Von Neumann Algebras form a Model for the Quantum Lambda Calculus

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Abstract

We present a model of Selinger and Valiron's quantum lambda calculus based on von Neumann algebras, and show that the model is adequate with respect to the operational semantics.

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1 Introduction

In 1925, Heisenberg realised, pondering upon the problem of the spectral lines of the hydrogen atom, that a physical quantity such as the x-position of an electron orbiting a proton is best described not by a real number but by an infinite array of complex numbers [12]. Soon afterwards, Born and Jordan noted that these arrays should be multiplied as matrices are [3]. Of course, multiplying such infinite matrices may lead to mathematically dubious situations, which spurred von Neumann to replace the infinite matrices by operators on a Hilbert space [44]. He organised these into *rings of operators* [25], that are now called von Neumann algebras, and thereby set off an explosion of research (also into related structures such as Jordan algebras [13], orthomodular lattices [2], C^*-algebras [34], AW^*-algebras [17], order unit spaces [14], Hilbert C^*-modules [28], operator spaces [31], effect algebras [8], ...), which continues even to this day.

One current line of research (with old roots [6, 7, 9, 19]) is the study von Neumann algebras from a categorical perspective (see e.g. [4, 5, 30]). One example relevant to this paper is Kornell's proof that the opposite of the category $\mathbf{vNA}_{\mathrm{MIU}}$ of von Neumann algebras with the obvious structure preserving maps (i.e. the unital normal $*$-homomorphisms) is monoidal closed when endowed with the spatial tensor product [18]. He argues that $\mathbf{vNA}^{\mathrm{op}}_{\mathrm{MIU}}$ should be thought of as the quantum version of **Set**. We would like to focus instead on the category of von Neumann algebras and completely positive normal subunital maps, $\mathbf{vNA}_{\mathrm{CPsU}}$, as it seems more appropriate for modelling quantum computation: the full subcategory of $\mathbf{vNA}^{\mathrm{op}}_{\mathrm{CPsU}}$ consisting of finite dimensional von Neumann algebras is equivalent to Selinger's category $\mathbf{Q}$ [35], which is used to model first order quantum programming languages.

On the syntactic side, in 2005, Selinger and Valiron [36, 37] proposed a typed[1] lambda calculus for quantum computation, and they studied it in a series of papers [38–40]. A striking feature of this quantum lambda calculus is that functions naturally appear as data in the description of the Deutch–Jozsa

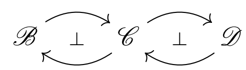

Figure 1 General shape of a model of the QLC

[1] An untyped quantum lambda calculus had already been proposed by Van Tonder [43].

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algorithm, teleportation algorithm and Bell's experiment. Although Selinger and Valiron gave a precise formulation of what might constitute a model of the quantum lambda calculus — basically a pair of adjunctions, see Figure 1, with some additional properties [40, §1.6] — the existence of such a model (other than the term model) was an open problem for several years until Malherbe constructed a model in his thesis using presheaves [21]. The construction of Malherbe's model is quite abstract, and it is (perhaps because of this) not yet known whether his model is adequate with respect to the operational semantics defined by Selinger and Valiron in [37] (see also [40]). While several adequate models for variations on the quantum lambda calculus have been proposed in the meantime (using the geometry of interaction in [10], and quantitative semantics in [27]), Malherbe's model remains the only model of the original quantum lambda calculus [37] known in the literature, and so the existence of an adequate model for the quantum lambda calculus is still open.

In this paper, we present the a model of Selinger and Valiron's quantum lambda calculus, based on von Neumann algebras, see Figure 2, and we show that the model is adequate with respect to the operational semantics. We should note that it is possible to extend the quantum lambda calculus with recursion and inductive types, but that we have not yet been able to include these features in our model.

Figure 2 A model of the QLC using von Neumann algebras

The paper is divided in six sections. We begin with a short review of quantum computation (in Section 2), and the quantum lambda calculus and its operational semantics (in Section 3). We give the denotational semantics for the quantum lambda calculus using von Neumann algebras and prove its adequacy in Section 4. For this we use several technical results about the categories $\mathbf{vNA}_{\mathrm{MIU}}$ and $\mathbf{vNA}_{\mathrm{CPsU}}$ of von Neumann algebras, which we will discuss in Section 5. We end with a conclusion in Section 6.

2 Quantum Computation

In a nutshell, one gets the quantum lambda calculus by taking the simply typed lambda calculus with products and coproducts and adding a qubit type. This single ingredient dramatically changes the flavour of the whole system e.g. forcing one to make the type system linear, so we will spend some words on the behaviour of qubits in this section. For more details on quantum computation, see [26].

A state of an isolated qubit is a vector $|\psi\rangle$ of length 1 in the Hilbert space $\mathbb{C}^2$, and can be written as a complex linear combination ("superposition") $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, since the vectors $|0\rangle = (1,0)$ and $|1\rangle = (0,1)$ form an orthonormal basis for $\mathbb{C}^2$.

When qubits are combined to form a larger system, one can sometimes no longer speak about the state of the individual qubits, but only of the state of the whole system (in which case the qubits are "entangled".) The state of a register of n qubits is a vector $|\psi\rangle$ of length 1 in the n-fold tensor product $(\mathbb{C}^2)^{\otimes n} \cong \mathbb{C}^{2^n}$, which has as an orthonormal basis the vectors of the form $|w\rangle \equiv |w_1\rangle \otimes \cdots \otimes |w_n\rangle$ where $w \equiv w_1 \cdots w_n \in 2^n$.

For the purposes of this paper there are three basic operations on registers of qubits.

1. One can add a new qubit in state $|0\rangle$ to a register of n qubits in state $|\psi\rangle$, turning it to a register of $n+1$ qubits in state $|\psi\rangle \otimes |0\rangle$. A qubit in state $|1\rangle$ can be added similarly.
2. One can apply a unitary $2^n \times 2^n$ matrix U to a register of n qubits in state $|\psi\rangle$ turning the state to $U|\psi\rangle$.

3. One can test the first qubit in the register. If the state of the register is written as $|\psi\rangle \equiv \alpha\,|0\rangle \otimes |\psi_0\rangle \,+\, \beta\,|1\rangle \otimes |\psi_1\rangle$ where the length of $|\psi_0\rangle$ and $|\psi_1\rangle$ is 1, then the test comes out negative and changes the state of the register to $|0\rangle \otimes |\psi_0\rangle$ with probability $|\alpha|^2$, and comes out positive with probability $|\beta|^2$ changing the state to $|1\rangle \otimes |\psi_1\rangle$.
   Measurement of the i-th qubit in the register is also possible and behaves similarly.

A predicate on a register of n qubits is a $2^n \times 2^n$ matrix P such that both P and $I - P$ are positive (which is the case when P is a projection). The probability that P holds in state $|\psi\rangle$ is $\langle\psi|P|\psi\rangle$. For example, given a state $|\psi\rangle$ of a qubit, the projection $|\psi\rangle\langle\psi|$ (which maps $|\xi\rangle$ to $\langle\psi|\xi\rangle|\psi\rangle$) represents the predicate "the qubit is in state $|\psi\rangle$".

Thus the predicates on a qubit are part of the algebra $\mathcal{M}_2$ of 2×2 complex matrices. There is also an algebra for the bit, namely $\mathbb{C}^2$. A predicate on a bit is an element $(x, y) \equiv v \in \mathbb{C}^2$ with $0 \leq v \leq 1$, which is interpreted as "the bit is true with probability y, false with probability x, and undefined with probability $1 - x - y$".

An operation on a register of qubits may not only be described by the effect it has on states (Schrödinger's view), but also by its action on predicates (Heisenberg's view).

1. The operation which takes a bit b and returns a qubit in state $|b\rangle$ is represented by the map $f_{\mathtt{new}} \colon \mathcal{M}_2 \to \mathbb{C}^2$ given by $f_{\mathtt{new}}(A) = (\,\langle 0|A|0\rangle,\ \langle 1|A|1\rangle\,)$.
2. The operation which applies a unitary U to a register of n qubits is represented by the map $f_U \colon \mathcal{M}_{2^n} \to \mathcal{M}_{2^n}$ given by $f_U(A) = U^*AU$.
3. The operation which tests a qubit and returns the outcome is represented by the map $f_{\mathtt{meas}} \colon \mathbb{C}^2 \to \mathcal{M}_2$ given by $f_{\mathtt{meas}}(\lambda, \varrho) = \lambda|0\rangle\langle 0| + \varrho|1\rangle\langle 1|$.

A general operation between *finite dimensional* quantum data types is usually taken to be a completely positive subunital linear map (see below) between direct sums of matrix algebras, $\bigoplus_{i=1}^{n} \mathcal{M}_{m_i}$. The category formed by these operations is equivalent to $\mathbf{Q}^{\mathrm{op}}$ [4, Th. 8.4].

Von Neumann algebras are a generalisation of direct sums of matrix algebras to infinite dimensions. Formally, a von Neumann algebra $\mathscr{A}$ is a linear subspace of the bounded operators on a Hilbert space $\mathscr{H}$, which contains the identity operator, 1, is closed under multiplication, involution, $(-)^*$, and is closed in the weak operator topology, i.e. the topology generated by the seminorms $|\langle x|{-}|x\rangle|$ where $x \in \mathscr{H}$ (cf. [16, 25]).

We believe that the opposite $\mathbf{vNA}_{\mathrm{CPsU}}^{\mathrm{op}}$ of the category of von Neumann algebras and normal completely positive subunital maps (definitions are given below) might turn out to be the most suitable extension of $\mathbf{Q}$ to describe operations between (possibly infinite dimensional) quantum data types. Indeed, to support this thesis, we will show that $\mathbf{vNA}_{\mathrm{CPsU}}^{\mathrm{op}}$ gives a model of the quantum lambda calculus.

Let us end this section with the definitions that are necessary to understand $\mathbf{vNA}_{\mathrm{CPsU}}$. An element a of a von Neumann algebra $\mathscr{A}$ is *self-adjoint* if $a^* = a$, and *positive* if $a \equiv b^*b$ for some $b \in \mathscr{A}$. The self-adjoint elements of a von Neumann algebra $\mathscr{A}$ are partially ordered by: $a \leq b$ iff $b - a$ is positive. Any upwards directed bounded subset D of self-adjoint elements of a von Neumann algebra $\mathscr{A}$ has a supremum $\bigvee D$ in the set of self-adjoint elements of $\mathscr{A}$ [16, Lem. 5.1.4]. (So a von Neumann algebra resembles a domain.)

The linear maps between von Neumann algebras which preserve the multiplication, involution, $(-)^*$, and unit, 1, are called unital $*$-homomorphisms in the literature and *MIU-maps* by us. A linear map f between von Neumann algebras is *positive* if it maps positive elements to positive elements, *unital* if it preserves the unit, *subunital* if $f(1) \leq 1$, and *normal* if f is positive and preserves suprema of bounded directed sets of self-adjoint elements. (If subunital maps are akin to partial maps between sets, then the unital maps are the total

$$\textit{Type}\quad A,B ::= \texttt{qbit} \mid \top \mid !A \mid A \multimap B \mid A\otimes B \mid A\oplus B$$
$$\textit{Term}\quad M,N,L ::= x^A \mid \texttt{new}^A \mid \texttt{meas}^A \mid U^A \mid \lambda^n x^A.M \mid MN \mid *^n \mid \texttt{let}\ \langle x^A,y^B\rangle^n = N\ \texttt{in}\ M$$
$$\mid \langle M,N\rangle^n \mid \texttt{inl}^n_{A,B}(M) \mid \texttt{inr}^n_{A,B}(N) \mid \texttt{match}\ L\ \texttt{with}^n(x^A\mapsto M \mid y^B \mapsto N)$$
$$\textit{Value}\quad V,W ::= x^A \mid \texttt{new}^A \mid \texttt{meas}^A \mid U^A \mid *^n \mid \lambda^n x^A.M \mid \langle V,W\rangle^n \mid \texttt{inl}^n_{A,B}(V) \mid \texttt{inr}^n_{A,B}(W)$$

Table 1 Types, terms and values of the quantum lambda calculus

maps. Normality is the incarnation of Scott continuity in this setting, and coincides with continuity with respect to the σ-weak = ultraweak = weak* topology [32, Th. 1.13.2].)

Given a von Neumann algebra $\mathscr{A}$ on a Hilbert space $\mathscr{H}$, and a von Neumann algebra $\mathscr{B}$ on a Hilbert space $\mathscr{K}$, the *spatial tensor product* $\mathscr{A}\otimes\mathscr{B}$ of $\mathscr{A}$ and $\mathscr{B}$ is the least von Neumann algebra on $\mathscr{H}\otimes\mathscr{K}$ which contains all operators of the form $a\otimes b$ where $(a\otimes b)(x,y) = a(x)\otimes b(y)$ for all $a\in\mathscr{A}$, $b\in\mathscr{B}$, $x\in\mathscr{H}$ and $y\in\mathscr{K}$ [15, §11.2]. (The tensor product $\mathscr{A}\otimes\mathscr{B}$ may be physically interpreted as the composition of the systems $\mathscr{A}$ and $\mathscr{B}$—recall that a register of two qubits is represented by the von Neumann algebra $\mathcal{M}_2\otimes\mathcal{M}_2$.)

Given normal positive $f\colon\mathscr{A}\to\mathscr{B}$ and $g\colon\mathscr{C}\to\mathscr{D}$ there might be a normal positive linear map $f\otimes g\colon\mathscr{A}\otimes\mathscr{C}\to\mathscr{B}\otimes\mathscr{D}$ given by $(f\otimes g)(a,c) = f(a)\otimes g(c)$. An interesting, and annoying, phenomenon is that such $f\otimes g$ need not exist for all f and g. This warrants the following definition: if $f\colon\mathscr{A}\to\mathscr{B}$ is a positive linear map such that for every natural number n the map $\mathcal{M}_n(f)\colon\mathcal{M}_n(\mathscr{A})\to\mathcal{M}_n(\mathscr{B})$ is positive, then f is called *completely positive* [29]. Here $\mathcal{M}_n(\mathscr{A})$ is the von Neumann algebra of $n\times n$ matrices with entries drawn from $\mathscr{A}$, and $\mathcal{M}_n(f)(A)_{ij} = f(A_{ij})$ for all i,j and $A\in\mathcal{M}_n(\mathscr{A})$. If f and g are normal and completely positive, then $f\otimes g$ exists, and is completely positive [41, Prop. IV/5.13].

3 The Quantum Lambda Calculus and its Operational Semantics

We review the quantum lambda calculus for which we will give a denotational semantics. The language and its operational semantics are basically the same as Selinger and Valiron's ones [37], but with sum type $\oplus$ [40] and 'indexed' terms [38], see Remark 1 and Notation 2 below. For space reasons we omit many details, and refer to [37, 38, 40].

3.1 Syntax and Typing Rules

The language consists of *types*, *terms* and *values* defined in Table 1. We use obvious shorthand $!^n A = !\cdots!A$ and $A^{\otimes n} = A\otimes\cdots\otimes A$. The *subtyping relation* $<:$ on types is defined by the rules shown in Table 2(a). In the definition of terms and values, $n\in\mathbb{N}$ is a natural number; x ranges over variables; and U ranges over $2^k\times 2^k$ unitary matrices for $k\geq 1$. The (nullary) constructors $\texttt{new}, \texttt{meas}, U$ are called *constants* and sometimes referred to by c. Clearly, values form a subclass of terms. As usual, we identify terms up to α-equivalence.

▶ **Remark 1.** The terms are *indexed terms* of [38], which have explicit type annotations (cf. Church-style vs. Curry-style in the simply-typed lambda calculus). A typing derivation for an indexed term is unique in a suitable sense, so that we can more easily obtain Lemma 12. In fact, for the language of [38] we can safely remove the type annotations [38, Corollary 1]. We conjecture that the same is true for our language, which is left as a future work.

▶ **Notation 2.** Following [40] (and [10, 27]), the language has sum type $\oplus$ instead of the bit type (which exists in [37]). The bit type and its constructors are emulated by $\texttt{bit} := \top\oplus\top$; $\texttt{ff}^n := \texttt{inl}^n_{\top,\top}(*^n)$; $\texttt{tt}^n := \texttt{inr}^n_{\top,\top}(*^n)$; and $\texttt{if}\ L\ \texttt{then}\ M\ \texttt{else}\ N := \texttt{match}\ L\ \texttt{with}^0(x^\top\mapsto M \mid y^\top\mapsto N)$, with fresh variables x,y.

$$\frac{}{!^n\texttt{qbit} <: !^m\texttt{qbit}} \qquad \frac{}{!^n\top <: !^m\top} \qquad \frac{A_1 <: B_1 \qquad A_2 <: B_2}{!^n(A_2 \multimap B_1) <: !^m(A_1 \multimap B_2)}$$

$$\frac{A_1 <: B_1 \qquad A_2 <: B_2}{!^n(A_1 \otimes A_2) <: !^m(B_1 \otimes B_2)} \qquad \frac{A_1 <: B_1 \qquad A_2 <: B_2}{!^n(A_1 \oplus A_2) <: !^m(B_1 \oplus B_2)}$$

(a) Rules for subtyping, with a condition $(n = 0 \Rightarrow m = 0)$ for each rule

$$\frac{\Delta, x : A, y : B, \Gamma \rhd M : C}{\Delta, y : B, x : A, \Gamma \rhd M : C}\,(ex) \qquad \frac{A <: B}{\Delta, x : A \rhd x^B : B}\,(ax_1) \qquad \frac{!A_c <: B}{\Delta \rhd c^B : B}\,(ax_2)$$

$$\frac{\Delta, x : A \rhd M : B}{\Delta \rhd \lambda^0 x^A.M : A \multimap B}\,(\multimap.I_1) \qquad \frac{\Gamma, !\Delta, x : A \rhd M : B \qquad \mathrm{FV}(M) \cap |\Gamma| = \varnothing}{\Gamma, !\Delta \rhd \lambda^{n+1} x^A.M : !^{n+1}(A \multimap B)}\,(\multimap.I_2)$$

$$\frac{!\Delta, \Gamma_1 \rhd M : A \multimap B \qquad !\Delta, \Gamma_2 \rhd N : A}{!\Delta, \Gamma_1, \Gamma_2 \rhd MN : B}\,(\multimap.E)$$

$$\frac{}{\Delta \rhd *^n : !^n\top}\,(\top) \qquad \frac{!\Delta, \Gamma_1 \rhd M : !^n A \qquad !\Delta, \Gamma_2 \rhd N : !^n B}{!\Delta, \Gamma_1, \Gamma_2 \rhd \langle M, N\rangle^n : !^n(A \otimes B)}\,(\otimes.I)$$

$$\frac{!\Delta, \Gamma_1, x : !^n A, y : !^n B \rhd M : C \qquad !\Delta, \Gamma_2 \rhd N : !^n(A \otimes B)}{!\Delta, \Gamma_1, \Gamma_2 \rhd \texttt{let}\ \langle x^A, y^B\rangle^n = N\ \texttt{in}\ M : C}\,(\otimes.E)$$

$$\frac{\Delta \rhd M : !^n A}{\Delta \rhd \texttt{inl}^n_{A,B}(M) : !^n(A \oplus B)}\,(\oplus.I_1) \qquad \frac{\Delta \rhd N : !^n B}{\Delta \rhd \texttt{inr}^n_{A,B}(N) : !^n(A \oplus B)}\,(\oplus.I_2)$$

$$\frac{!\Delta, \Gamma_1, x : !^n A \rhd M : C \qquad !\Delta, \Gamma_1, y : !^n B \rhd N : C \qquad !\Delta, \Gamma_2 \rhd L : !^n(A \oplus B)}{!\Delta, \Gamma_1, \Gamma_2 \rhd \texttt{match}\ L\ \texttt{with}^n (x^A \mapsto M \mid y^B \mapsto N) : C}\,(\oplus.E)$$

(b) Typing rules

Table 2 Subtyping relation and typing rules

The set $\mathrm{FV}(M)$ of *free variables* is defined in the usual way. A *context* is a list $\Delta = x_1 : A_1, \ldots, x_n : A_n$ of variables x_i and types A_i where the variables x_i are distinct. We write $|\Delta| = \{x_1, \ldots, x_n\}$ and $!\Delta = x_1 : !A_1, \ldots, x_n : !A_n$. We also write $\Delta|_M = \Delta \cap \mathrm{FV}(M)$ for the context restricted to the free variables of M.

A *typing judgement*, written as $\Delta \rhd M : A$, consists of a context Δ, a term M and a type A. A typing judgement is *valid* if it can be derived by the *typing rules* shown in Table 2(b). In the rule (ax_2), c ranges over `new`, `meas` and $2^k \times 2^k$ unitary matrices U; and the types A_c are defined as follows: $A_{\texttt{new}} = \texttt{bit} \multimap \texttt{qbit}$, $A_{\texttt{meas}} = \texttt{qbit} \multimap !\texttt{bit}$, $A_U = \texttt{qbit}^{\otimes k} \multimap \texttt{qbit}^{\otimes k}$.

The type system is affine (weak linear). Each variable may occur at most once, unless it has a duplicable type $!A$. Substitution of the following form is admissible.

▶ **Lemma 3** (Substitution). *If* $!\Delta, \Gamma_1, x : A \rhd M : B$ *and* $!\Delta, \Gamma_2 \rhd V : A$*, where* V *is a value and* $|\Gamma_1| \cap |\Gamma_2| = \varnothing$*, then* $!\Delta, \Gamma_1, \Gamma_2 \rhd M[V/x] : B$*.* ◀

Note, however, that we need to define the substitution $M[V/x]$ with care. For example, if $A <: A'$, $M = y^{A' \multimap B} x^{A'}$ and $V = z^A$, then we substitute $z^{A'}$ (not z^A) for $x^{A'}$ in M. See [38, §2.5] or [42, §9.1.4] for details.

3.2 Operational Semantics

The operational semantics is taken from [37, 40], but is adapted for indexed terms.

▶ **Definition 4.** A *quantum closure* is a triple $[\,|\psi\rangle, \Psi, M\,]$ with $m \in \mathbb{N}$ where:

- $|\psi\rangle$ is a normalised vector of the Hilbert space $(\mathbb{C}^2)^{\otimes m} \cong \mathbb{C}^{2^m}$.

$$[\,|\psi\rangle, \Psi, (\lambda^0 x^A.M)V\,] \to_1 [\,|\psi\rangle, \Psi, M[V/x]\,] \qquad (\multimap)$$
$$[\,|\psi\rangle, \Psi, \mathtt{let}\ \langle x^A, y^B\rangle^n = \langle V, W\rangle^n\ \mathtt{in}\ M\,] \to_1 [\,|\psi\rangle, \Psi, M[V/x, W/y]\,] \qquad (\otimes)$$
$$[\,|\psi\rangle, \Psi, \mathtt{match}\ \mathtt{inl}^n_{A,B}(V)\ \mathtt{with}^n(x^A \mapsto M \mid y^B \mapsto N)\,] \to_1 [\,|\psi\rangle, \Psi, M[V/x]\,] \qquad (\oplus_1)$$
$$[\,|\psi\rangle, \Psi, \mathtt{match}\ \mathtt{inr}^n_{A,B}(W)\ \mathtt{with}^n(x^A \mapsto M \mid y^B \mapsto N)\,] \to_1 [\,|\psi\rangle, \Psi, N[W/y]\,] \qquad (\oplus_2)$$

(a) Classical control

$$[\,|\psi\rangle, \Psi, U^{\mathtt{qbit}^{\otimes k} \multimap \mathtt{qbit}^{\otimes k}} \langle x_1^{\mathtt{qbit}}, \dots, x_k^{\mathtt{qbit}}\rangle^0\,] \to_1 [\,|\psi'\rangle, \Psi, \langle x_1^{\mathtt{qbit}}, \dots, x_k^{\mathtt{qbit}}\rangle^0\,] \qquad (U)$$
$$[\,|\psi\rangle, |x_1 \dots x_m\rangle, \mathtt{meas}^{\mathtt{qbit} \multimap !^n \mathtt{bit}}\ x_i^{\mathtt{qbit}}\,] \to_{p_0} [\,|\psi_0\rangle, |x_1 \dots x_m\rangle, \mathtt{ff}^n\,] \qquad (\mathtt{meas}_0)$$
$$[\,|\psi\rangle, |x_1 \dots x_m\rangle, \mathtt{meas}^{\mathtt{qbit} \multimap !^n \mathtt{bit}}\ x_i^{\mathtt{qbit}}\,] \to_{p_1} [\,|\psi_1\rangle, |x_1 \dots x_m\rangle, \mathtt{tt}^n\,] \qquad (\mathtt{meas}_1)$$
$$[\,|\psi\rangle, |x_1 \dots x_m\rangle, \mathtt{new}^{A \multimap \mathtt{qbit}}\ \tilde{\mathtt{ff}}\,] \to_1 [\,|\psi\rangle|0\rangle, |x_1 \dots x_m y\rangle, y^{\mathtt{qbit}}\,] \qquad (\mathtt{new}_0)$$
$$[\,|\psi\rangle, |x_1 \dots x_m\rangle, \mathtt{new}^{A \multimap \mathtt{qbit}}\ \tilde{\mathtt{tt}}\,] \to_1 [\,|\psi\rangle|1\rangle, |x_1 \dots x_m y\rangle, y^{\mathtt{qbit}}\,] \qquad (\mathtt{new}_1)$$

(b) Quantum data

If $[\,|\psi\rangle, \Psi, M\,] \to_p [\,|\psi'\rangle, \Psi', M'\,]$, the following are valid reductions (if well-formed).

$$[\,|\psi\rangle, \Psi, MN\,] \to_p [\,|\psi'\rangle, \Psi', M'N\,] \qquad [\,|\psi\rangle, \Psi, VM\,] \to_p [\,|\psi'\rangle, \Psi', VM'\,]$$
$$[\,|\psi\rangle, \Psi, \langle M, N\rangle^n\,] \to_p [\,|\psi'\rangle, \Psi', \langle M', N\rangle^n\,] \qquad [\,|\psi\rangle, \Psi, \langle V, M\rangle^n\,] \to_p [\,|\psi'\rangle, \Psi', \langle V, M'\rangle^n\,]$$
$$[\,|\psi\rangle, \Psi, \mathtt{let}\ \langle x^A, y^B\rangle^n = M\ \mathtt{in}\ N\,] \to_p [\,|\psi'\rangle, \Psi', \mathtt{let}\ \langle x^A, y^B\rangle^n = M'\ \mathtt{in}\ N\,]$$
$$[\,|\psi\rangle, \Psi, \mathtt{inl}^n_{A,B}(M)\,] \to_p [\,|\psi'\rangle, \Psi', \mathtt{inl}^n_{A,B}(M')\,]$$
$$[\,|\psi\rangle, \Psi, \mathtt{inr}^n_{A,B}(M)\,] \to_p [\,|\psi'\rangle, \Psi', \mathtt{inr}^n_{A,B}(M')\,]$$
$$[\,|\psi\rangle, \Psi, \mathtt{match}\ M\ \mathtt{with}^n(x^A \mapsto N \mid y^B \mapsto L)\,] \to_p [\,|\psi'\rangle, \Psi', \mathtt{match}\ M'\ \mathtt{with}^n(x^A \mapsto N \mid y^B \mapsto L)\,]$$

(c) Congruence rules

Table 3 Reduction rules

- Ψ is a list of m distinct variables, written as $|x_1 \dots x_m\rangle$. We write $|\Psi| = \{x_1, \dots, x_m\}$, and $\Psi(x_i) = i$ for the position of a variable in the list.
- M is a term with $\mathrm{FV}(M) \subseteq |\Psi|$.

We say a quantum closure $P = [\,|\psi\rangle, |x_1 \dots x_m\rangle, M\,]$ is *well-typed of type* A, written as $P : A$, if the typing judgement $x_1 : \mathtt{qbit}, \dots, x_m : \mathtt{qbit} \rhd M : A$ is valid. We call $[\,|\psi\rangle, \Psi, V\,]$ a *value closure* if V is a value.

▶ **Definition 5.** A (small-step) *reduction* $P \to_p Q$ consists of quantum closures P, Q and $p \in [0, 1]$, meaning that P reduces to Q with probability p. The valid reductions $P \to_p Q$ are given inductively by the *reduction rules* shown in Table 3. In the rules, V and W refer to values. The 'quantum data' rules (b) correspond to the three basic operations explained in §2. In the rule (U), $|\psi'\rangle$ is the state obtained by applying the $2^k \times 2^k$ unitary matrix U to the k qubits of the position $\Psi(x_1), \dots, \Psi(x_k)$ in $|\psi\rangle$. In the rule $(\mathtt{meas}_0)$, p_0 is the probability that we obtain 0 ('negative' in terms of §2) by measuring the i-th qubit of $|\psi\rangle$; and $|\psi_0\rangle$ is the state after that. The rule $(\mathtt{meas}_1)$ is similar. In the rule $(\mathtt{new}_0)$, we denote by $\tilde{\mathtt{ff}}$ any term of the form $\mathtt{inl}^n_{!^k\top, !^k\top}(*^{n+k})$ (cf. Notation 2). The term $\tilde{\mathtt{tt}}$ in $(\mathtt{new}_1)$ is similar.

Reduction satisfies the following properties.

▶ **Lemma 6** (Subject reduction). *If $P : A$ and $P \to_p Q$, then $Q : A$.* ◀

▶ **Lemma 7** (Progress). *Let $P : A$ be a well-typed quantum closure. Then either P is a value closure, or there exists a quantum closure Q such that $P \to_p Q$. In the latter case, there are* *at most two (up to α-equivalence) single-step reductions from P, and the total probability of all the single-step reductions from P is 1.* ◀

The next definitions follow [37, 39].

▶ **Definition 8.** We define the *small-step reduction probability* $\mathrm{prob}(P,Q) \in [0,1]$ for well-typed quantum closures P,Q by: $\mathrm{prob}(P,Q) = p$ if $P \to_p Q$; $\mathrm{prob}(V,V) = 1$ if V is a value closure; $\mathrm{prob}(P,Q) = 0$ otherwise. Lemma 7 guarantees that prob is a probabilistic system in a suitable sense. For a well-typed quantum closure P and a well-typed value closure Z, the *big-step reduction probability* $\mathrm{Prob}(P,Z) \in [0,1]$ is defined by $\mathrm{Prob}(P,Z) = \lim_{n\to\infty} \mathrm{prob}^n(P,Z)$, where $\mathrm{prob}^1(P,Z) = \mathrm{prob}(P,Z)$ and $\mathrm{prob}^{n+1}(P,Z) = \sum_Q \mathrm{prob}(P,Q)\,\mathrm{prob}^n(Q,Z)$.

▶ **Definition 9.** For each $b \in \{\mathtt{ff}^0, \mathtt{tt}^0\}$, we define $P \Downarrow b = \sum_{Z\in U_b} \mathrm{Prob}(P,Z)$, where U_b is the set of well-typed quantum closures of the form $[\,|\psi\rangle, \Psi, b\,]$.

We will use a strong normalisation result. The proof is similar to [27, Lemma 33].

▶ **Lemma 10** (Strong normalisation). *Let $P : A$ be a well-typed quantum closure. Then there is no infinite sequence of reductions $P \to_{p_1} P_1 \to_{p_2} P_2 \to_{p_3} \cdots$.*

Proof (Sketch). Clearly it suffices to prove the strong normalisation for the underlying (non-deterministic) reductions $M \to N$ on terms. We add a constant $c^{\mathtt{qbit}}$ to replace free variables $x^{\mathtt{qbit}}$. We then define a translation $(-)^\dagger$ from the quantum lambda calculus (with $c^{\mathtt{qbit}}$) to a simply-typed lambda calculus with product, unit, sum types and constants $\mathtt{new}, \mathtt{meas}, U, c^{\mathtt{qbit}}$. The translation $(-)^\dagger$ forgets the ! modality, and translates the let constructor via $(\mathtt{let}\ \langle x,y\rangle = N\ \mathtt{in}\ M)^\dagger = (\lambda z.(\lambda x.\lambda y.M^\dagger)\,\mathtt{fst}(z)\,\mathtt{snd}(z))N^\dagger$. We can prove the strong normalisation for the simply-typed lambda calculus via standard techniques. ◀

4 Denotational Semantics Using von Neumann Algebras

4.1 Facts about von Neumann Algebras

We need the following notation and facts concerning von Neumann algebras. Those facts for which we could not find proof in the literature will be discussed in the next section.

Let $(\mathbf{vNA}_{\mathrm{MIU}}, \otimes, \mathbb{C})$ be the symmetric monoidal category (SMC) of von Neumann algebras and normal MIU-maps [18, Prop. 7.2], and $(\mathbf{vNA}_{\mathrm{CPsU}}, \otimes, \mathbb{C})$ the SMC of von Neumann algebras and normal CPsU-maps (where $\otimes$ is the spatial tensor product) [4]. Note that the unit $\mathbb{C}$ is initial in $\mathbf{vNA}_{\mathrm{MIU}}$ (but not in $\mathbf{vNA}_{\mathrm{CPsU}}$). Both categories have products given by direct sums $\oplus$ (with the supremum norm [41, Def. 3.4]). To interpret the quantum lambda calculus, we will use the following pair of (lax) symmetric monoidal adjunctions,

$$(\mathbf{Set}^{\mathrm{op}}, \times, 1) \underset{\ell^\infty}{\overset{\mathrm{sp}}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} (\mathbf{vNA}_{\mathrm{MIU}}, \otimes, \mathbb{C}) \underset{\mathcal{J}}{\overset{\mathcal{F}}{\underset{\hookrightarrow}{\overset{\longleftarrow}{\bot}}}} (\mathbf{vNA}_{\mathrm{CPsU}}, \otimes, \mathbb{C}) \qquad (1)$$

where $\mathbf{Set}^{\mathrm{op}}$ is the opposite of the category $\mathbf{Set}$ of sets and functions, considered as a SMC via cartesian products (i.e. coproducts in $\mathbf{Set}^{\mathrm{op}}$). The functor $\mathcal{J}$ is the inclusion functor; the other functors are explained in the next section. Note that $\mathcal{J}$ is strict symmetric monoidal and strictly preserves products. The following facts are important:

- $\mathbf{vNA}_{\mathrm{MIU}}$ is a co-closed SMC. This means the endofunctor $(-)\otimes\mathscr{A}$ on $\mathbf{vNA}_{\mathrm{MIU}}$ has a left adjoint $(-)^{*\mathscr{A}}$. The von Neumann algebra $\mathscr{B}^{*\mathscr{A}}$ is called the *free exponential* in [18].

- The counit of the adjunction $\mathrm{sp} \dashv \ell^\infty$ is an isomorphism (see Corollary 21).
- The functors sp, ℓ^∞ and the adjunction $\mathrm{sp} \dashv \ell^\infty$ are *strong* monoidal (see Corollary 23).
- Moreover, the functors sp and ℓ^∞ preserves products (see Cor. 20 and Lem. 22).

The tensor product $\otimes$ distributes over products $\oplus$ in $\mathbf{vNA}_{\mathrm{MIU}}$, as $\mathscr{A} \otimes (\mathscr{B} \oplus \mathscr{C}) \cong (\mathscr{A} \otimes \mathscr{B}) \oplus (\mathscr{A} \otimes \mathscr{C})$, since $\mathscr{A} \otimes (-)$ is a right adjoint and thus preserves products. We denote the canonical isomorphism by $\theta_{\mathscr{A},\mathscr{B},\mathscr{C}} \colon (\mathscr{A} \otimes \mathscr{B}) \oplus (\mathscr{A} \otimes \mathscr{C}) \to \mathscr{A} \otimes (\mathscr{B} \oplus \mathscr{C})$.

We define a 'Kleisli co-exponential' $\multimap$ by $\mathscr{A} \multimap \mathscr{B} := (\mathcal{F}\mathscr{B})^{*\mathscr{A}}$. We have the bijective correspondence as shown on the right. We write $\Lambda f = g$ for the MIU-map $\mathscr{A} \multimap \mathscr{B} \to \mathscr{C}$ corresponding to f. We also write $\varepsilon_{\mathscr{A},\mathscr{B}} = \Lambda^{-1}\mathrm{id} \colon \mathscr{B} \to (\mathscr{A} \multimap \mathscr{B}) \otimes \mathscr{A}$ for the co-evaluation map, i.e. the CPsU-map corresponding to $\mathrm{id} \colon \mathscr{A} \multimap \mathscr{B} \to \mathscr{A} \multimap \mathscr{B}$. Then $(\Lambda f \otimes \mathrm{id}) \circ \varepsilon = f$ by the naturality of the bijective correspondence.

$$\cfrac{\cfrac{\mathscr{B} \xrightarrow{f} \mathscr{C} \otimes \mathscr{A} \text{ in } \mathbf{vNA}_{\mathrm{CPsU}}}{\mathcal{F}\mathscr{B} \longrightarrow \mathscr{C} \otimes \mathscr{A} \text{ in } \mathbf{vNA}_{\mathrm{MIU}}}}{(\mathcal{F}\mathscr{B})^{*\mathscr{A}} = \mathscr{A} \multimap \mathscr{B} \xrightarrow{g} \mathscr{C} \text{ in } \mathbf{vNA}_{\mathrm{MIU}}}$$

We write $\mathcal{L} = \ell^\infty \circ \mathrm{sp}$ for the strong symmetric monoidal monad on $\mathbf{vNA}_{\mathrm{MIU}}$ induced by the left-hand adjunction of (1). The unit and multiplication are denoted by η and μ respectively. From the fact that the counit of $\mathrm{sp} \dashv \ell^\infty$ is an isomorphism, it easily follows that $\mathcal{L}$ is an idempotent monad, i.e. the multiplication $\mu \colon \mathcal{L}^2 \Rightarrow \mathcal{L}$ is an isomorphism. Note also that $\mathcal{L}$ preserves products. We denote the structure isomorphisms by: $d^{\mathcal{L}}_{\mathbb{C}} \colon \mathbb{C} \to \mathcal{L}\mathbb{C}$; $d^{\mathcal{L}}_{\mathscr{A},\mathscr{B}} \colon \mathcal{L}\mathscr{A} \otimes \mathcal{L}\mathscr{B} \to \mathcal{L}(\mathscr{A} \otimes \mathscr{B})$; and $e^{\mathcal{L}}_{\mathscr{A},\mathscr{B}} \colon \mathcal{L}\mathscr{A} \oplus \mathcal{L}\mathscr{B} \to \mathcal{L}(\mathscr{A} \oplus \mathscr{B})$.

Because the adjunction $\mathrm{sp} \dashv \ell^\infty$ satisfies a dual condition to a linear-non-linear model [1] (see also [23,33]), the monad $\mathcal{L}$ has a property which is dual to a *linear exponential comonad.* Thus each object of the form $\mathcal{L}\mathscr{A}$ is equipped with a map $\triangledown_{\mathscr{A}} \colon \mathcal{L}\mathscr{A} \otimes \mathcal{L}\mathscr{A} \to \mathcal{L}\mathscr{A}$ which, with a unique map $\mathsf{i}_{\mathcal{L}\mathscr{A}} \colon \mathbb{C} \to \mathcal{L}\mathscr{A}$, makes $\mathcal{L}\mathscr{A}$ into a $\otimes$-monoid in $\mathbf{vNA}_{\mathrm{MIU}}$.

▶ **Remark 11.** One can summarise these facts by saying that the opposite $\mathbf{vNA}^{\mathrm{op}}_{\mathrm{MIU}}$ is a *(weak) linear category for duplication* [38, 40]; and moreover $\mathbf{vNA}^{\mathrm{op}}_{\mathrm{MIU}}$ is a *concrete model of the quantum lambda calculus* defined by Selinger and Valiron [40, §1.6.8]. Although they gave the definition of concrete models of the quantum lambda calculus, results on them (e.g. how to interpret the quantum lambda calculus; adequacy of models) have never been given. In the remainder of the section, therefore, we will give the interpretation of the language in von Neumann algebras concretely, and then prove its adequacy.

4.2 The Interpretation of Types and Typing Judgements

We interpret types as von Neumann algebras, i.e. objects in $\mathbf{vNA}_{\mathrm{MIU}}$ / $\mathbf{vNA}_{\mathrm{CPsU}}$, as follows.

$$[\![\mathtt{qbit}]\!] = \mathcal{M}_2 \qquad [\![\top]\!] = \mathbb{C} \qquad [\![!A]\!] = \mathcal{L}[\![A]\!]$$
$$[\![A \multimap B]\!] = [\![A]\!] \multimap [\![B]\!] \qquad [\![A \otimes B]\!] = [\![A]\!] \otimes [\![B]\!] \qquad [\![A \oplus B]\!] = [\![A]\!] \oplus [\![B]\!]$$

▶ Remark. One familiar with Fock space might be surprised to realise that $[\![!\mathtt{qbit}]\!] = \{0\}$, because there is no normal MIU-map $\varphi \colon \mathcal{M}_2 \to \mathbb{C}$. The intuition here may be that no part of a qubit can be duplicated, and so the assumption of a duplicable qubit amounts to nothing. This is also the interpretation of !qbit intended by Selinger and Valiron, see [38, §5].

▶ Remark. The interpretation of a function type $A \multimap B$ is obtained by abstract means, and at this point we know very little about it. (Might it be as intangible as an ultrafilter?) However, applying ! makes the function type almost trivial: after §4, it will be clear that

$$[\![!(A \multimap B)]\!] \;=\; \ell^\infty(\{\, f \colon [\![B]\!] \overset{\mathrm{CPsU}}{\longrightarrow} [\![A]\!] \,\}).$$

The interpretation of the subtyping relation $A <: B$ is a 'canonical' map $[\![B]\!] \to [\![A]\!]$ in $\mathbf{vNA}_{\mathrm{MIU}}$, which exists uniquely by a coherence property for an idempotent (co)monad; see [42, §8.3.2] for details. For instance, we have $[\![A \multimap !B <: !A \multimap !!B]\!] = \eta_{[\![A]\!]} \multimap \mu_{[\![B]\!]}$.

Contexts $\Delta = x_1 : A_1, \ldots, x_n : A_n$ are interpreted as $[\![\Delta]\!] = [\![A_1]\!] \otimes \cdots \otimes [\![A_n]\!]$. We shall treat the monoidal structure $(\otimes, \mathbb{C})$ as if it were strict monoidal, which is justified by the coherence theorem for monoidal categories.

The interpretations $[\![\mathtt{new}]\!]$, $[\![\mathtt{meas}]\!]$ and $[\![U]\!]$ of constants are defined using the maps $f_{\mathtt{new}} \colon \mathcal{M}_2 \to \mathbb{C}^2$, $f_{\mathtt{meas}} \colon \mathbb{C}^2 \to \mathcal{M}_2$ and $f_U \colon \mathcal{M}_2^{\otimes k} \to \mathcal{M}_2^{\otimes k}$ given in §2, as follows.

$$\begin{aligned}
[\![\mathtt{new}]\!] &= \eta_{\mathbb{C}}^{-1} \circ \mathcal{L}\Lambda f_{\mathtt{new}} \colon [\![!A_{\mathtt{new}}]\!] = \mathcal{L}(\mathbb{C}^2 \multimap \mathcal{M}_2) \longrightarrow \mathbb{C} \\
[\![\mathtt{meas}]\!] &= \eta_{\mathbb{C}}^{-1} \circ \mathcal{L}\Lambda(f_{\mathtt{meas}} \circ \eta_{\mathbb{C}^2}^{-1}) \colon [\![!A_{\mathtt{meas}}]\!] = \mathcal{L}(\mathcal{M}_2 \multimap \mathcal{L}\mathbb{C}^2) \longrightarrow \mathbb{C} \\
[\![U]\!] &= \eta_{\mathbb{C}}^{-1} \circ \mathcal{L}\Lambda f_U \colon [\![!A_U]\!] = \mathcal{L}(\mathcal{M}_2^{\otimes k} \multimap \mathcal{M}_2^{\otimes k}) \longrightarrow \mathbb{C}
\end{aligned}$$

We now give the interpretation $[\![\Delta \rhd M : A]\!]$ of a typing judgement as a map $[\![A]\!] \to [\![\Delta]\!]$ in $\mathbf{vNA}_{\mathrm{CPsU}}$. The definition is similar to [11]. First we define a normal CPsU-map $[\![\Delta \rhd M : A]\!]^{\mathrm{FV}} \colon [\![A]\!] \to [\![\Delta|_M]\!]$ (recall that $\Delta|_M = \Delta \cap \mathrm{FV}(M)$) by induction on the derivation of the typing judgement as shown in Table 4. We then define $[\![\Delta \rhd M : A]\!] \coloneqq ([\![A]\!] \xrightarrow{[\![\Delta \rhd M:A]\!]^{\mathrm{FV}}} [\![\Delta|_M]\!] \xrightarrow{\iota} [\![\Delta]\!])$. Here and in Table 4, the following notations are used (we often suppress subscripts). We denote the symmetry isomorphism by $\gamma_{\mathscr{A},\mathscr{B}} \colon \mathscr{A} \otimes \mathscr{B} \to \mathscr{B} \otimes \mathscr{A}$. For contexts $\Delta \subseteq \Gamma$, we write $\iota \colon [\![\Delta]\!] \to [\![\Gamma]\!]$ for the 'injection' map defined via unique MIU-maps $\mathfrak{i}_{\mathscr{A}} \colon \mathbb{C} \to \mathscr{A}$. For a context $!\Delta, \Gamma_1, \Gamma_2$, we define the map merge: $[\![!\Delta, \Gamma_1]\!] \otimes [\![!\Delta, \Gamma_2]\!] \to [\![!\Delta, \Gamma_1, \Gamma_2]\!]$ via monoid structures $\triangledown_{[\![A]\!]} \colon [\![!A]\!] \otimes [\![!A]\!] \to [\![!A]\!]$ and symmetry maps γ. We can obtain the map $d^{\mathcal{L}}_{\Delta} \colon [\![!\Delta]\!] \to \mathcal{L}[\![\Delta]\!]$ via $d^{\mathcal{L}}_{\mathscr{A},\mathscr{B}} \colon \mathcal{L}\mathscr{A} \otimes \mathcal{L}\mathscr{B} \to \mathcal{L}(\mathscr{A} \otimes \mathscr{B})$. We write $\mu_\Delta \colon [\![!!\Delta]\!] \to [\![!\Delta]\!]$ for $\mu_{[\![A_1]\!]} \otimes \cdots \otimes \mu_{[\![A_n]\!]}$; $d^{\mathcal{L}^n} \colon \mathcal{L}^n\mathscr{A} \otimes \mathcal{L}^n\mathscr{B} \to \mathcal{L}^n(\mathscr{A} \otimes \mathscr{B})$ for $\mathcal{L}^{n-1}d^{\mathcal{L}} \circ \cdots \circ d^{\mathcal{L}}$; and $e^{\mathcal{L}^n} \colon \mathcal{L}^n\mathscr{A} \oplus \mathcal{L}^n\mathscr{B} \to \mathcal{L}^n(\mathscr{A} \oplus \mathscr{B})$ similarly. Projection maps and tupling for direct sums, products in $\mathbf{vNA}_{\mathrm{CPsU}}$, are denoted by $\pi_i \colon \mathscr{A}_1 \oplus \mathscr{A}_2 \to \mathscr{A}_i$ and $\langle f, g \rangle \colon \mathscr{A} \to \mathscr{B} \oplus \mathscr{C}$.

Note that the interpretation $[\![\Delta \rhd M : A]\!]$ is defined by induction on typing derivations. Because we use indexed terms, it is not hard to prove the following fact by induction on a typing derivation Π.

▶ **Lemma 12.** *Suppose that $\Delta \rhd M : A$ is valid with a derivation Π, and so is $\Delta' \rhd M : A$ with Π'. Then $[\![\Pi']\!]^{\mathrm{FV}} = \sigma \circ [\![\Pi]\!]^{\mathrm{FV}}$, where $\sigma \colon [\![\Delta|_M]\!] \to [\![\Delta'|_M]\!]$ is a (unique by coherence) isomorphism that permutes $\Delta|_M$ to $\Delta'|_M$. In particular, $[\![\Delta \rhd M : A]\!]$ is well-defined, not depending on derivations.* ◀

Let $[\,|\psi\rangle, |x_1 \ldots x_n\rangle, M\,] : A$ be a well-typed quantum closure. The mapping $A \mapsto \langle\psi|A|\psi\rangle$ defines a normal CPU-map $\langle\psi|-|\psi\rangle \colon \mathcal{M}_2^{\otimes m} \to \mathbb{C}$. The interpretation of the quantum closure is defined by:

$$[\![[\,|\psi\rangle, |x_1 \ldots x_n\rangle, M\,] : A]\!] \;\coloneqq\; [\![A]\!] \xrightarrow{[\![x_1:\mathtt{qbit},\ldots,x_n:\mathtt{qbit} \rhd M:A]\!]} \mathcal{M}_2^{\otimes n} \xrightarrow{\langle\psi|-|\psi\rangle} \mathbb{C}$$

4.3 Adequacy of the Denotational Semantics

The next soundness/invariance for the small-step reduction is a key result to obtain adequacy. Note that for normal CPsU-maps $f_1, \ldots, f_n \colon \mathscr{A} \to \mathscr{B}$ and $r_i \in [0,1]$ with $\sum_i r_i \leq 1$, the (convex) sum $\sum_i r_i f_i$ of maps is defined in the obvious pointwise manner and is a normal CPsU-map.

▶ **Proposition 13** (Soundness for the small-step reduction)**.** *Let $P : A$ be a well-typed quantum closure. Then $[\![P : A]\!] = \sum_Q \mathrm{prob}(P, Q)[\![Q : A]\!]$.*

$$\frac{[\![\Delta, x : A, y : B, \Gamma \rhd M : C]\!]^{\mathrm{FV}} = [\![C]\!] \xrightarrow{f} [\![(\Delta, x : A, y : B, \Gamma)|_M]\!]}{[\![\Delta, y : B, x : A, \Gamma \rhd M : C]\!]^{\mathrm{FV}} = (\mathrm{id}_{[\![\Delta|_M]\!]} \otimes \gamma \otimes \mathrm{id}_{[\![\Gamma|_M]\!]}) \circ f \;\; (\text{if } x, y \in \mathrm{FV}(M));\;\; f \;\; (\text{otherwise})}$$

$$[\![\Delta, x : A \rhd x^B : B]\!]^{\mathrm{FV}} = [\![B]\!] \xrightarrow{[\![A <: B]\!]} [\![A]\!] \qquad [\![\Delta \rhd c^B : B]\!]^{\mathrm{FV}} = [\![B]\!] \xrightarrow{[\![!A_c <: B]\!]} [\![!A_c]\!] \xrightarrow{[\![c]\!]} \mathbb{C}$$

$$[\![\Delta \rhd *^n : !^n\top]\!]^{\mathrm{FV}} = [\![!^n\top]\!] \xrightarrow{[\![!\top <: !^n\top]\!]} \mathcal{L}\mathbb{C} \xrightarrow{(d^{\mathcal{L}}_{\mathbb{C}})^{-1}} \mathbb{C}$$

$$\frac{[\![\Delta, x : A \rhd M : B]\!]^{\mathrm{FV}} = [\![B]\!] \xrightarrow{f} [\![(\Delta, x : A)|_M]\!]}{[\![\Delta \rhd \lambda^0 x^A.M : A \multimap B]\!]^{\mathrm{FV}} = [\![A]\!] \multimap [\![B]\!] \xrightarrow{\Lambda f'} [\![\Delta|_M]\!]} \quad \text{where:} \quad \begin{array}{ccc} [\![B]\!] & \xrightarrow{f} & [\![(\Delta, x : A)|_M]\!] \\ & \searrow^{f'} & \downarrow \iota \\ & & [\![\Delta|_M]\!] \otimes [\![A]\!] \end{array}$$

$$\frac{[\![\Gamma, !\Delta, x : A \rhd M : B]\!]^{\mathrm{FV}} = [\![B]\!] \xrightarrow{f} [\![(\Gamma, !\Delta, x : A)|_M]\!]}{[\![\Gamma, !\Delta \rhd \lambda^{n+1} x^A.M : A \multimap B]\!]^{\mathrm{FV}} = [\![!^{n+1}(A \multimap B)]\!] \xrightarrow{[\![!(A \multimap B) <: !^{n+1}(A \multimap B)]\!]} \mathcal{L}([\![A]\!] \multimap [\![B]\!])}$$

$$\xrightarrow{\mathcal{L}(\Lambda f')} \mathcal{L}[\![!\Delta|_M]\!] \xrightarrow{(d^{\mathcal{L}})^{-1}} [\![!!\Delta|_M]\!] \xrightarrow{\mu} [\![!\Delta|_M]\!] = [\![(!\Delta, \Gamma)|_M]\!] \quad (f' \text{ defined similarly})$$

$$\frac{\begin{array}{c}[\![!\Delta, \Gamma_1 \rhd M : A \multimap B]\!]^{\mathrm{FV}} = [\![A \multimap B]\!] \xrightarrow{f} [\![(!\Delta, \Gamma_1)|_M]\!] \\ [\![!\Delta, \Gamma_2 \rhd N : A]\!]^{\mathrm{FV}} = [\![A]\!] \xrightarrow{g} [\![(!\Delta, \Gamma_2)|_N]\!]\end{array}}{[\![!\Delta, \Gamma_1, \Gamma_2 \rhd MN : B]\!]^{\mathrm{FV}} = [\![B]\!] \xrightarrow{\varepsilon} [\![A \multimap B]\!] \otimes [\![A]\!] \xrightarrow{f \otimes g} [\![(!\Delta, \Gamma_1)|_M]\!] \otimes [\![(!\Delta, \Gamma_2)|_N]\!]}$$

$$\xrightarrow{\iota \otimes \iota} [\![(!\Delta, \Gamma_1)|_{MN}]\!] \otimes [\![(!\Delta, \Gamma_2)|_{MN}]\!] \xrightarrow{\mathrm{merge}} [\![(!\Delta, \Gamma_1, \Gamma_2)|_{MN}]\!]$$

$$\frac{[\![!\Delta, \Gamma_1 \rhd M : !^n A]\!]^{\mathrm{FV}} = [\![!^n A]\!] \xrightarrow{f} [\![(!\Delta, \Gamma_1)|_M]\!] \qquad [\![!\Delta, \Gamma_2 \rhd N : !^n B]\!]^{\mathrm{FV}} = [\![!^n B]\!] \xrightarrow{g} [\![(!\Delta, \Gamma_2)|_N]\!]}{[\![!\Delta, \Gamma_1, \Gamma_2 \rhd \langle M, N\rangle^n : !^n(A \otimes B)]\!]^{\mathrm{FV}} = [\![!^n(A \otimes B)]\!] \xrightarrow{(d^{\mathcal{L}^n})^{-1}} [\![!^n A]\!] \otimes [\![!^n B]\!] \xrightarrow{f \otimes g}}$$

$$[\![(!\Delta, \Gamma_1)|_M]\!] \otimes [\![(!\Delta, \Gamma_2)|_N]\!] \xrightarrow{\iota \otimes \iota} [\![(!\Delta, \Gamma_1)|_{\langle M, N\rangle}]\!] \otimes [\![(!\Delta, \Gamma_2)|_{\langle M, N\rangle}]\!] \xrightarrow{\mathrm{merge}} [\![(!\Delta, \Gamma_1, \Gamma_2)|_{\langle M, N\rangle}]\!]$$

$$\frac{\begin{array}{c}[\![!\Delta, \Gamma_1, x : !^n A, y : !^n B \rhd M : C]\!]^{\mathrm{FV}} = [\![C]\!] \xrightarrow{f} [\![(!\Delta, \Gamma_1, x : !^n A, y : !^n B)|_M]\!] \\ [\![!\Delta, \Gamma_2 \rhd N : !^n(A \otimes B)]\!]^{\mathrm{FV}} = [\![!^n(A \otimes B)]\!] \xrightarrow{g} [\![(!\Delta, \Gamma_2)|_M]\!]\end{array}}{[\![!\Delta, \Gamma_1, \Gamma_2 \rhd \mathtt{let}\ \langle x^A, y^B\rangle^n = N\ \mathtt{in}\ M : C]\!]^{\mathrm{FV}} = [\![C]\!] \xrightarrow{f} [\![(!\Delta, \Gamma_1, x : !^n A, y : !^n B)|_M]\!]}$$

$$\xrightarrow{\iota} [\![(!\Delta, \Gamma_1)|_{\mathtt{let}\ldots}]\!] \otimes [\![!^n A]\!] \otimes [\![!^n B]\!] \xrightarrow{\mathrm{id} \otimes d^{\mathcal{L}^n}} [\![(!\Delta, \Gamma_1)|_{\mathtt{let}\ldots}]\!] \otimes [\![!^n(A \otimes B)]\!] \xrightarrow{\mathrm{id} \otimes g}$$

$$[\![(!\Delta, \Gamma_1)|_{\mathtt{let}\ldots}]\!] \otimes [\![(!\Delta, \Gamma_2)|_N]\!] \xrightarrow{\mathrm{id} \otimes \iota} [\![(!\Delta, \Gamma_1)|_{\mathtt{let}\ldots}]\!] \otimes [\![(!\Delta, \Gamma_2)|_{\mathtt{let}\ldots}]\!] \xrightarrow{\mathrm{merge}} [\![(!\Delta, \Gamma_1, \Gamma_2)|_{\mathtt{let}\ldots}]\!]$$

$$\frac{[\![\Delta \rhd M : !^n A]\!]^{\mathrm{FV}} = [\![!^n A]\!] \xrightarrow{f} [\![\Delta|_M]\!]}{[\![\Delta \rhd \mathtt{inl}^n_{A,B}(M) : !^n(A \oplus B)]\!]^{\mathrm{FV}} = [\![!^n(A \oplus B)]\!] \xrightarrow{\mathcal{L}^n \pi_1} [\![!^n A]\!] \xrightarrow{f} [\![\Delta|_M]\!]}$$

$$\frac{[\![\Delta \rhd N : !^n B]\!]^{\mathrm{FV}} = [\![!^n B]\!] \xrightarrow{g} [\![\Delta|_N]\!]}{[\![\Delta \rhd \mathtt{inr}^n_{A,B}(N) : !^n(A \oplus B)]\!]^{\mathrm{FV}} = [\![!^n(A \oplus B)]\!] \xrightarrow{\mathcal{L}^n \pi_2} [\![!^n B]\!] \xrightarrow{g} [\![\Delta|_N]\!]}$$

$$\frac{\begin{array}{c}[\![!\Delta, \Gamma_1, x : !^n A \rhd M : C]\!]^{\mathrm{FV}} = [\![C]\!] \xrightarrow{f} [\![(!\Delta, \Gamma_1, x : !^n A)|_M]\!] \\ [\![!\Delta, \Gamma_1, y : !^n B \rhd N : C]\!]^{\mathrm{FV}} = [\![C]\!] \xrightarrow{g} [\![(!\Delta, \Gamma_1, y : !^n B)|_N]\!] \\ [\![!\Delta, \Gamma_2 \rhd L : !^n(A \oplus B)]\!]^{\mathrm{FV}} = [\![!^n(A \oplus B)]\!] \xrightarrow{h} [\![(!\Delta, \Gamma_2)|_L]\!]\end{array}}{[\![!\Delta, \Gamma_1, \Gamma_2 \rhd \mathtt{match}\ L\ \mathtt{with}^n (x^A \mapsto M \mid y^B \mapsto N) : C]\!]^{\mathrm{FV}} =}$$

$$[\![C]\!] \xrightarrow{\langle f, g\rangle} [\![(!\Delta, \Gamma_1, x : !^n A)|_M]\!] \oplus [\![(!\Delta, \Gamma_1, y : !^n B)|_N]\!] \xrightarrow{\iota \oplus \iota}$$

$$([\![(!\Delta, \Gamma_1)|_{\mathtt{match}\ldots}]\!] \otimes [\![!^n A]\!]) \oplus ([\![(!\Delta, \Gamma_1)|_{\mathtt{match}\ldots}]\!] \otimes [\![!^n B]\!]) \xrightarrow{\theta} [\![(!\Delta, \Gamma_1)|_{\mathtt{match}\ldots}]\!] \otimes ([\![!^n A]\!] \oplus [\![!^n B]\!])$$

$$\xrightarrow{\mathrm{id} \otimes e^{\mathcal{L}^n}} [\![(!\Delta, \Gamma_1)|_{\mathtt{match}\ldots}]\!] \otimes [\![!^n(A \oplus B)]\!] \xrightarrow{\mathrm{id} \otimes h} [\![(!\Delta, \Gamma_1)|_{\mathtt{match}\ldots}]\!] \otimes [\![(!\Delta, \Gamma_2)|_L]\!]$$

$$\xrightarrow{\mathrm{id} \otimes \iota} [\![(!\Delta, \Gamma_1)|_{\mathtt{match}\ldots}]\!] \otimes [\![(!\Delta, \Gamma_2)|_{\mathtt{match}\ldots}]\!] \xrightarrow{\mathrm{merge}} [\![(!\Delta, \Gamma_1, \Gamma_2)|_{\mathtt{match}\ldots}]\!]$$

Table 4 Inductive definition of the interpretation of typing judgements

Proof. See Appendix A. ◀

▶ **Proposition 14** (Soundness for the big-step reduction). *Let $P : A$ be a well-typed quantum closure. Then $[\![P : A]\!] = \sum_Z \mathrm{Prob}(P, Z)[\![Z : A]\!]$, where Z runs over well-typed value closures.*

Proof. By Lemmas 7 and 10, $\mathrm{Prob}(P, Z) \overset{\text{def}}{=} \lim_{n\to\infty} \mathrm{prob}^n(P, Z) = \mathrm{prob}^m(P, Z)$ for some m. It is then easy to obtain $[\![P : A]\!] = \sum_Q \mathrm{prob}^m(P, Q)[\![Q : A]\!]$ by induction on m, using Proposition 13. ◀

▶ **Theorem 15** (Adequacy). *Let $P : \mathtt{bit}$ be a quantum closure of type* bit*. For the interpretation $[\![P : \mathtt{bit}]\!] \colon \mathbb{C} \oplus \mathbb{C} \to \mathbb{C}$, we have $P \Downarrow \mathtt{ff} = [\![P : \mathtt{bit}]\!](1, 0)$ and $P \Downarrow \mathtt{tt} = [\![P : \mathtt{bit}]\!](0, 1)$.*

Proof. By Proposition 14 we have $[\![P : \mathtt{bit}]\!] = \sum_Z \mathrm{Prob}(P, Z)[\![Z : \mathtt{bit}]\!]$. Note that for each well-typed value closure $[\,|\psi\rangle, \Psi, V\,] : \mathtt{bit}$, either $V = \mathtt{ff}^0$ or $V = \mathtt{tt}^0$. Then the assertion follows since $[\![[\,|\psi\rangle, \Psi, \mathtt{ff}^0\,] : \mathtt{bit}]\!](\lambda, \rho) = \lambda$ and $[\![[\,|\psi\rangle, \Psi, \mathtt{tt}^0\,] : \mathtt{bit}]\!](\lambda, \rho) = \rho$. ◀

5 Technical Results about von Neumann Algebras

Let us sketch how we obtained the two monoidal adjunctions in (1).

▶ **Definition 16.** Let $\ell^\infty(X)$ denote the von Neumann algebra of bounded maps $f \colon X \to \mathbb{C}$ on a set X. Addition, multiplication, involution, suprema, and so on, are computed coordinatewise in $\ell^\infty(X)$. In fact, $\ell^\infty(X)$ is simply the X-fold product in $\mathbf{vNA}_{\mathrm{MIU}}$ of $\mathbb{C}$ with $\varphi \mapsto \varphi(x)$ as x-th projection. We extend $X \mapsto \ell^\infty(X)$ to a functor $\ell^\infty \colon \mathbf{Set}^{\mathrm{op}} \to \mathbf{vNA}_{\mathrm{MIU}}$ by defining $\ell^\infty(f)(\varphi) = \varphi \circ f$ for every map $f \colon X \to Y$ (in $\mathbf{Set}$) and $\varphi \in \ell^\infty(Y)$.

Let $\mathrm{sp}(\mathscr{A})$ be the set of normal MIU-maps $\varphi \colon \mathscr{A} \to \mathbb{C}$ on a von Neumann algebra $\mathscr{A}$. We extend $\mathscr{A} \mapsto \mathrm{sp}(\mathscr{A})$ to a functor $\mathrm{sp} \colon \mathbf{vNA}_{\mathrm{MIU}} \to \mathbf{Set}^{\mathrm{op}}$ by defining $\mathrm{sp}(f)(\varphi) = \varphi \circ f$ for every normal MIU-map $f \colon \mathscr{A} \to \mathscr{B}$ and $\varphi \in \mathrm{sp}(\mathscr{B})$.

Note that any normal MIU-map $f \colon \mathscr{A} \to \ell^\infty(X)$ gives a map $g \colon X \to \mathrm{sp}(\mathscr{A})$ by "swapping arguments" — $g(x)(\varphi) = f(\varphi)(x)$ —, and with a little bit more work, we get:

▶ **Lemma 17.** *There is an adjunction $\mathrm{sp} \dashv \ell^\infty$.* ◀

The following two lemmas describe the normal spectrum of direct products and tensors of von Neumann algebras, and can be proven using standard techniques.

▶ **Lemma 18.** *Let I be a set, and for each $i \in I$, let $\mathscr{A}_i$ be a von Neumann algebra. For each $\omega \in \mathrm{sp}(\bigoplus_{i\in I} \mathscr{A}_i)$, there is $i \in I$ and $\tilde{\omega} \in \mathrm{sp}(\mathscr{A}_i)$ with $\omega = \tilde{\omega} \circ \pi_i$.* ◀

▶ **Lemma 19.** *Let $\mathscr{A}_1$ and $\mathscr{A}_2$ be von Neumann algebras. Then for every $\omega \in \mathrm{sp}(\mathscr{A}_1 \otimes \mathscr{A}_2)$ there are unique $\omega_1 \in \mathscr{A}_1$ and $\omega_2 \in \mathscr{A}_2$ with $\omega(a_1 \otimes a_2) = \omega_1(a_1) \cdot \omega_2(a_2)$ for all $a_i \in \mathscr{A}_i$.* ◀

▶ **Corollary 20.** *The functor $\mathrm{sp} \colon \mathbf{vNA}_{\mathrm{MIU}} \longrightarrow \mathbf{Set}^{\mathrm{op}}$ preserves products, and tensors.* ◀

Using that $\ell^\infty(X)$ is the X-fold product of $\mathbb{C}$ in $\mathbf{vNA}_{\mathrm{MIU}}$ we get:

▶ **Corollary 21.** *The counit of the adjunction $\mathrm{sp} \dashv \ell^\infty$ is an isomorphism.* ◀

▶ **Lemma 22.** *Let X and Y be sets. There is a normal MIU-isomorphism*

$$\varphi \colon \ell^\infty(X) \otimes \ell^\infty(Y) \longrightarrow \ell^\infty(X \times Y) \qquad \text{given by} \qquad \varphi(f \otimes g)(x, y) \;=\; f(x) \cdot g(y).$$

Proof. Use the proof of Proposition 9.2 from [4]. ◀

▶ **Corollary 23.** *The adjunction* $\mathrm{sp} \dashv \ell^\infty$ *is strong monoidal.* ◀

Let us turn to the second adjunction in (1). In [45] it is shown how the following result follows from Freyd's Adjoint Functor Theorem (see Theorem V.6.2 of [20]).

▶ **Theorem 24** ([45])**.** *The inclusion* $\mathcal{J}\colon \mathbf{vNA}_{\mathrm{MIU}} \to \mathbf{vNA}_{\mathrm{CPsU}}$ *has a left adjoint.* ◀

▶ **Corollary 25.** *The category* $\mathbf{vNA}_{\mathrm{CPsU}}$ *is isomorphic to the co-Kleisli category of the comonad* $\mathcal{F} \circ \mathcal{J}$ *on* $\mathbf{vNA}_{\mathrm{MIU}}$ *induced by* $\mathcal{F} \dashv \mathcal{J}$*.* ◀

Proof. See Theorem 9 of [45], or do Exercise VI.5.2 of [20] (and use the fact that an equivalence of categories which is bijective on objects is an isomorphism.) ◀

▶ **Corollary 26.** *The adjunction* $\mathcal{F} \dashv \mathcal{J}$ *is symmetric monoidal.*

Proof. Clearly, $\mathcal{J}\colon \mathbf{vNA}_{\mathrm{MIU}} \to \mathbf{vNA}_{\mathrm{CPsU}}$ is strict symmetric monoidal. From this fact alone, it follows that the adjunction $\mathcal{F} \dashv \mathcal{J}$ is symmetric monoidal, see Prop. 14 of [24]. ◀

In our model of the quantum lambda calculus the von Neumann algebras of the form $\ell^\infty(X)$ serve as the interpretation of the *duplicable types* (of the form $!A$), because $\ell^\infty(X)$ carries a $\otimes$-monoid structure. Among all von Neumann algebras $\ell^\infty(X)$ is arguably quite special and one might wonder if there is a broader class of von Neumann algebras that might serve as the interpretation of duplicable types (such as the class of all commutative von Neumann algebras, which includes $L^\infty[0,1]$.) The following result settles this matter: no. Due to space constraints, the proof will appear somewhere else.

▶ **Theorem 27.** *For a von Neumann algebra* $\mathscr{A}$ *the following are equivalent.*

1. *There is a* duplicator *on* $\mathscr{A}$*, that is, a normal positive unital map* $\mu\colon \mathscr{A} \otimes \mathscr{A} \to \mathscr{A}$ *such that* $\mu(1 \otimes a) = a = \mu(a \otimes 1)$ *and* $\mu(a \otimes \mu(b \otimes c)) = \mu(\mu(a \otimes b) \otimes c)$ *for all* $a, b, c \in \mathscr{A}$*.*
2. $\mathscr{A}$ *is isomorphic to* $\ell^\infty(X)$ *for some set* X*.*

Moreover, there is at most one duplicator on $\mathscr{A}$*.* ◀

▶ **Corollary 28.** $\ell^\infty(\mathrm{sp}(\mathscr{A}))$ *is the free* $\otimes$*-monoid on* $\mathscr{A}$ *from* $\mathbf{vNA}_{\mathrm{MIU}}$*.* ◀

6 Final Remarks

We have given a rather concrete proof of adequacy for the sake of clarity. However, it seems that we only used the fact that $\mathbf{vNA}_{\mathrm{MIU}}$ is a 'concrete model of the quantum lambda calculus' (see Remark 11), and that $\mathbf{vNA}_{\mathrm{CPsU}}$ is 'suitably' enriched over convex sets. Thus an abstract result might be distilled from our work stating that any concrete model of the quantum lambda calculus is adequate when suitably enriched over convex sets, but we have not pursued this.

We believe selling points of our model are that it is a model for Selinger and Valiron's original quantum lambda calculus [37] (in Selinger and Valiron's linear fragment [39] the ! modality is absent; in Hasuo and Hoshino's language [10] the tensor type `qbit` $\otimes$ `qbit` does not represent two qubits; and only function types may be duplicable, $!(A \multimap B)$, in the language of Pagani et al. [27]); that it is adequate (Malherbe's model [21, 22] is not known to be); that the interpretation of ! is rather simple; and that it is formed using von Neumann algebras, a mathematical classic.

We believe our model could be improved by a more concrete description of $[\![A \multimap B]\!]$ (as all the other models have), and by features such as recursion and inductive types (present in e.g. Hasuo and Hoshino's and Pagani's models), which leaves us with ample material for future research.

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A Proof of Soundness for the Small-Step Reduction

We need some results on the denotational semantics.

► **Lemma 29.** *Suppose that* $!\Delta, \Gamma_1, x : A \rhd M : B$ *and* $!\Delta, \Gamma_2 \rhd V : A$*, so that* $!\Delta, \Gamma_1, \Gamma_2 \rhd M[V/x] : B$ *by Lemma 3. Then the following diagram commute.*

$$\begin{array}{ccc} [\![B]\!] & \xrightarrow{[\![!\Delta,\Gamma_1,\Gamma_2 \rhd M[V/x]:B]\!]} & [\![!\Delta,\Gamma_1,\Gamma_2]\!] \\ {\scriptstyle [\![!\Delta,\Gamma_2,x:A \rhd M:B]\!]}\downarrow & & \uparrow{\scriptstyle \mathrm{merge}} \\ [\![!\Delta,\Gamma_1]\!] \otimes [\![A]\!] & \xrightarrow{\mathrm{id} \otimes [\![!\Delta,\Gamma_2 \rhd V:A]\!]} & [\![!\Delta,\Gamma_1]\!] \otimes [\![!\Delta,\Gamma_2]\!] \end{array}$$

Proof. By induction on M. Note that the interpretation of a value is MIU. ◀

▶ **Lemma 30.** *We have the following equations, when terms M, N and values V, W are appropriately well-typed.*

$$\begin{aligned}
[\![(\lambda^0 x^A.M)V]\!] &= [\![M[V/x]]\!] \\
[\![\mathtt{let}\ \langle x^A, y^B\rangle^n = \langle V, W\rangle^n\ \mathtt{in}\ M]\!] &= [\![M[V/x, W/y]]\!] \\
[\![\mathtt{match}\ \mathtt{inl}^n_{A,B}(V)\ \mathtt{with}^n(x^A \mapsto M \mid y^B \mapsto N)]\!] &= [\![M[V/x]]\!] \\
[\![\mathtt{match}\ \mathtt{inr}^n_{A,B}(W)\ \mathtt{with}^n(x^A \mapsto M \mid y^B \mapsto N)]\!] &= [\![N[W/y]]\!]
\end{aligned}$$

Here we abbreviate $[\![\Delta \rhd M : A]\!]$ to $[\![M]\!]$.

Proof. Straightforward, using Lemma 29. ◀

To prove Proposition 13 by induction, we need to strengthen the statement as Lemma 32. Note that $[\![P : A]\!]^{(0)} = [\![P : A]\!]$.

▶ **Definition 31.** Let $[\,|\psi\rangle, |x_1 \ldots x_m\rangle, M\,] : A$ be a well-typed quantum closure such that $x_i \notin \mathrm{FV}(M)$ for all $i \leq l$. Then we define:

$$[\![[\,|\psi\rangle, |x_1 \ldots x_m\rangle, M\,] : A]\!]^{(l)} = \mathcal{M}_2^{\otimes l} \otimes [\![A]\!] \xrightarrow{\mathrm{id}\otimes[\![x_{l+1}:\mathtt{qbit},\ldots,x_m:\mathtt{qbit}\rhd M:A]\!]} \mathcal{M}_2^{\otimes m} \xrightarrow{\langle\psi|-|\psi\rangle} \mathbb{C}$$

▶ **Lemma 32.** *Let $P = [\,|\psi\rangle, |x_1 \ldots x_m\rangle, M\,] : A$ be a well-typed quantum closure such that $x_i \notin \mathrm{FV}(M)$ for all $i \leq l$. Then $[\![P : A]\!]^{(l)} = \sum_Q \mathrm{prob}(P, Q)[\![Q : A]\!]^{(l)}$.*

Proof. We prove it by induction on terms M. If M is a value, then it holds by the definition of prob. In the other induction steps, we prove the assertion by cases.

Consider the induction step for MN, and the case where M is not a value. Then the only possible reductions from $P = [\,\psi, \Psi, MN\,]$ are $[\,\psi, \Psi, MN\,] \to_p [\,\psi', \Psi', M'N\,]$ when $[\,\psi, \Psi, M\,] \to_p [\,\psi', \Psi', M'\,]$. Without loss of generality,[2] we may assume that

$$l = |y_1 \ldots y_l z_1 \ldots z_k x_1 \ldots x_h\rangle$$

such that $x_1 : \mathtt{qbit}, \ldots, x_h : \mathtt{qbit} \rhd M : A \multimap B$ and $z_1 : \mathtt{qbit}, \ldots, z_k : \mathtt{qbit} \rhd N : A$. We will simply write $[\![M]\!]$ for $[\![x_1 : \mathtt{qbit}, \ldots, x_h : \mathtt{qbit} \rhd M : A \multimap B]\!]$ and $[\![N]\!]$ similarly. Then

$$\begin{aligned}
&[\![[\,\psi, \Psi, MN\,] : B]\!]^{(l)} \\
&= \mathcal{M}_2^{\otimes l} \otimes [\![B]\!] \xrightarrow{\mathrm{id}\otimes\varepsilon} \mathcal{M}_2^{\otimes l} \otimes [\![A \multimap B]\!] \otimes [\![A]\!] \xrightarrow{\mathrm{id}\otimes[\![M]\!]\otimes[\![N]\!]} \mathcal{M}_2^{\otimes l} \otimes \mathcal{M}_2^{\otimes h} \otimes \mathcal{M}_2^{\otimes k} \\
&\quad \xrightarrow{\mathrm{id}\otimes\gamma} \mathcal{M}_2^{\otimes l} \otimes \mathcal{M}_2^{\otimes k} \otimes \mathcal{M}_2^{\otimes h} \xrightarrow{\langle\psi|-|\psi\rangle} \mathbb{C} \\
&= \mathcal{M}_2^{\otimes l} \otimes [\![B]\!] \xrightarrow{\mathrm{id}\otimes\varepsilon} \mathcal{M}_2^{\otimes l} \otimes [\![A \multimap B]\!] \otimes [\![A]\!] \xrightarrow{\mathrm{id}\otimes\gamma} \mathcal{M}_2^{\otimes l} \otimes [\![A]\!] \otimes [\![A \multimap B]\!] \\
&\quad \xrightarrow{\mathrm{id}\otimes[\![N]\!]\otimes\mathrm{id}} \mathcal{M}_2^{\otimes(l+k)} \otimes [\![A \multimap B]\!] \xrightarrow{\mathrm{id}\otimes[\![M]\!]} \mathcal{M}_2^{\otimes(l+k)} \otimes \mathcal{M}_2^{\otimes h} \xrightarrow{\langle\psi|-|\psi\rangle} \mathbb{C}
\end{aligned}$$

[2] A permutation of variables in Ψ which keeps the first l variables, with the permutation of the corresponding qubits in $|\psi\rangle$, does not change $[\![P : A]\!]^{(l)}$. The same is true for the operational semantics [39, §3.2].

Let $[\,\psi, \Psi, M\,] \to_{p_i} [\,\psi_i, \Psi_i, M_i\,]$ $(i \in I)$ be all the reductions from $[\,\psi, \Psi, M\,]$. By IH, we have

$$\begin{aligned}
&\mathcal{M}_2^{\otimes(l+k)} \otimes [\![A \multimap B]\!] \xrightarrow{\mathrm{id}\otimes[\![M]\!]} \mathcal{M}_2^{\otimes(l+k)} \otimes \mathcal{M}_2^{\otimes h} \xrightarrow{\langle\psi|-|\psi\rangle} \mathbb{C} \\
&= [\![[\,\psi, \Psi, M\,] : A \multimap B]\!]^{(l+k)} \\
&= \sum_{i\in I} p_i [\![[\,\psi_i, \Psi_i, M_i\,] : A \multimap B]\!]^{(l+k)} \\
&= \sum_{i\in I} p_i \Big(\mathcal{M}_2^{\otimes(l+k)} \otimes [\![A \multimap B]\!] \xrightarrow{\mathrm{id}\otimes[\![M_i]\!]} \mathcal{M}_2^{\otimes(l+k)} \otimes \mathcal{M}_2^{\otimes h_i} \xrightarrow{\langle\psi_i|-|\psi_i\rangle} \mathbb{C}\Big)
\end{aligned}$$

It is then straightforward to see that $[\![[\,\psi, \Psi, MN\,] : B]\!]^{(l)} = \sum_i p_i [\![[\,\psi_i, \Psi_i, M_i N\,] : B]\!]^{(l)}$.

Next consider the case where $M = U$ and $N = \langle x_1, \ldots, x_k\rangle^0$. Without loss of generality we may assume $P = [\,|\psi\rangle, \Psi, U\langle x_1, \ldots, x_k\rangle\,]$ with $l = |y_1 \ldots y_l x_1 \ldots x_k z_1 \ldots z_h\rangle$. The only reduction from P is $[\,|\psi\rangle, \Psi, U\langle x_1, \ldots, x_k\rangle\,] \to_1 [\,|\psi'\rangle, \Psi, \langle x_1, \ldots, x_k\rangle\,] =: Q$, where $|\psi'\rangle = (\mathcal{I}_l \otimes U \otimes \mathcal{I}_h)|\psi\rangle$ ($\mathcal{I}_n$ denotes the $2^n \times 2^n$ identity matrix). We need to show that $[\![P : \mathtt{qbit}^{\otimes k}]\!]^{(l)} = [\![Q : \mathtt{qbit}^{\otimes k}]\!]^{(l)}$. Note that

$$[\![x_1 : \mathtt{qbit}, \ldots, x_k : \mathtt{qbit} \rhd U\langle x_1, \ldots, x_k\rangle : \mathtt{qbit}^{\otimes k}]\!] = f_U \colon \mathcal{M}_2^{\otimes k} \to \mathcal{M}_2^{\otimes k}$$

Thus we have

$$[\![P : \mathtt{qbit}^{\otimes k}]\!]^{(l)} = \mathcal{M}_2^{\otimes l} \otimes \mathcal{M}_2^{\otimes k} \xrightarrow{\mathrm{id}\otimes f_U} \mathcal{M}_2^{\otimes l} \otimes \mathcal{M}_2^{\otimes k} \xrightarrow{\mathrm{id}\otimes\iota} \mathcal{M}_2^{\otimes(l+k+h)} \xrightarrow{\langle\psi|-|\psi\rangle} \mathbb{C}$$

On the other hand, we have

$$[\![x_1 : \mathtt{qbit}, \ldots, x_k : \mathtt{qbit} \rhd \langle x_1, \ldots, x_k\rangle : \mathtt{qbit}^{\otimes k}]\!] = \mathrm{id} \colon \mathcal{M}_2^{\otimes k} \to \mathcal{M}_2^{\otimes k}$$

and hence

$$[\![Q : \mathtt{qbit}^{\otimes k}]\!]^{(l)} = \mathcal{M}_2^{\otimes l} \otimes \mathcal{M}_2^{\otimes k} \xrightarrow{\mathrm{id}\otimes\iota} \mathcal{M}_2^{\otimes(l+k+h)} \xrightarrow{\langle\psi'|-|\psi'\rangle} \mathbb{C}$$

For each elementary tensor $A \otimes B \in \mathcal{M}_2^{\otimes l} \otimes \mathcal{M}_2^{\otimes k}$,

$$\begin{aligned}
[\![P : \mathtt{qbit}^{\otimes k}]\!]^{(l)}(A \otimes B) &= \langle\psi|(\mathrm{id}\otimes\iota)((\mathrm{id}\otimes f_U)(A \otimes B))|\psi\rangle \\
&= \langle\psi|A \otimes (U^\dagger B U) \otimes \mathcal{I}_h|\psi\rangle \\
&= \langle\psi|(\mathcal{I}_l \otimes U^\dagger \otimes \mathcal{I}_h)(A \otimes B \otimes \mathcal{I}_h)(\mathcal{I}_l \otimes U \otimes \mathcal{I}_h)|\psi\rangle \\
&= \langle\psi'|(\mathrm{id}\otimes\iota)(A \otimes B)|\psi'\rangle \\
&= [\![Q : \mathtt{qbit}^{\otimes k}]\!]^{(l)}(A \otimes B)
\end{aligned}$$

We conclude that $[\![P : \mathtt{qbit}^{\otimes k}]\!]^{(l)} = [\![Q : \mathtt{qbit}^{\otimes k}]\!]^{(l)}$.

Consider the case where MN is of the form $(\lambda x.M)V$. Only the reduction is $[\,|\psi\rangle, \Psi, (\lambda x.M)V\,] \to_1 [\,|\psi\rangle, \Psi, M[V/x]\,]$. The assertion holds immediately by Lemma 30.

The other cases in the induction step MN can be shown similarly. We can prove the other induction steps similarly by cases. ◀