An Effect-Theoretic Account of Lebesgue Integration

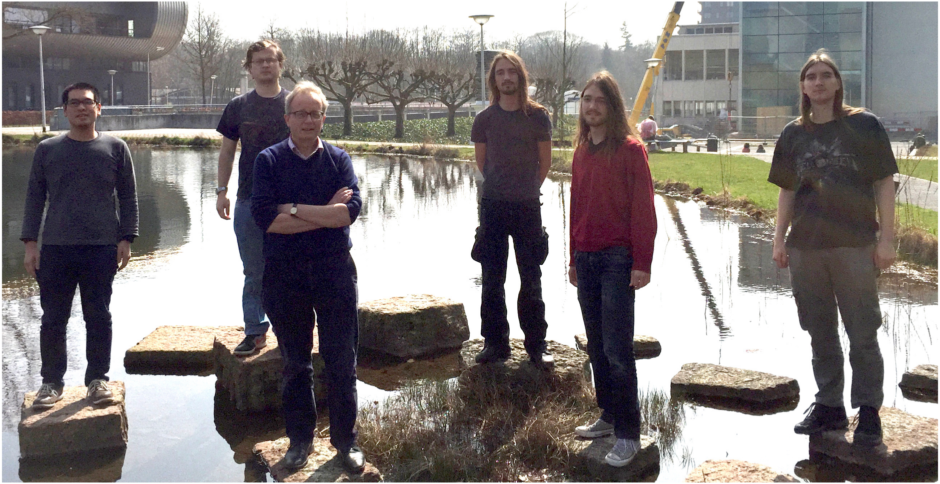
Bart Jacobs
bart@cs.ru.nl

Bram Westerbaan
awesterb@cs.ru.nl

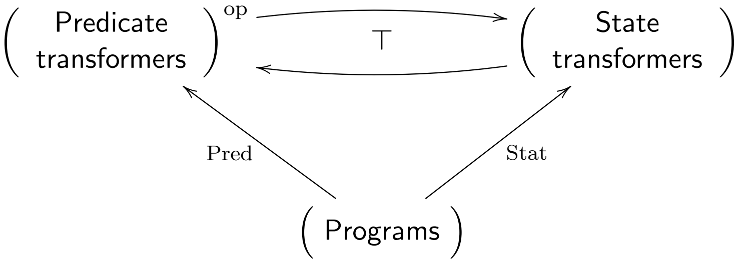
Radboud University Nijmegen

June 23, 2015

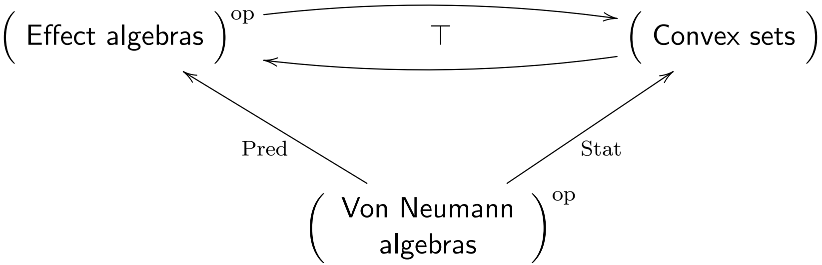
Some locals



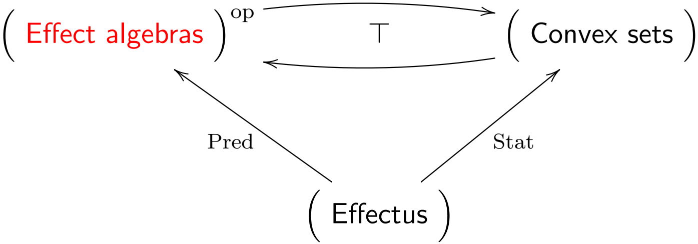
Our usual business: categorical program semantics



Our usual business: semantics of quantum programs



Our usual business: effectus theory



Some related work

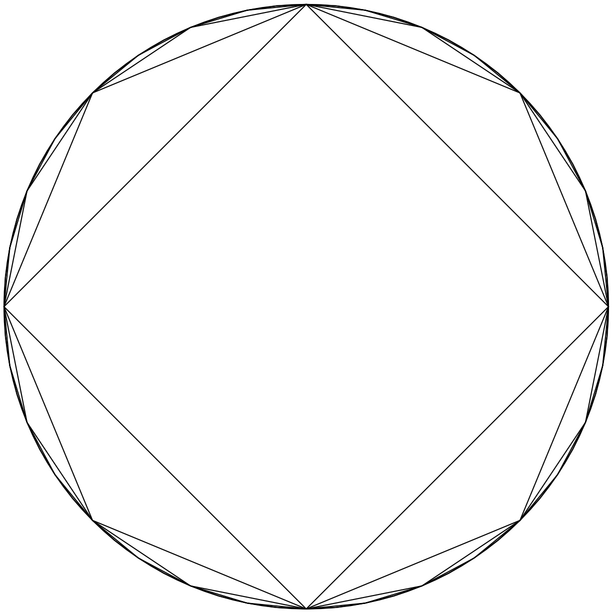
Some related work

1. *method of exhaustion* by Eudoxos, $\sim$390, BC

Short history of integration

1. *method of exhaustion* by Eudoxos, $\sim$390, BC

Method of exhaustion



Short history of integration

1. *method of exhaustion* by Eudoxos, $\sim$390, BC

Short history of integration

1. *method of exhaustion* by Eudoxos, $\sim$390, BC
2. integration of functions *Newton*, $\sim$1665, . . .

Short history of integration

1. *method of exhaustion* by Eudoxos, $\sim$390, BC
2. integration of functions *Newton*, $\sim$1665, ...
3. formalised by *Riemann*, 1854

Short history of integration

1. *method of exhaustion* by Eudoxos, $\sim$390, BC
2. integration of functions *Newton*, $\sim$1665, ...
3. formalised by *Riemann*, 1854
4. completed by *Lebesgue*, 1902

Short history of integration

1. *method of exhaustion* by Eudoxos, $\sim$390, BC
2. integration of functions *Newton*, $\sim$1665, ...
3. formalised by *Riemann*, 1854
4. completed by *Lebesgue*, 1902
5. generalised by *Daniell* in 1918, *Bochner* in 1933, *Haar* in 1940, *Pettis* around 1943, *Stone* in 1948, ...

Short history of integration

1. *method of exhaustion* by Eudoxos, $\sim$390, BC
2. integration of functions *Newton*, $\sim$1665, ...
3. formalised by *Riemann*, 1854
4. completed by *Lebesgue*, 1902
5. generalised by *Daniell* in 1918, *Bochner* in 1933, *Haar* in 1940, *Pettis* around 1943, *Stone* in 1948, ...
6. we present another generalisation* based on **effect algebras**

Short history of integration

1. *method of exhaustion* by Eudoxos, $\sim$390, BC
2. integration of functions *Newton*, $\sim$1665, ...
3. formalised by *Riemann*, 1854
4. completed by *Lebesgue*, 1902
5. generalised by *Daniell* in 1918, *Bochner* in 1933, *Haar* in 1940, *Pettis* around 1943, *Stone* in 1948, ...
6. we present another generalisation* based on **effect algebras**

* of integration of $[0,1]$-*valued functions*
with respect to *probability measures* ($\approx [0,1]$-valued measures)

But why yet another !?

But why yet another !?

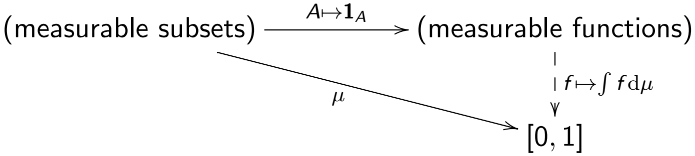
The theory of integration, because of its central rôle in mathematical analysis and geometry, continues to afford opportunities for serious investigation.

— M.H. Stone, 1948

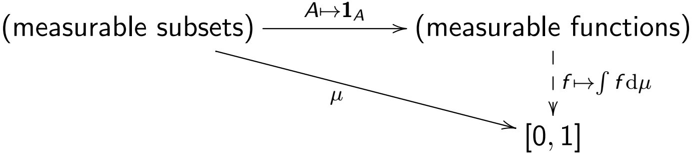
But why yet another !?

> The theory of integration, because of its central rôle in mathematical analysis and geometry, continues to afford opportunities for serious investigation.
>
> — M.H. Stone, 1948

Universal property



Universal property



key observation: both μ and $\int(-)\mathrm{d}\mu$ are homomorphisms of ω-*complete effect algebras*

Effect algebras

An **effect algebra** is a set E with 0, 1, $(-)^{\perp}$, and *partial* $\varovee$

Examples:

1. $[0, 1]$ $\quad a \varovee b = a + b \quad \text{if} \quad a + b \leq 1$
2. $\wp(X)$ $\quad A \varovee B = A \cup B \quad \text{if} \quad A \cap B = \varnothing$

Effect algebras

An **effect algebra** is a set E with 0, 1, $(-)^\perp$, and *partial* $\ovee$ with

1. $a \ovee b = b \ovee a$
2. $a \ovee (b \ovee c) = (a \ovee b) \ovee c$
3. $a \ovee 0 = a$
4. $a \ovee a^\perp = 1$

Examples:

1. $[0, 1]$ $\quad a \ovee b = a + b \quad \text{if} \quad a + b \leq 1$
2. $\wp(X)$ $\quad A \ovee B = A \cup B \quad \text{if} \quad A \cap B = \varnothing$

Effect algebras

An **effect algebra** is a set E with 0, 1, $(-)^{\perp}$, and *partial* $\ovee$ with

1. $a \ovee b = b \ovee a$
2. $a \ovee (b \ovee c) \;=\; (a \ovee b) \ovee c$
3. $a \ovee 0 = a$
4. $a \ovee a^{\perp} = 1$
5. $a \ovee b = 0 \quad \Longrightarrow \quad a = b = 0$
6. $a \ovee b = a \ovee c \quad \Longrightarrow \quad b = c$

Examples:

1. $[0,1]$ $\qquad a \ovee b = a + b \quad \text{if} \quad a + b \leq 1$
2. $\wp(X)$ $\qquad A \ovee B = A \cup B \quad \text{if} \quad A \cap B = \varnothing$

Effect algebras

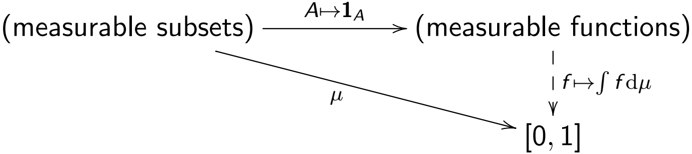
An **effect algebra** is a set E with 0, 1, $(-)^\perp$, and *partial* $\ovee$ with

1. $a \ovee b = b \ovee a$
2. $a \ovee (b \ovee c) \;=\; (a \ovee b) \ovee c$
3. $a \ovee 0 = a$
4. $a \ovee a^\perp = 1$
5. $a \ovee b = 0 \quad \Longrightarrow \quad a = b = 0$
6. $a \ovee b = a \ovee c \quad \Longrightarrow \quad b = c$

Examples:

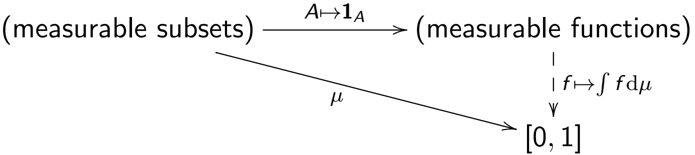
1. $[0, 1]$ $\qquad a \ovee b = a + b \quad \text{if} \quad a + b \leq 1$
2. $\wp(X)$ $\qquad A \ovee B = A \cup B \quad \text{if} \quad A \cap B = \varnothing$
3. $\mathcal{E}f(\mathcal{H})$ $\qquad A \ovee B = A + B \quad \text{if} \quad A + B \leq I$

Universal property



key observation: both μ and $\int(-)\mathrm{d}\mu$ are homomorphisms of ω-*complete effect algebras*

Universal property



key observation: both μ and $\int(-)\mathrm{d}\mu$ are homomorphisms of *ω-complete effect algebras*

ω-complete effect algebras

ω-complete effect algebras

Let E be an effect algebra.

$$a \leq b \quad \iff \quad \exists d \quad a \ovee d = b$$

ω-complete effect algebras

Let E be an effect algebra.

$$a \leq b \iff \exists d \quad a \ovee d = b$$

The effect algebra E is ω-**complete** if each chain

$$a_1 \;\leq\; a_2 \;\leq\; \cdots$$

has a supremum, $\bigvee_n a_n$.

ω-complete effect algebras

Let E be an effect algebra.

$$a \leq b \quad \iff \quad \exists d \quad a \ovee d = b$$

The effect algebra E is ω-**complete** if each chain

$$a_1 \;\leq\; a_2 \;\leq\; \cdots$$

has a supremum, $\bigvee_n a_n$.

Examples:

1. $[0, 1]$
2. $\wp(X) \qquad \bigvee_n A_n = \bigcup_n A_n$
3. $\mathcal{E}f(\mathcal{H})$

ω-complete effect algebras

Let E be an effect algebra.

$$a \leq b \iff \exists d \quad a \ovee d = b$$

The effect algebra E is ω-**complete** if each chain

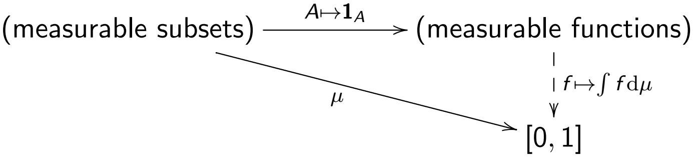
$$a_1 \;\leq\; a_2 \;\leq\; \cdots$$

has a supremum, $\bigvee_n a_n$.

Examples:

1. $[0,1]$
2. $\wp(X)$ $\qquad \bigvee_n A_n = \bigcup_n A_n$
3. $\mathcal{E}f(\mathcal{H})$
4. σ-algebra on X $\quad = \quad$ sub-(ω-complete EA) of $\wp(X)$!

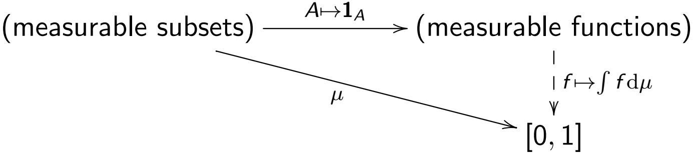
Universal property



key observation: both μ and $\int(-)\mathrm{d}\mu$ are homomorphisms of ω-*complete effect algebras*

Universal property

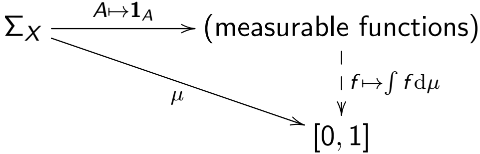
Let Σ_X be a σ-algebra on a set X.



key observation: both μ and $\int(-)\mathrm{d}\mu$ are homomorphisms of ω-*complete effect algebras*

Universal property

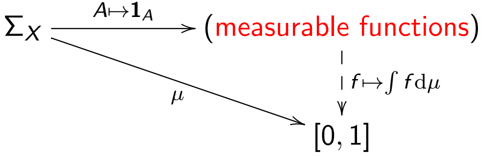
Let Σ_X be a σ-algebra on a set X.



key observation: both μ and $\int (-) \mathrm{d}\mu$ are homomorphisms of ω-*complete effect algebras*

Universal property

Let Σ_X be a σ-algebra on a set X.



key observation: both μ and $\int (-) \mathrm{d}\mu$ are homomorphisms of ω-*complete effect algebras*

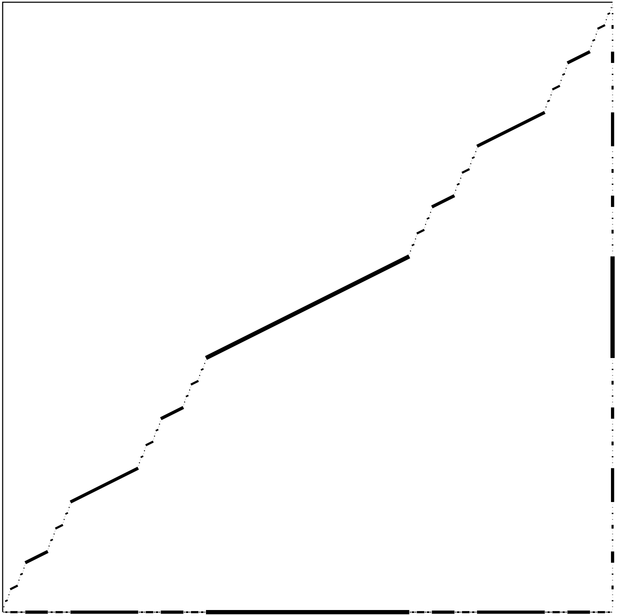
Measurable functions

Let Σ_X be a σ-algebra on a set X

A map $f \colon X \to [0,1]$ is **measurable** if

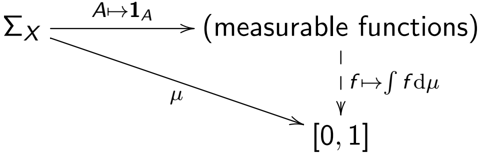
$$f^{-1}([a,b]) \in \Sigma_X \qquad \text{for all } \; a \leq b \; \text{ in } [0,1]$$

$$\mathrm{Meas}(X,[0,1]) \quad = \quad \{\, f \colon X \to [0,1] \colon \; f \text{ is measurable} \,\}$$



Universal property

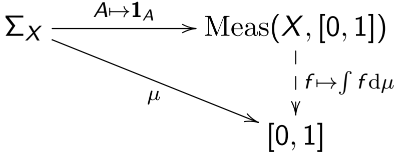
Let Σ_X be a σ-algebra on a set X.



key observation: both μ and $\int (-) \mathrm{d}\mu$ are homomorphisms of ω-*complete effect algebras*

Universal property

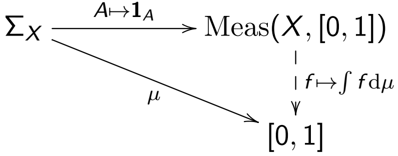
Let Σ_X be a σ-algebra on a set X.



key observation: both μ and $\int (-) \mathrm{d}\mu$ are homomorphisms of ω-*complete effect algebras*

Universal property

Let Σ_X be a σ-algebra on a set X.



key observation: both μ and $\int (-) \mathrm{d}\mu$ are homomorphisms of ω-*complete effect algebras*

Homomorphisms of (ω-complete) effect algebras

$f : F \to E$ is a **homomorphism of effect algebras** if

1. $f(0) = 0 \quad f(1) = 1 \quad f(a^\perp) = f(a)^\perp$
2. if $a \ovee b$ is defined, then $\qquad f(a \ovee b) = f(a) \ovee f(b)$

Homomorphisms of (ω-complete) effect algebras

$f\colon F \to E$ is a **homomorphism of effect algebras** if

1. $f(0) = 0 \quad f(1) = 1 \quad f(a^\perp) = f(a)^\perp$
2. if $a \ovee b$ is defined, then $\qquad f(a \ovee b) = f(a) \ovee f(b)$

f is a **homomorphism of ω-complete effect algebras** if

3. $\bigvee_n f(a_n) = f(\bigvee_n a_n) \quad$ for $a_1 \leq a_2 \leq \cdots$ in F

Homomorphisms of (ω-complete) effect algebras

$f\colon F \to E$ is a **homomorphism of effect algebras** if

1. $f(0) = 0 \quad f(1) = 1 \quad f(a^\perp) = f(a)^\perp$
2. if $a \ovee b$ is defined, then $\quad f(a \ovee b) = f(a) \ovee f(b)$

f is a **homomorphism of ω-complete effect algebras** if

3. $\bigvee_n f(a_n) = f(\bigvee_n a_n) \quad$ for $a_1 \leq a_2 \leq \cdots$ in F

Examples:

1. $\mathbf{1}_{(-)}\colon \Sigma_X \longrightarrow \mathrm{Meas}(X, [0,1])$

Homomorphisms of (ω-complete) effect algebras

$f\colon F \to E$ is a **homomorphism of effect algebras** if

1. $f(0) = 0 \quad f(1) = 1 \quad f(a^\perp) = f(a)^\perp$
2. if $a \ovee b$ is defined, then $\qquad f(a \ovee b) \;=\; f(a) \ovee f(b)$

f is a **homomorphism of ω-complete effect algebras** if

3. $\bigvee_n f(a_n) \;=\; f(\bigvee_n a_n) \quad$ for $a_1 \leq a_2 \leq \cdots$ in F

Examples:

1. $\mathbf{1}_{(-)}\colon \Sigma_X \longrightarrow \mathrm{Meas}(X, [0,1])$
2. homomorphisms of ω-complete EA $\quad \mu\colon \Sigma_X \to [0,1]$

   $=$ probability measures on X (!)

Universal property

Let Σ_X be a σ-algebra on a set X.

$$\begin{array}{ccc} \Sigma_X & \xrightarrow{A \mapsto \mathbf{1}_A} & \mathrm{Meas}(X, [0,1]) \\ & \searrow^{\mu} & \Big\downarrow{\scriptstyle f \mapsto \int f \mathrm{d}\mu} \\ & & [0,1] \end{array}$$

key observation: both μ and $\int (-) \mathrm{d}\mu$ are homomorphisms of ω-*complete effect algebras*

Universal property

Let Σ_X be a σ-algebra on a set X.

$$
\begin{array}{ccc}
\Sigma_X & \xrightarrow{A \mapsto \mathbf{1}_A} & \mathrm{Meas}(X, [0,1]) \\
& \underset{\mu}{\searrow} & \Big\downarrow{\scriptstyle f \mapsto \int f \mathrm{d}\mu} \\
& & [0,1]
\end{array}
$$

For every homomorphism of ω-complete effect algebras μ
there is a unique hom. of (...?...) $\int(-)\mathrm{d}\mu$
such that $\int \mathbf{1}_A \mathrm{d}\mu = \mu(A)$.

Universal property

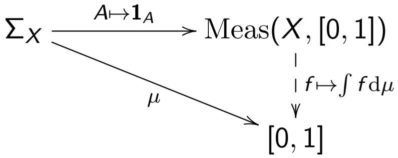
Let Σ_X be a σ-algebra on a set X.

$$\begin{array}{ccc} \Sigma_X & \xrightarrow{A \mapsto \mathbf{1}_A} & \mathrm{Meas}(X,[0,1]) \\ & \underset{\mu}{\searrow} & \Big\downarrow{\scriptstyle f \mapsto \int f \mathrm{d}\mu} \\ & & [0,1] \end{array}$$

For every homomorphism of ω-complete effect algebras μ
there is a unique hom. of (...?...) $\int(-)\mathrm{d}\mu$
such that $\int \mathbf{1}_A \mathrm{d}\mu = \mu(A)$.

Universal property

Let Σ_X be a σ-algebra on a set X.



For every homomorphism of ω-complete effect algebras μ there is a unique hom. of ω-complete effect modules $\int(-)\mathrm{d}\mu$ such that $\int \mathbf{1}_A \mathrm{d}\mu = \mu(A)$.

Effect modules

An **effect module** is an effect algebra E with scalar multiplication $\lambda \cdot x$ ($\lambda \in [0, 1]$, $x \in E$)

Effect modules

An **effect module** is an effect algebra E with scalar multiplication $\lambda \cdot x$ ($\lambda \in [0,1]$, $x \in E$) such that

1. $1 \cdot a = a$
2. $\lambda \cdot (\mu \cdot a) = (\lambda \cdot \mu) \cdot a$
3. $\lambda \cdot (-)$ preserves $\ovee$ and 0
4. $(-) \cdot a$ preserves $\ovee$ and 0

Effect modules

An **effect module** is an effect algebra E with scalar multiplication $\lambda \cdot x$ ($\lambda \in [0,1]$, $x \in E$) such that

1. $1 \cdot a = a$
2. $\lambda \cdot (\mu \cdot a) = (\lambda \cdot \mu) \cdot a$
3. $\lambda \cdot (-)$ preserves $\ovee$ and 0
4. $(-) \cdot a$ preserves $\ovee$ and 0

Examples:

1. $[0,1]$, $\mathcal{E}f(\mathcal{H})$, $\mathrm{Meas}(X, [0,1])$ are
2. $\wp(X)$ is not

Effect modules

An **effect module** is an effect algebra E with scalar multiplication $\lambda \cdot x$ ($\lambda \in [0,1]$, $x \in E$) such that

1. $1 \cdot a = a$
2. $\lambda \cdot (\mu \cdot a) = (\lambda \cdot \mu) \cdot a$
3. $\lambda \cdot (-)$ preserves $\ovee$ and 0
4. $(-) \cdot a$ preserves $\ovee$ and 0

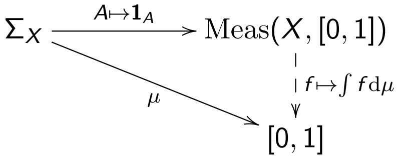
Examples:

1. $[0,1]$, $\mathcal{E}f(\mathcal{H})$, $\mathrm{Meas}(X, [0,1])$ are
2. $\wp(X)$ is not

A **homomorphism** of effect modules is what you expect

Universal property

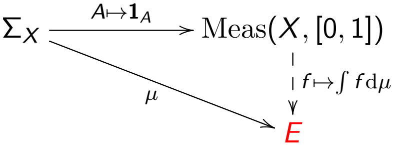
Let Σ_X be a σ-algebra on a set X.



For every homomorphism of ω-complete effect algebras μ
there is a unique hom. of ω-complete effect modules $\int(-)\mathrm{d}\mu$
such that $\int \mathbf{1}_A \mathrm{d}\mu = \mu(A)$.

Universal property

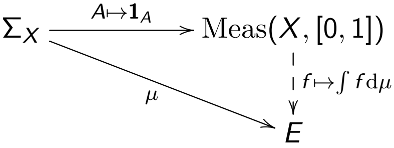
Let E be an ω-complete effect module.
Let Σ_X be a σ-algebra on a set X.



For every homomorphism of ω-complete effect algebras μ
there is a unique hom. of ω-complete effect modules $\int(-)\mathrm{d}\mu$
such that $\int \mathbf{1}_A \mathrm{d}\mu = \mu(A)$.

Universal property

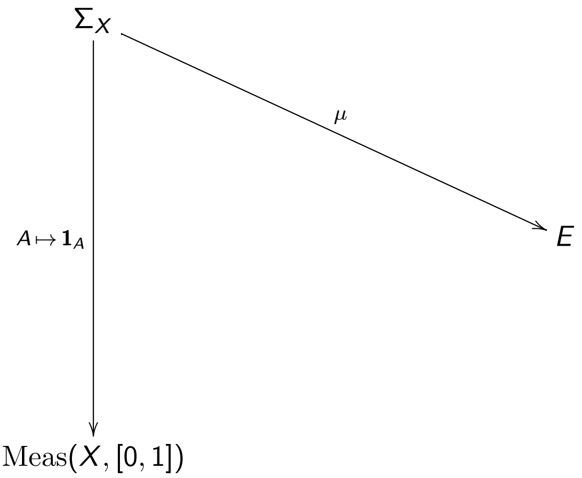
Let E be an ω-complete effect module.
Let Σ_X be a σ-algebra on a set X.



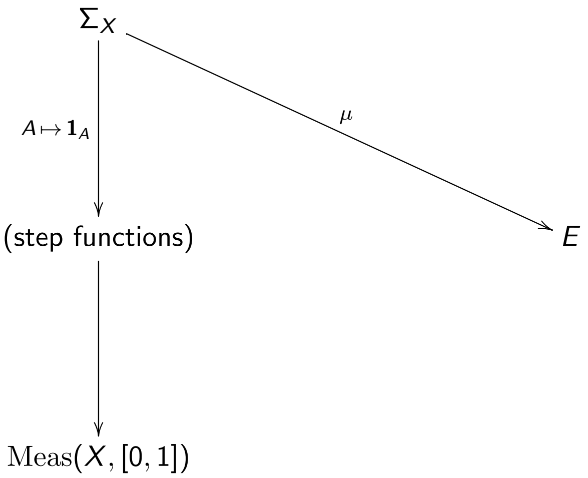
For every homomorphism of ω-complete effect algebras μ there is a unique hom. of ω-complete effect modules $\int(-)\mathrm{d}\mu$ such that $\int \mathbf{1}_A \mathrm{d}\mu = \mu(A)$.

Conclusion: $\mathrm{Meas}(X, [0,1])$ is the free ω-complete effect module over Σ_X via $A \mapsto \mathbf{1}_A$.

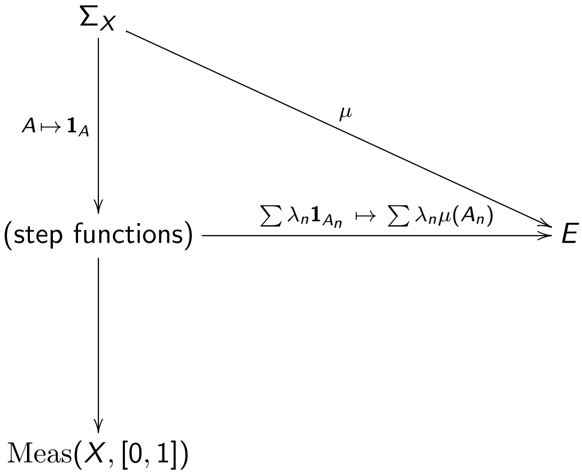
Sketch of the proof



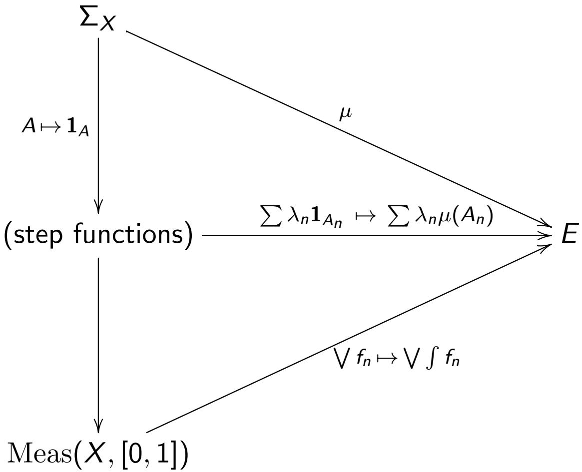
Sketch of the proof



Sketch of the proof



Sketch of the proof



Example: formulation of the Spectral Theorem

Let $\mathcal{H}$ be a Hilbert space.

Example: formulation of the Spectral Theorem

Let $\mathcal{H}$ be a Hilbert space.
Let A be an effect on $\mathcal{H}$ — $A \in \mathcal{E}f(\mathcal{H})$.

Example: formulation of the Spectral Theorem

Let $\mathcal{H}$ be a Hilbert space.
Let A be an effect on $\mathcal{H}$ — $A \in \mathcal{E}f(\mathcal{H})$.
Let $\sigma(A) = \{\lambda \in \mathbb{C} \colon A - \lambda \text{ is not invertible}\}$

Example: formulation of the Spectral Theorem

Let $\mathcal{H}$ be a Hilbert space.
Let A be an effect on $\mathcal{H}$ — $A \in \mathcal{E}f(\mathcal{H})$.
Let $\sigma(A) = \{\lambda \in \mathbb{C} \colon A - \lambda \text{ is not invertible}\}$

Spectral theorem: there is a unique homomorphism of ω-complete effect algebras $\phi \colon \Sigma_{\sigma_A} \longrightarrow \mathcal{E}f(\mathcal{H})$ such that

1. $A = \int \mathrm{id}\, \mathrm{d}\phi$
2. $\phi(S)$ is a projection for all $S \in \Sigma_{\sigma_A}$

Example: formulation of the Spectral Theorem

Let $\mathcal{H}$ be a Hilbert space.
Let A be an effect on $\mathcal{H}$ — $A \in \mathcal{E}f(\mathcal{H})$.
Let $\sigma(A) = \{\lambda \in \mathbb{C} \colon A - \lambda \text{ is not invertible}\}$

Spectral theorem: there is a unique homomorphism of ω-complete effect algebras $\phi \colon \Sigma_{\sigma_A} \longrightarrow \mathcal{E}f(\mathcal{H})$ such that

1. $A = \int \mathrm{id} \, \mathrm{d}\phi$
2. $\phi(S)$ is a projection for all $S \in \Sigma_{\sigma_A}$
3. if $\phi(G) = 0$ for an open subset of σ_A, then $G = \varnothing$

Example: formulation of the Spectral Theorem

Let $\mathcal{H}$ be a Hilbert space.
Let A be an effect on $\mathcal{H}$ — $A \in \mathcal{E}f(\mathcal{H})$.
Let $\sigma(A) = \{\lambda \in \mathbb{C} \colon A - \lambda \text{ is not invertible}\}$

Spectral theorem: there is a unique homomorphism of ω-complete effect algebras $\phi \colon \Sigma_{\sigma_A} \longrightarrow \mathcal{E}f(\mathcal{H})$ such that

1. $A = \int \mathrm{id}\, \mathrm{d}\phi$
2. $\phi(S)$ is a projection for all $S \in \Sigma_{\sigma_A}$
3. if $\phi(G) = 0$ for an open subset of σ_A, then $G = \varnothing$

Moreover, we have

4. $\int f \mathrm{d}\phi \cdot \int g \mathrm{d}\phi \;=\; \int f \cdot g\, \mathrm{d}\phi$

Example: formulation of the Spectral Theorem

Let $\mathcal{H}$ be a Hilbert space.
Let A be an effect on $\mathcal{H}$ — $A \in \mathcal{E}f(\mathcal{H})$.
Let $\sigma(A) = \{\lambda \in \mathbb{C} \colon A - \lambda \text{ is not invertible}\}$

Spectral theorem: there is a unique homomorphism of ω-complete effect algebras $\phi \colon \Sigma_{\sigma_A} \longrightarrow \mathcal{E}f(\mathcal{H})$ such that

1. $A = \int \mathrm{id}\, \mathrm{d}\phi$
2. $\phi(S)$ is a projection for all $S \in \Sigma_{\sigma_A}$
3. if $\phi(G) = 0$ for an open subset of σ_A, then $G = \varnothing$

Moreover, we have

4. $\int f \mathrm{d}\phi \cdot \int g \mathrm{d}\phi \;=\; \int f \cdot g \,\mathrm{d}\phi$
5. B commutes with A iff B commutes with all $\phi(S)$

Example: formulation of the Spectral Theorem

Let $\mathcal{H}$ be a Hilbert space.
Let A be an effect on $\mathcal{H}$ — $A \in \mathcal{E}f(\mathcal{H})$.
Let $\sigma(A) = \{\lambda \in \mathbb{C} \colon A - \lambda \text{ is not invertible}\}$

Spectral theorem: there is a unique homomorphism of ω-complete effect algebras $\phi \colon \Sigma_{\sigma_A} \longrightarrow \mathcal{E}f(\mathcal{H})$ such that

1. $A = \int \mathrm{id}\, \mathrm{d}\phi$
2. $\phi(S)$ is a projection for all $S \in \Sigma_{\sigma_A}$
3. if $\phi(G) = 0$ for an open subset of σ_A, then $G = \varnothing$

Moreover, we have

4. $\int f \mathrm{d}\phi \cdot \int g \mathrm{d}\phi \;=\; \int f \cdot g \, \mathrm{d}\phi$
5. B commutes with A iff B commutes with all $\phi(S)$

Motto: effects behave somewhat like measurable functions; the integral $\int(-)\mathrm{d}\phi \colon \;\mathrm{Meas}(X, [0,1]) \to \mathcal{E}f(\mathcal{H})$ translates.

Recap and outlook

You have seen:

1. Lebesgue integration and effect algebras.
2. A universal property of the extension of measure to integral.

Agenda:

1. Fubini's Theorem
2. Carathéodory's Extension Theorem
3. Gleason's Theorem