that Lebesgue integral $\varphi_\mathcal{L}\colon F_\mathcal{L} \to \mathbb{R}$ is a valuation. For all $n \in \mathbb{N}$, define

$$f_n \;=\; (-1)^n \cdot \tfrac{1}{n} \cdot \mathbf{1}_{[n,n+1]}.$$

Then $f_n \in F_\mathcal{L}$ for all n, and the sequence $f_1, f_2, \dots$ $\varphi_\mathcal{L}$-converges to $\mathbf{0}$. As one expects, the sequence $-f_1, -f_2, \dots$ also $\varphi_\mathcal{L}$-converges to $\mathbf{0}$. However, the sequence

$$f_1 \vee (-f_1),\ f_2 \vee (-f_2),\ \dots$$

does not $\varphi_\mathcal{L}$-converge to $\mathbf{0}$, because $f_n \vee (-f_n) = \frac{1}{n} \cdot \mathbf{1}_{[n,n+1]}$ (see Example 190).

Lemma 196. *Let E be an ordered Abelian group.*
Let Φ be a fitting uniformity on E.
Let $V \supseteq L \xrightarrow{\varphi} E$ be an extendible valuation system. Let $a \in \overline{L}$ be given.
Then there is a sequence $a_1, a_2, \dots$ in L that weakly $\overline{\varphi}$-converges to a.

Proof. By Corollary 145 we know that $\overline{L} = \Pi_{\aleph_1} L$. So it suffices to prove that the following statement holds for every ordinal number α.

$$\left[\begin{array}{l} \text{Let } a \in \Pi_\alpha L \cup \Sigma_\alpha L \text{ be given.} \\ \text{There is a sequence } a_1, a_2, \dots \text{ in } L \text{ that } \overline{\varphi}\text{-converges to } a. \end{array}\right.$$

Let us name the above statement $P(\alpha)$. We prove $\forall \alpha\ P(\alpha)$ with induction.

Clearly, $P(0)$ holds, since $\Pi_0 L = L = \Sigma_0 L$.

Let α be an ordinal such that $P(\alpha)$ holds. We prove that $P(\alpha + 1)$ holds. Let $a \in \Pi_{\alpha+1} L \cup \Sigma_{\alpha+1} L$ be given. We must find a sequence in L that $\overline{\varphi}$-converges to a.

Assume that $a \in \Pi_{\alpha+1} L$. There is a $\Sigma_\alpha \varphi$-convergent sequence $b_1 \geq b_2 \geq \cdots$ in $\Sigma_\alpha L$ such that $\bigwedge_n b_n = a$. In particular, $a_1, a_2, \dots$ $\overline{\varphi}$-converges to a. Since $P(\alpha)$

holds, we can find for each $N \in \mathbb{N}$ a sequence $b_1^N, b_2^N, \dots$ in L that $\overline{\varphi}$-converges to a_N. Then by Lemma 193 there are $j_1 < j_2 < \cdots$ in $\mathbb{N}$ such that $b_{j_1}^1, b_{j_2}^2, \dots$ $\overline{\varphi}$-converges to a. So we see that there is a sequence in L that $\overline{\varphi}$-converges to a.

By a similar reasoning we see that if $a \in \Sigma_{\alpha+1}L$ then there is a sequence in L that $\overline{\varphi}$-converges to a. Hence $P(\alpha+1)$.

Let λ be a limit ordinal such that $P(\alpha)$ holds for all $\alpha < \lambda$. We must prove that $P(\lambda)$ holds. Let $a \in \Pi_\lambda L \cup \Sigma_\lambda L$ be given. We must find a sequence in L that $\overline{\varphi}$-converges to a. By definition of $\Pi_\lambda L$ and $\Sigma_\lambda L$, there is an $\alpha < \lambda$ such that $a \in \Pi_\alpha L \cup \Sigma_\alpha L$ (see Definition 138). Since we know that $P(\alpha)$ holds, there must be a sequence in L that $\overline{\varphi}$-converges to a. Hence $P(\lambda)$. ☐

Corollary 197. *Let E be an ordered Abelian group.*
Let Φ be a fitting uniformity on E.
Let $V \supseteq L \xrightarrow{\varphi} E$ be an extendible valuation system. Let $a, b \in \overline{L}$ with $a \leq b$ be given.
Then there is a sequence $a_1, a_2, \dots$ in L that weakly $\overline{\varphi}$-converges to a,
and there is a sequence $b_1, b_2, \dots$ in L that weakly $\overline{\varphi}$-converges to b,
such that $a_n \leq b_n$ for all $n \in \mathbb{N}$.

Proof. Let $a, b \in \overline{L}$ with $a \leq b$ be given. Using Lemma 196 find a sequence $a_1', a_2', \dots$ in L that weakly $\overline{\varphi}$-converges to a and find a sequence $b_1', b_2', \dots$ in L that weakly $\overline{\varphi}$-converges to b. Consider the sequences $a_1, a_2, \dots$ and $b_1, b_2, \dots$ in L given by

$$a_n := a_n' \wedge b_n' \qquad \text{and} \qquad b_n := a_n' \vee b_n' \qquad\qquad (n \in \mathbb{N}).$$

Clearly $a_n \leq b_n$ for all $n \in \mathbb{N}$. Moreover, by Lemma 194 we know that $a_1, a_2, \dots$ weakly $\overline{\varphi}$-converges to $a = a \wedge b$ and weakly $b_1, b_2, \dots$ $\overline{\varphi}$-converges to $b = a \vee b$. ☐

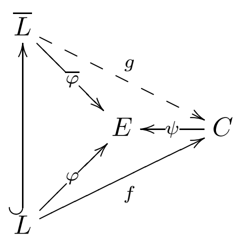
Proposition 198. *Let E be an ordered Abelian group.*
Let Φ be a fitting uniformity on E.
Let $V \supseteq L \xrightarrow{\varphi} E$ be an extendible valuation system.

(i) Given $a \in \overline{L}$, there is a sequence $a_1, a_2, \dots$ in L which $\overline{\varphi}$-converges to a.
(ii) Let $a, b \in \overline{L}$ with $a \leq b$ be given.
There is a sequence $a_1, a_2, \dots$ in L which φ-converges to a, and there is a sequence $b_1, b_2, \dots$ in L which φ-converges to b, such that

$$a_n \;\leq\; b_n \qquad\qquad (n \in \mathbb{N}).$$

Proof. Combine Lemma 196, Corollary 197, and Proposition 192. ☐

Theorem 199. *Let E be a lattice ordered Abelian group.*
Let Φ be a fitting uniformity on E.
Let $V \supseteq L \xrightarrow{\varphi} E$ be an extendible valuation system.
Let $\psi\colon C \to E$ be a complete Hausdorff valuation.
Let $f\colon L \to C$ be an order preserving map such that $\psi \circ f = \varphi$.
Then there is a unique order preserving extension $g\colon \overline{L} \to C$ of f such that $\psi \circ g = \overline{\varphi}$.



Proof. (Uniqueness) Let $g_1, g_2 \colon \overline{L} \to C$ be order preserving extensions of f such that $\psi \circ g_i = \overline{\varphi}$. We prove that $g_1 = g_2$. Let $a \in \overline{L}$ be given. By Proposition 198 there is a φ-convergent sequence $a_1, a_2, \ldots$ in L which $\overline{\varphi}$-converges to a.

We must show that $g_1(a) = g_2(a)$. To this end we prove that $f(a_1),\, f(a_2),\, \ldots$ weakly ψ-converges to $g_i(a)$. Then $g_1(a) = g_2(a)$ because ψ is Hausdorff.

Let $i \in \{1, 2\}$ be given. Note that $g_i(a_n) = f(a_n)$ because g_i extends f. So we must prove that $g_i(a_1),\, g_i(a_2),\, \ldots$ weakly ψ-converges to $g_i(a)$. Let $\varepsilon \in \Phi$ be given. We must find an $N \in \mathbb{N}$ such that

$$d_\psi(\, g_i(a_n),\, g_i(a)\,) \quad \varepsilon \quad 0 \qquad\qquad (n \geq N). \tag{88}$$

Since $a_1, a_2, \ldots$ $\overline{\varphi}$-converges to a we know that $a_1, a_2, \ldots$ weakly $\overline{\varphi}$-converges to a so we know there is an $N \in \mathbb{N}$ with

$$d_{\overline{\varphi}}(a_n, a) \quad \varepsilon \quad 0 \qquad\qquad (n \geq N).$$

We will prove that Statement (88) holds for this N.

Let $n \geq N$ be given. Since g is order preserving, we have

$$g_i(a_n \wedge a) \;\leq\; g_i(a_n) \wedge g_i(a) \qquad\qquad g_i(a_n) \vee g_i(a) \;\leq\; g_i(a_n \vee a).$$

In particular (recall that $\psi \circ g_i = \overline{\varphi}$),

$$\begin{aligned} d_\psi(\, g_i(a_n),\, g_i(a)\,) \;&=\; \psi(g_i(a_n) \vee g_i(a)) \,-\, \psi(g_i(a_n) \wedge g_i(a)) \\ &\leq\; \psi(g_i(a_n \vee a)) \,-\, \psi(g_i(a_n \wedge a)) \\ &=\; \overline{\varphi}(a_n \vee a) \,-\, \overline{\varphi}(a_n \wedge a) \;=\; d_{\overline{\varphi}}(a_n, a). \end{aligned} \tag{89}$$

So we know that $0 \;\varepsilon\; d_{\overline{\varphi}}(a_n, a)$ and we have the following inequalities.

$$0 \;\leq\; d_\psi(\, g_i(a_n),\, g_i(a)\,) \;\leq\; d_{\overline{\varphi}}(a_n, a)$$

Hence $0 \;\varepsilon\; d_\psi(g_i(a_n), g_i(a))$ by property (iv) of a fitting uniformity. So we have shown that Statement (88) holds. Thus, $g_1 = g_2$.

(Existence) We will prove the following statement.

$$\left[\;\begin{array}{l} \text{Let } a \in \overline{L} \text{ be given. There is a unique } b \in C \text{ such that for} \\ \text{every sequence } a_1, a_2, \ldots \text{ in } L \text{ that } \overline{\varphi}\text{-converges to } a \text{, we} \\ \text{have } f(a_1),\, f(a_2),\, \ldots\ \psi\text{-converges to } b. \end{array}\right. \tag{90}$$

Of course, we will later define $g \colon \overline{L} \to C$ by $g(a) = b$.

Let $a \in \overline{L}$ be given. For each $i \in \{1, 2\}$, let $b_i \in C$ and $a^i_1, a^i_2, \ldots \in L$ be given, such that $a^i_1, a^i_2, \ldots$ $\overline{\varphi}$-converges to a, and $f(a^i_1),\, f(a^i_2), \ldots$ ψ-converges to b_i.

We must prove that $b_1 = b_2$. Let $\varepsilon \in \Phi$ be given. Since ψ is Hausdorff, it suffices to show that $0 \;\varepsilon\; d_\psi(b_1, b_2)$ (see 22).

Note that by points (i) and (iv) of Lemma 21 we have

$$0 \;\leq\; d_\psi(b_1, b_2) \quad\leq\quad d_\psi(b_1,\, f(a^1_n)\,) \;+\; d_\psi(\, f(a^1_n),\, f(a^2_n)\,) \;+\; d_\psi(\, f(a^2_n),\, b_2).$$

So to prove $0 \;\varepsilon\; d_\psi(b_1, b_2)$, it sufficient to find $N \in \mathbb{N}$ such that for all $n \geq N$ the following statement holds (see Definition 168, points (iv), (iii) and (viii)).

$$\begin{array}{lll} 0 & \varepsilon/4 & d_\psi(b_1,\, f(a^1_n)\,), \quad \text{and} \\ 0 & \varepsilon/4 & d_\psi(\, f(a^1_n),\, f(a^2_n)\,), \quad \text{and} \\ 0 & \varepsilon/4 & d_\psi(\, f(a^2_n),\, b_2). \end{array}$$

Recall that $f(a^i_1),\, f(a^i_2),\, \ldots$ ψ-converges to b_i for all i. Hence $f(a^i_1),\, f(a^i_2),\, \ldots$ weakly ψ-converges to b_i for all i. So we know there is an $N \in \mathbb{N}$ such that $0 \;\varepsilon/4\; d_\psi(f(a^i_n),\, b_i)$ for all $n \geq N$ and i.

It remains to be shown that there is an $N \in \mathbb{N}$ such that $0 \ \varepsilon/4 \ d_\psi(f(a_n^1),\, f(a_n^2))$ for all $n \geq N$. To this end, note that f is order preserving and that $\psi \circ f = \varphi$. So with a similar reasoning as before (see Statement (89)), we see that

$$d_\psi(\, f(a_n^1),\, f(a_n^2)\,) \ \leq\ d_\varphi(a_n^1, a_n^2).$$

So to complete the proof Statement (90) it suffices to find an $N \in \mathbb{N}$ such that

$$0 \quad \varepsilon/4 \quad d_\varphi(a_n^1, a_n^2) \qquad\qquad (n \geq N). \tag{91}$$

Note that by points (i) and (iv) of Lemma 21 we have

$$0 \ \leq\ d_\varphi(a_n^1, a_n^2) \ \leq\ d_\varphi(a_n^1, a) + d_\varphi(a, a_n^2).$$

Since the sequence $a_1^i, a_2^i, \ldots$ $\overline{\varphi}$-converges to a (and hence also weakly) we can find an $N \in \mathbb{N}$ such that $0 \ \varepsilon/4 \ d_\varphi(a_n^i, a)$ for all $n \geq N$ and $i \in \{1, 2\}$.

So by points (iv), (iii) and (viii) of Definition 168 we see that Statement (91) holds.

Hence we have proven Statement (90). So we now know there is a unique map $g\colon \overline{L} \to C$ such that for every $a \in \overline{L}$ and every sequence $a_1, a_2, \ldots$ in L that $\overline{\varphi}$-converges to a we have $f(a_1),\, f(a_2),\, \ldots$ ψ-converges to $g(a)$.

To complete the proof of this theorem, we show that g extends f, we show that g is order preserving, and that $\psi \circ g = \overline{\varphi}$.

Let $a \in L$ be given. To prove that g extends f we show that $g(a) = f(a)$. Note that $a, a, \ldots$ $\overline{\varphi}$-converges to a. So by definition of g we know that $f(a),\, f(a),\, \ldots$ ψ-converges to $g(a)$. But $f(a),\, f(a),\, \ldots$ ψ-converges to $f(a)$ too, and ψ is Hausdorff. So we see that $f(a) = g(a)$.

Let $a, b \in L$ with $a \leq b$ be given. To prove that g is order preserving we must show that $g(a) \leq g(b)$. By Proposition 198 we can find a sequence $a_1, a_2, \ldots$ in L that $\overline{\varphi}$-converges to a and a sequence $b_1, b_2, \ldots$ in L that $\overline{\varphi}$-converges to b such that we have $a_n \leq b_n$ for all $n \in \mathbb{N}$. Now, note that by Lemma 194 we know that

$$f(a_1) \wedge f(b_1), \quad f(a_2) \wedge f(b_2), \quad \ldots \qquad \textit{weakly } \psi\text{-converges to} \quad g(a) \wedge g(b).$$

Let $n \in \mathbb{N}$ be given. Since f is order preserving and $a_n \leq b_n$ we have $f(a_n) \leq f(b_n)$ and so $f(a_n) \wedge f(b_n) = f(a_n)$. Hence $f(a_1),\, f(a_2),\, \ldots$ weakly ψ-converges to both $g(a)$ and $g(a) \wedge g(b)$. So we see that $g(a) = g(a) \wedge g(b)$ and thus $g(a) \leq g(b)$.

Let $a \in \overline{L}$ be given. We show that $\psi(g(a)) = \overline{\varphi}(a)$. Find a sequence $a_1, a_2, \ldots$ in L that $\overline{\varphi}$-converges to a (see Lemma 196).

Recall that E is a *lattice* ordered Abelian group. By Theorem 63 we see that

$$\overline{\varphi}(a) \ =\ \lim_n \varphi(a_n). \tag{92}$$

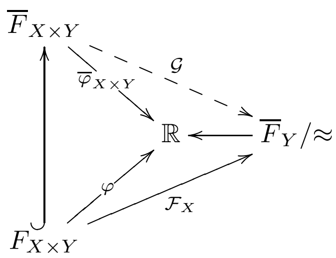
By definition of g we have $f(a_1),\, f(a_2),\, \ldots$ ψ-converges to $g(a)$. So by Theorem 63 we have $\psi(g(a)) = \lim_n \psi(f(a_n))$. But $\psi(f(a_n)) = \varphi(a_n)$ for all $n \in \mathbb{N}$, so we have

$$\psi(f(a)) \ =\ \lim_n \varphi(a_n). \tag{93}$$

If we combine Equalities (92) and (93) we get $\overline{\varphi}(a) = \psi(f(a))$. □

Theorem 200. *Statement* (71) *holds.*

Proof. We only give hints and leave the details to the reader. With the notation of Subsection 9.1 apply Theorem (199) to the following situation.



Now, note that $\mathcal{G}$ is a complete valuation which extends $\mathcal{F}_X$. □

10. Epilogue

Starting from the similarity between the Lebesgue measure and the Lebesgue integral as shown on page 9 I have tried to rebuild a small part of the theory of measure and integration in a more general setting. When I look back at the result I am most pleased that it was possible to introduce the Lebesgue measure and the Lebesgue integral with such natural and old primitives. Indeed, completeness and convexity together is nothing more than the method of exhaustion used by the ancient Greeks to determine the area of the disk (see title page).

The price for simple primitives seems to be that much more effort is required to prove even the simplest statements, as attested by the size of this text. Of course, the number of pages could be greatly reduced if we worked with $\mathbb{R}$ instead of any E, but even then I doubt that the approach taken in this thesis would be suitable for a first course on the Lebesgue measure and the Lebesgue integral.

Whether the theory in this thesis will bear any fruit I cannot tell, but nevertheless I am content, because I have enjoyed writing it, and I hope that you have enjoyed reading it as well.

Appendix A. Ordered Abelian Groups

In this thesis we do not only consider $\mathbb{R}$-valued measures and integrals, but also E-valued ones, where E is an *ordered Abelian group.* Since we do not expect reader to be familiar with this particular generalisation of $\mathbb{R}$, we have collected the relevant definitions and basic results in this appendix.

Definition 201. An **ordered Abelian group** E is a set that is endowed with an Abelian group operation, $+$, and a partial order, $\leq$, such that

$$x \;\leq\; y \qquad\Longrightarrow\qquad w+x \;\leq\; w+y \qquad\qquad (w,x,y\in E).$$

Examples 202. (i) The integers, $\mathbb{Z}$, the rationals, $\mathbb{Q}$, and the reals, $\mathbb{R}$, endowed with the usual addition and order are ordered Abelian groups.

(ii) Let $\mathbb{Q}^\circ$ be the set of rational numbers q with $q > 0$. Order $\mathbb{Q}^\circ$ by

$$q \;\preccurlyeq\; r \qquad\Longleftrightarrow\qquad \exists n\in\mathbb{N}\;[\;q\cdot n = r\;].$$

Then $\mathbb{Q}^\circ$ with the usual multiplication is an ordered Abelian group.

(iii) Let E_1 and E_2 be ordered Abelian groups. Then $E_1 \times E_2$ with pointwise order and pointwise group operation is an ordered Abelian group.

(iv) Consider $\mathbb{R}^2$ with the pointwise addition. By point (iii), $\mathbb{R}^2$ with the usual order is an ordered Abelian group. Further, $\mathbb{R}^2$ with the *lexicograpgic order*,

$$(x_1,x_2) \;\leq\; (y_1,y_2) \qquad\Longleftrightarrow\qquad \left[\begin{array}{lll} x_1 < y_1 & \text{or} & \\ x_1 = y_1 & \text{and} & x_2\leq y_2 \end{array}\right.,$$

is also an ordered Abelian group, called the **lexicographic plane**, $\mathbb{L}$.

Let us prove some simple statements concerning ordered Abelian groups.

Lemma 203. *Let E be an ordered Abelian group. Then, for $x,y,w\in E$,*

$$x \leq y \qquad\Longleftrightarrow\qquad w+x \;\leq\; w+y.$$

Proof. "$\Longrightarrow$" By the definition of ordered Abelian group.
"$\Longleftarrow$" If $w+x\leq w+y$, then $x \;=\; -w \;+\; (w+x) \;\leq\; -w \;+\; (w+y) \;=\; y.$ □

Lemma 204. *Let E be an ordered Abelian group. Let $A\subseteq E$ and $x\in E$ be given.*

(i) If A has an infimum, then so has $x+A := \{x+a\colon a\in A\}$, and

$$\bigwedge x+A \;=\; x+\bigwedge A.$$

(ii) If A has a supremum, then so has $x+A$, and

$$\bigvee x+A \;=\; x+\bigvee A.$$

Proof. It suffices to prove that the map $E\to E$ given by $u\mapsto x+u$ is an order isomorphism. This follows easily using Lemma 203. □

Lemma 205. *Let E be an ordered Abelian group. Then, for $x,y\in E$,*

$$x \leq y \qquad\Longleftrightarrow\qquad -x \;\geq\; -y.$$

Proof. "$\Longrightarrow$" If $x\leq y$, then $-y \;=\; (-x-y) \;+\; x \;\leq\; (-x-y) \;+\; y \;=\; -x.$
"$\Longleftarrow$" If $-y\leq -x$, then $x \;=\; -(-x) \;\leq\; -(-y) \;=\; y$ by "$\Longrightarrow$". □

Lemma 206. *Let E be an ordered Abelian group, and $A\subseteq E$.*

(i) If A has an infimum, then $-A := \{-a\colon a\in A\}$ has a supremum, and

$$-\bigwedge A \;=\; \bigvee -A.$$

(ii) If A has an supremum, then $-A$ has an infimum, and

$$-\bigvee A \;=\; \bigwedge -A.$$

Proof. The map $E\to E$ given by $u\mapsto -u$ is an order reversing isomorphism. □

We use the following lemma regularly.

Lemma 207. *Let E be an ordered Abelian group.*
Let $x_1 \leq x_2 \leq \cdots$ be from E such that $\bigvee_n x_n$ exists.
Let $y_1 \leq y_2 \leq \cdots$ be from E such that $\bigvee_n y_n$ exists. Then

$$(\textstyle\bigvee_n x_n) + (\bigvee_n y_n) \ = \ \bigvee_k x_k + y_k. \tag{94}$$

Proof. By Lemma 204 we know that

$$(\textstyle\bigvee_n x_n) + (\bigvee_m y_m) \ = \ \bigvee_{n,m} x_n + y_m.$$

So to prove Equation (94) holds, it suffices to show that

$$\textstyle\bigvee_{n,m} x_n + y_m \ = \ \bigvee_k x_k + y_k.$$

That is, writing $z := \bigvee_{n,m} x_n + y_m$, we must show that z is the supremum of

$$S \ := \ \{\ x_1 + y_1,\ x_2 + y_2,\ \ldots\ \}.$$

That is, we must show that z is the smallest upper bound of S.

Given $s \in S$, we have $s \equiv x_k + y_k$ for some $k \in \mathbb{N}$, and

$$x_k + y_k \ \leq \ \textstyle\bigvee_{n,m} x_n + y_m \ \equiv \ z.$$

So we see that z is an upper bound of S.

Let $u \in E$ be an upper bound of S. To prove that z is the smallest upper bound of S, we must show that $z \leq u$. It suffices to prove that, for all $n, m \in \mathbb{N}$,

$$x_n + y_m \ \leq \ u. \tag{95}$$

Let $n, m \in \mathbb{N}$ be given, and define $k := \max\{n, m\}$. Then we see that

$$x_n + y_m \ \leq \ x_k + y_k \ \leq \ u.$$

Hence Statement (95) holds, and we are done. □

Of course, we have a similar statement concerning infima.

Lemma 208. *Let E be an ordered Abelian group.*
Let $x_1 \geq x_2 \geq \cdots$ be from E such that $\bigwedge_n x_n$ exists.
Let $y_1 \geq y_2 \geq \cdots$ be from E such that $\bigwedge_n y_n$ exists. Then

$$(\textstyle\bigwedge_n x_n) + (\bigwedge_n y_n) \ = \ \bigwedge_k x_k + y_k.$$

Proof. Similar to the proof of Lemma 207. □

We will occasionally use the following notation.

Definition 209. Let E be an ordered Abelian group. We write

$$E^+ \ := \ \{\ a \in E\colon\ a \geq 0\ \}, \qquad\qquad E^- \ := \ \{\ a \in E\colon\ a \leq 0\ \}.$$

Let us now turn to a special class of ordered Abelian groups.

Definition 210. A **lattice ordered Abelian group** is an ordered Abelian group E, such that the order $\leq$ makes E a lattice, i.e., each pair $x, y \in E$ has an infimum, $x \wedge y$, and a supremum, $x \vee y$.

Examples 211.
(i) The sets $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$ are lattices under the usual order. The supremum of two elements is their maximum, the infimum is the minimum.
(ii) More generally, any partially ordered set E that is *totally ordered*, i.e.,

$$\text{either} \quad x \leq y \quad \text{or} \quad y \leq x \quad \text{for all } x, y \in E,$$

is a lattice. The supremum of $x, y \in E$ is simply the maximum of x and y, the infimum x and y is the minimum of x and y.

(iii) The space $\mathbb{L}$ (see Example 202(iv)) is totally ordered and hence a lattice.
(iv) The set $\mathbb{Q}^\circ$ ordered by $\preccurlyeq$ (see Examples 202(ii)) is a lattice.
Let $m, n \in \mathbb{Q}^\circ$ be given. If $m, n \in \mathbb{Z}$, then the supremum of m and n is the least common multiple of m and n, and the infimum of m and n is the greatest common divisor of m and n.

The following result is quite suprising.

Lemma 212. *Let E be a lattice ordered Abelian group. Then we have*

$$a \wedge b \,+\, a \vee b \;=\; a \,+\, b \qquad (a, b \in E).$$

Proof. $a \vee b - a - b = (a - a - b) \vee (b - a - b) = (-b) \vee (-a) = -(a \wedge b)$. □

Examples 213. (i) Let $x, y \in \mathbb{R}$ be given. Then Lemma 212 gives us

$$x + y \;=\; \min\{x, y\} \,+\, \max\{x, y\}.$$

Of course, this is trivial.
(ii) Let $m, n \in \mathbb{Z}$ with $m, n \geq 0$ be given. Then Lemma 212 gives us

$$m \cdot n \;=\; \mathrm{gcd}\{m, n\} \,\cdot\, \mathrm{lcm}\{m, n\}.$$

The above equality is more difficult to derive directly.

We now turn to 'complete' ordered Abelian groups.

Definition 214. Let E be an ordered Abelian group.
We say E is **σ-Dedekind complete** if the following statement holds.

> Let $x_1, x_2, \ldots$ be a sequence in E.
> Assume $x_1, x_2, \ldots$ has an upper bound.
> Then $\bigvee_n x_n$ exists.

Examples 215. (i) The ordered Abelian group $\mathbb{R}$ is σ-Dedekind complete.
(ii) The ordered Abelian $\mathbb{Q}$ is *not* σ-Dedekind complete.
(iii) The lexicographic plane $\mathbb{L}$ (see Examples 202(iv)) is *not* σ-Dedekind complete.
Indeed, consider the following elements of $\mathbb{L}$.

$$(0,0) \;\leq\; (0,1) \;\leq\; (0,2) \;\leq\; \cdots \;\leq\; (1,0)$$

If $\mathbb{L}$ were σ-Dedekind complete, then $S := \{\, (0, n) \colon\; n \in \mathbb{N} \,\}$ would have a supremum; we will prove that S does not have a supremum.
Suppose (towards a contradiction) that S has a supremum, (x, y).
Then we have $(0, n) \leq (x, y)$ for all $n \in \mathbb{N}$. In other words, for all $n \in \mathbb{N}$,

$$0 < x \qquad \text{or} \qquad (\ 0 = x \quad \text{and} \quad n \leq y\ ).$$

Hence $0 < x$, because there is no $y \in \mathbb{R}$ such that $n \leq y$ for all $n \in \mathbb{N}$.
But then $(x, y - 1)$ is an upper bound of S as well.
Since (x, y) is the smallest upper bound of S, we have $(x, y) \leq (x, y - 1)$.
So $y \leq y - 1$, which is absurd. Hence S has no supremum.

Remark 216. The requirement in Definition 214 that $x_1, x_2, \ldots$ has an upper bound is essential to make the notion of σ-Dedekind completeness non-trivial.

Indeed, if E is an ordered Abelian group in which *every* sequence $x_1, x_2, \ldots$ has a supremum $\bigvee_n x_n$, then we have $E = \{0\}$!

Let $a \in E^+$ be given. We prove that $a = 0$. Note that the sequence

$$1 \cdot a \;\leq\; 2 \cdot a \;\leq\; 3 \cdot a \;\leq\; \cdots$$

has a supremum, $\bigvee_n n \cdot a$. Note that by Lemma 204, we have

$$(\textstyle\bigvee_n n \cdot a) \,-\, a \;=\; \textstyle\bigvee_n (n-1) \cdot a \;=\; \textstyle\bigvee_n n \cdot a.$$

So we see that $b - a = b$, where $b := \bigvee_n n \cdot a$. Hence $a = 0$.

Let $a \in E$ be given. We must prove that $a = 0$. Note that by Lemma 212,

$$a \;=\; 0 \wedge a \,+\, 0 \vee a. \tag{96}$$

We have $0 \wedge a = 0$, since $0 \wedge a \in E^+$. We also have $0 \vee a = 0$, because $0 \vee a \in E^-$, so $-(0 \vee a) \in E^+$, so $-(0 \vee a) = 0$, and thus $0 \vee a = 0$.

So we see that $a = 0$ by Equation (96). Hence $E = \{0\}$.

Remark 217. Let E be an ordered Abelian group.
Using the order reversing isomorphism $x \mapsto -x$, the reader can easily verify that E is σ-Dedekind complete if and only if the following statement holds.

> Let $x_1, x_2, \ldots$ be a sequence in E.
> Assume $x_1, x_2, \ldots$ has a lower bound.
> Then $\bigwedge_n x_n$ exists.

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