

Quotient–Comprehension Chains

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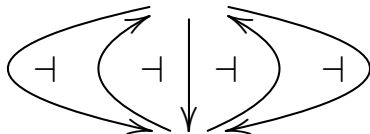
16 July 2015

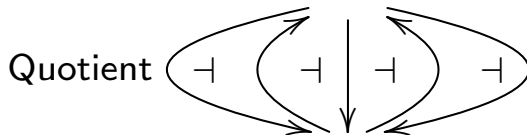
A Tree

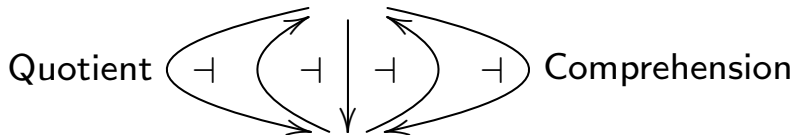


A Chain of Adjunctions

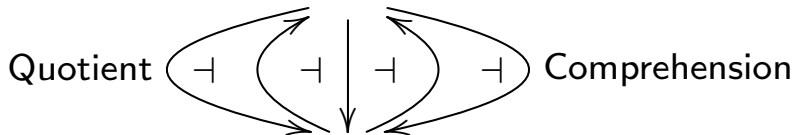








Quotient–Comprehension Chains



Example: Linear Subspaces

LSub



Vect

Example: Linear Subspaces

$$\begin{array}{c} \mathbf{LSub} \\ \downarrow \\ (V, S) \mapsto V \\ \downarrow \\ \mathbf{Vect} \end{array}$$

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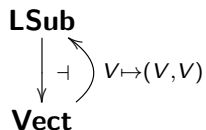
$$\frac{f: (V, S) \longrightarrow (W, T) \quad \text{in } \mathbf{LSub}}{f: V \longrightarrow W \quad \text{in } \mathbf{Vect} \quad \text{with } f(S) \subseteq T}$$

Example: Linear Subspaces

$$\begin{array}{c} \mathbf{LSub} \\ \downarrow \\ (V, S) \mapsto V \\ \downarrow \\ \mathbf{Vect} \end{array}$$

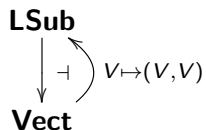
$$\frac{f: (V, S) \longrightarrow (W, W) \text{ in } \mathbf{LSub}}{f: V \longrightarrow W \text{ in } \mathbf{Vect}}$$

Example: Linear Subspaces



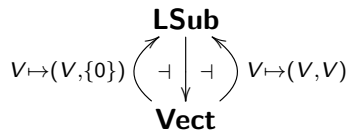
$$\frac{f: (V, S) \longrightarrow (W, W) \quad \text{in } \mathbf{LSub}}{f: V \longrightarrow W \quad \text{in } \mathbf{Vect}}$$

Example: Linear Subspaces



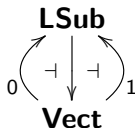
$$\frac{f: (V, \{0\}) \longrightarrow (W, T) \text{ in } \mathbf{LSub}}{f: V \rightarrow W \text{ in } \mathbf{Vect}}$$

Example: Linear Subspaces



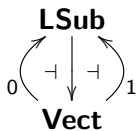
$$\frac{f: (V, \{0\}) \longrightarrow (W, T) \quad \text{in } \mathbf{LSub}}{f: V \longrightarrow W \quad \text{in } \mathbf{Vect}}$$

Example: Linear Subspaces

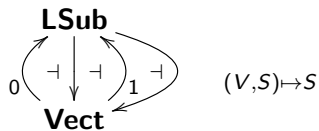


$$\frac{f: (V, \{0\}) \longrightarrow (W, T) \text{ in } \mathbf{LSub}}{f: V \rightarrow W \text{ in } \mathbf{Vect}}$$

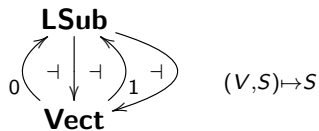
Example: Linear Subspaces



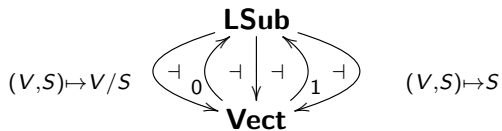
Example: Linear Subspaces



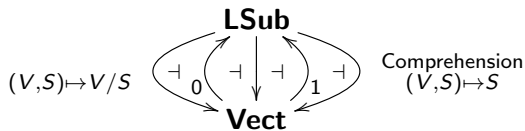
Example: Linear Subspaces



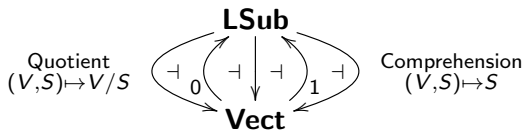
Example: Linear Subspaces



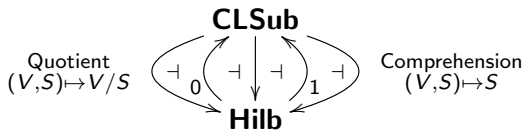
Example: Linear Subspaces



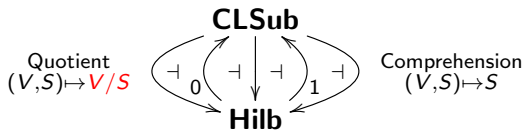
Example: Linear Subspaces



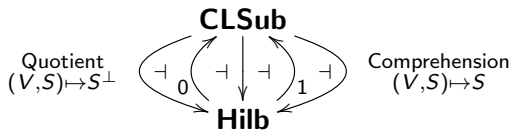
Example: Closed Linear Subspaces



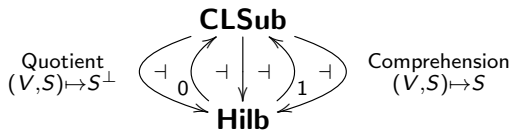
Example: Closed Linear Subspaces



Example: Closed Linear Subspaces

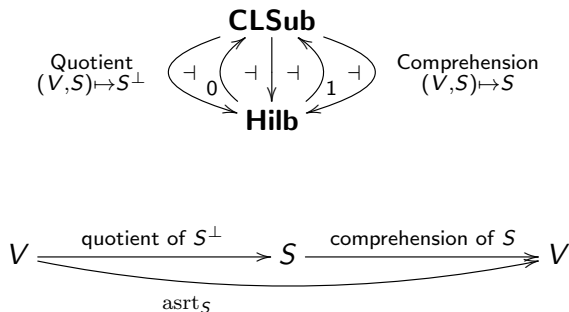


Example: Closed Linear Subspaces

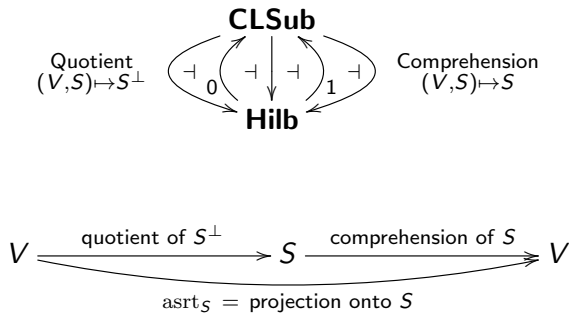


$$V \xrightarrow{\text{quotient of } S^\perp} S \xrightarrow{\text{comprehension of } S} V$$

Example: Closed Linear Subspaces



Example: Closed Linear Subspaces



Teaser

$$A \mapsto \sqrt{B} A \sqrt{B}$$

Categories with Quotient–Comprehension chain

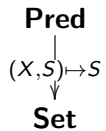
Categories with Quotient–Comprehension chain

1. **Vect, Hilb**

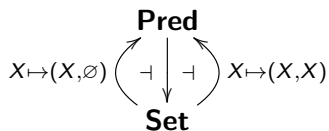
Categories with Quotient–Comprehension chain

1. **Vect, Hilb**
2. *(Boolean)*
3. *(Probabilistic)*
4. *(Quantum)*

Boolean Example: Subsets

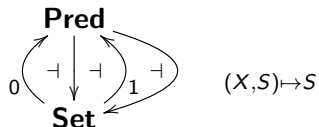


Boolean Example: Subsets



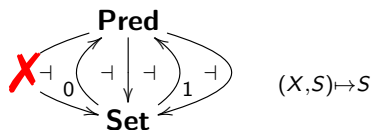
$$\frac{f: (X, S) \rightarrow (Y, T) \text{ in } \mathbf{Pred}}{f: X \rightarrow Y \text{ in } \mathbf{Set} \text{ with } f(S) \subseteq T}$$

Boolean Example: Subsets



$$\frac{f: (X, S) \rightarrow (Y, T) \text{ in } \mathbf{Pred}}{f: X \rightarrow Y \text{ in } \mathbf{Set} \text{ with } f(S) \subseteq T}$$

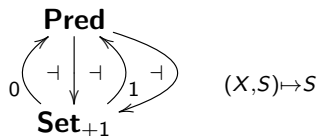
Boolean Example: Subsets



$X \mapsto (X, \emptyset)$ does not preserve limits

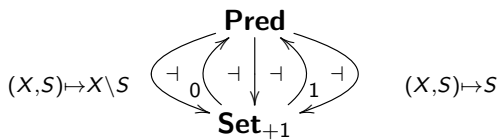
(because $(1, \emptyset)$ is not final in **Pred**.)

Boolean Example: Subsets

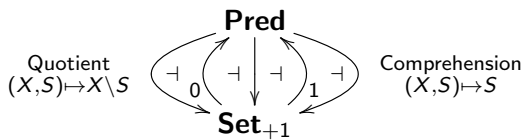


Set₊₁ = Kleisli category of $(-)+1$
= Sets with partial maps

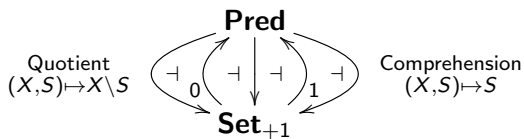
Boolean Example: Subsets



Boolean Example: Subsets

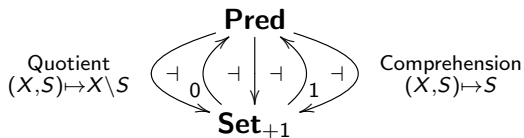


Boolean Example: Subsets



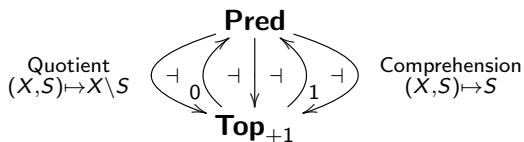
$$X \xrightarrow{\text{quotient of } X \setminus S} S \xrightarrow{\text{comprehension of } S} X$$

Boolean Example: Subsets



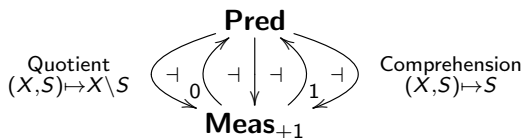
$$\begin{array}{c}
 X \xrightarrow{\text{quotient of } X \setminus S} S \xrightarrow{\text{comprehension of } S} X \\
 \text{asrt}_S: x \mapsto x \text{ for } x \in S \text{ and otherwise undefined}
 \end{array}$$

Boolean Example: Clopen Subsets



(X, S) in **Pred**
clopen S of a topological space X

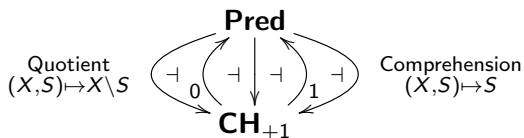
Boolean Example: Measurable Subsets



(X, S) in **Pred**

measurable subset S of a measurable space X

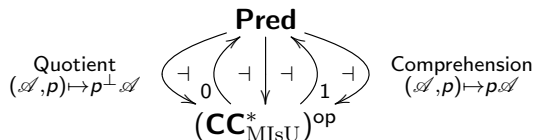
Boolean Example: Clopen Subsets



(X, S) in **Pred**

clopen S of a compact Hausdorff space X

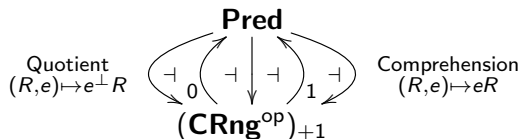
Boolean Example: Projections



$(\mathcal{A}, p)$ in $\mathbf{Pred}$

projection p of a commutative unital C^* -algebra $\mathcal{A}$

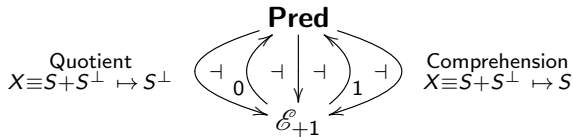
Boolean Example: Idempotents



(R, e) in **Pred**

idempotent p of a commutative unital ring $\mathcal{A}$

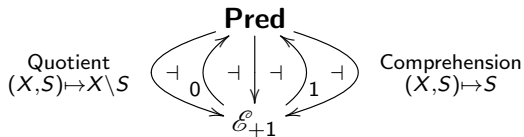
Boolean Example: Extensive Category



$$\frac{(X, S) \text{ in } \mathbf{Pred}}{S + S^\perp \equiv X}$$

where $\mathcal{E}$ is an *extensive category* with final object, 1

Boolean Example: Boolean Subobjects in a Topos



$$\begin{array}{c}
 (X, S) \text{ in } \mathbf{Pred} \\
 \hline
 S \xrightarrow{\text{Boolean subobject}} X
 \end{array}$$

where $\mathcal{E}$ is a *topos*

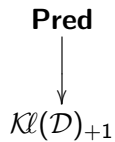
Categories with Quotient–Comprehension chain

1. **Vect, Hilb**
2. *(Boolean)*
3. *(Probabilistic)*
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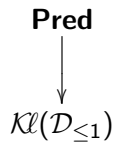
Categories with Quotient–Comprehension chain

1. **Vect**, **Hilb**
2. *(Boolean)* **Set**, **Top**, **Meas**, **CH**, $(\mathbf{CC}_{\text{MIU}}^*)^{\text{op}}$, $\mathbf{CRng}^{\text{op}}$,
any extensive category (with final object, such as a topos)
3. *(Probabilistic)*
4. *(Quantum)*

Probabilistic Example: $\mathcal{Kl}(\mathcal{D})$



Probabilistic Example: $\mathcal{Kl}(\mathcal{D})$



Probabilistic Example: $\mathcal{Kl}(\mathcal{D})$

$$\begin{array}{c} \mathbf{Pred} \\ \downarrow \\ \mathcal{Kl}(\mathcal{D}_{\leq 1}) \end{array}$$

$$\mathcal{D}_{\leq 1}(X) = \{ \sum p_i |x_i\rangle : \sum p_i \leq 1 \}$$

Probabilistic Example: $\mathcal{Kl}(\mathcal{D})$

$$\begin{array}{c} \mathbf{Pred} \\ \downarrow \\ (X, p) \mapsto X \\ \downarrow \\ \mathcal{Kl}(\mathcal{D}_{\leq 1}) \end{array}$$

$$\frac{(X, p) \text{ in } \mathbf{Pred}}{p: X \rightarrow [0, 1]}$$

$$\frac{f: (X, p) \rightarrow (Y, q) \text{ in } \mathbf{Pred}}{f: X \rightarrow Y \text{ in } \mathcal{Kl}(\mathcal{D}_{\leq 1}) \text{ with } p \leq q \circ f}$$

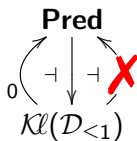
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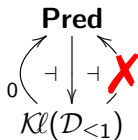
Probabilistic Example: $\mathcal{Kl}(\mathcal{D})$



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Probabilistic Example: $\mathcal{Kl}(\mathcal{D})$



(X, p) in **Pred**

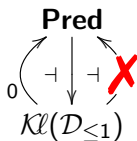
$p: X \rightarrow [0, 1]$

$f: (X, p) \rightarrow (Y, q)$ in **Pred**

$f: X \rightarrow Y$ in $\mathcal{Kl}(\mathcal{D}_{\leq 1})$ with $p \leq q \circ f$

since $\emptyset$ is final in $\mathcal{Kl}(\mathcal{D}_{\leq 1})$...

Probabilistic Example: $\mathcal{Kl}(\mathcal{D})$



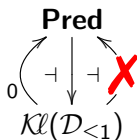
$$\frac{(X, p) \text{ in } \mathbf{Pred}}{p: X \rightarrow [0, 1]}$$

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since $\emptyset$ is final in $\mathcal{Kl}(\mathcal{D}_{\leq 1}) \dots$

$(\emptyset, q)$ is final in **Pred** for some q ,

Probabilistic Example: $\mathcal{Kl}(\mathcal{D})$



$$\frac{(X, p) \text{ in } \mathbf{Pred}}{\underline{\underline{p: X \rightarrow [0, 1]}}} \qquad \frac{f: (X, p) \rightarrow (Y, q) \text{ in } \mathbf{Pred}}{\underline{\underline{f: X \rightarrow Y \text{ in } \mathcal{Kl}(\mathcal{D}_{\leq 1}) \text{ with } p \leq q \circ f}}}$$

since $\emptyset$ is final in $\mathcal{Kl}(\mathcal{D}_{\leq 1})$...

$(\emptyset, q)$ is final in **Pred** for some q ,

but it is not, since $q \circ f = 0$ for all $f: X \rightarrow \emptyset$.

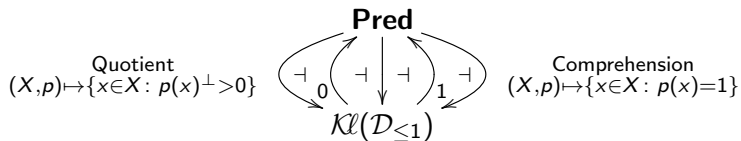
Probabilistic Example: $\mathcal{Kl}(\mathcal{D})$

$$\begin{array}{c} \mathbf{Pred} \\ \downarrow \\ \mathcal{Kl}(\mathcal{D}_{\leq 1}) \end{array}$$

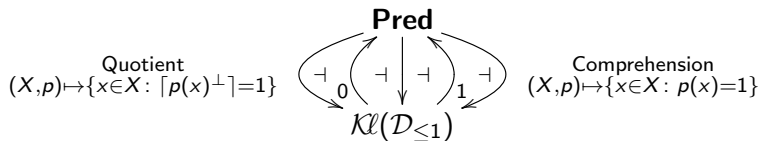
$$\frac{(X, p) \text{ in } \mathbf{Pred}}{p: X \rightarrow [0, 1]}$$

$$\frac{f: (X, p) \rightarrow (Y, q) \text{ in } \mathbf{Pred}}{f: X \rightarrow Y \text{ in } \mathcal{Kl}(\mathcal{D}_{\leq 1}) \text{ with } p \leq q \circ f + (1 \circ f)^\perp}$$

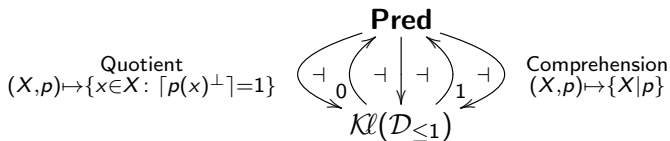
Probabilistic Example: $\mathcal{KL}(\mathcal{D})$



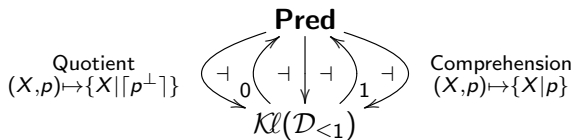
Probabilistic Example: $\mathcal{KL}(\mathcal{D})$



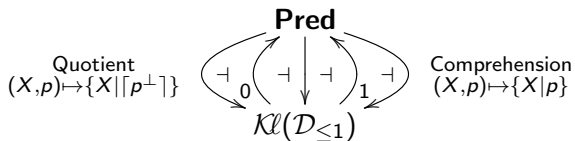
Probabilistic Example: $\mathcal{KL}(\mathcal{D})$



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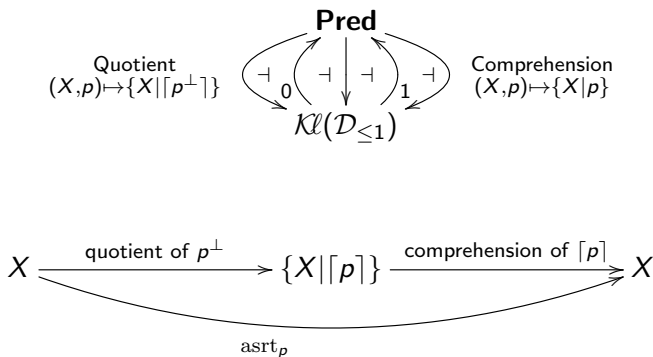


Probabilistic Example: $\mathcal{KL}(\mathcal{D})$

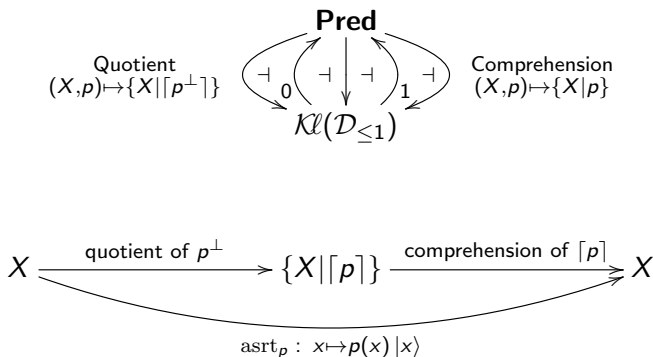


$$X \xrightarrow{\text{quotient of } p^\perp} \{X \mid \lceil p \rceil\} \xrightarrow{\text{comprehension of } \lceil p \rceil} X$$

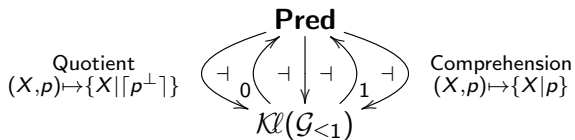
Probabilistic Example: $\mathcal{KL}(\mathcal{D})$



Probabilistic Example: $\mathcal{KL}(\mathcal{D})$



Probabilistic Example: $\mathcal{KL}(\mathcal{G})$



Categories with Quotient–Comprehension chain

1. **Vect**, **Hilb**
2. *(Boolean)* **Set**, **Top**, **Meas**, **CH**, $(\mathbf{CC}_{\text{MIU}}^*)^{\text{op}}$, $\mathbf{CRng}^{\text{op}}$,
any extensive category (with final object, such as a topos)
3. *(Probabilistic)*
4. *(Quantum)*

Categories with Quotient–Comprehension chain

1. **Vect**, **Hilb**
2. (*Boolean*) **Set**, **Top**, **Meas**, **CH**, $(\mathbf{CC}_{\text{MIU}}^*)^{\text{op}}$, $\mathbf{CRng}^{\text{op}}$,
any extensive category (with final object, such as a topos)
3. (*Probabilistic*) $\mathcal{KL}(\mathcal{D})$, $\mathcal{KL}(\mathcal{G})$, ...
4. (*Quantum*)

Quantum Example: Operator Algebras

$$\begin{array}{c} \mathbf{Pred} \\ \downarrow \\ (\mathbf{C}_{\text{PsU}}^*)^{\text{op}} \end{array}$$

Quantum Example: Operator Algebras

$$\begin{array}{c} \mathbf{Pred} \\ \downarrow \\ (\mathcal{A}, p) \mapsto \mathcal{A} \\ \downarrow \\ (\mathbf{C}_{\text{PsU}}^*)^{\text{op}} \end{array}$$

$$\frac{(\mathcal{A}, p) \text{ in } \mathbf{Pred}}{p \in [0, 1]_{\mathcal{A}}}$$

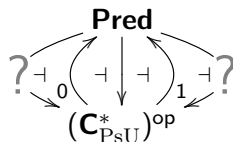
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$$\frac{f: (\mathcal{A}, p) \rightarrow (\mathcal{B}, q) \text{ in } \mathbf{Pred}}{f: \mathcal{B} \rightarrow \mathcal{A} \text{ in } \mathbf{C}_{\text{PsU}}^* , p \leq f(q) + f(1)^{\perp}}$$

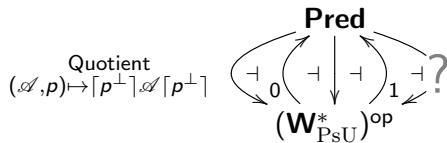
Quantum Example: Operator Algebras



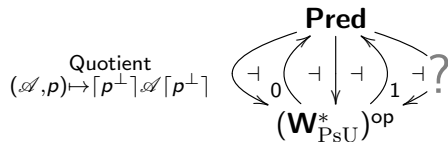
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Quantum Example: Operator Algebras

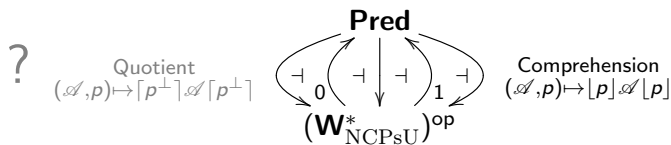


Quantum Example: Operator Algebras

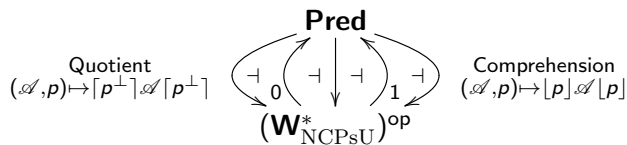


$[q] = (\text{least projection above } q)$

Quantum Example: Operator Algebras



Quantum Example: Operator Algebras



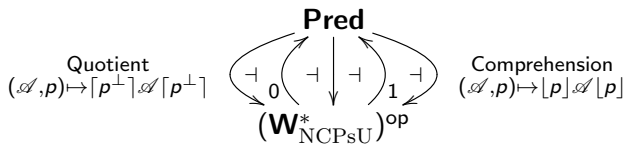
Quantum Example: Operator Algebras

$$\begin{array}{ccc}
 \text{Quotient} & \text{Pred} & \text{Comprehension} \\
 (\mathcal{A}, p) \mapsto [p^\perp] \mathcal{A} [p^\perp] & \begin{array}{c} \begin{array}{c} \curvearrowright \quad \curvearrowleft \\ \downarrow \\ \downarrow \end{array} \\ \text{0} \quad \text{1} \end{array} & (\mathcal{A}, p) \mapsto [p] \mathcal{A} [p] \\
 & (\mathbf{W}_{\text{NCPsU}}^*)^{\text{op}} &
 \end{array}$$

Cauchy–Schwarz inequality for a 2-positive map $f: \mathcal{A} \rightarrow \mathcal{B}$:

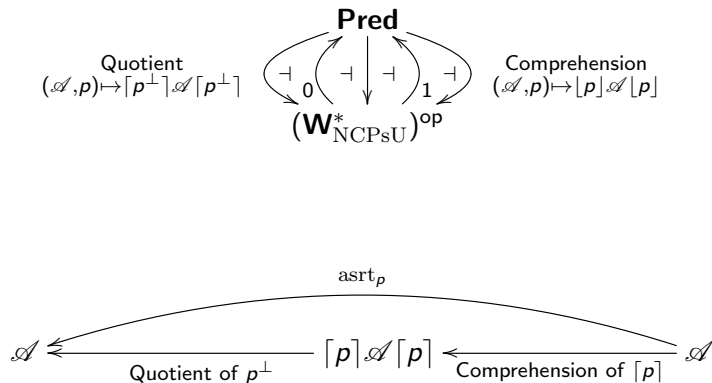
$$\|f(a^*b)\|^2 \leq \|f(a^*a)\| \cdot \|f(b^*b)\|$$

Quantum Example: Operator Algebras

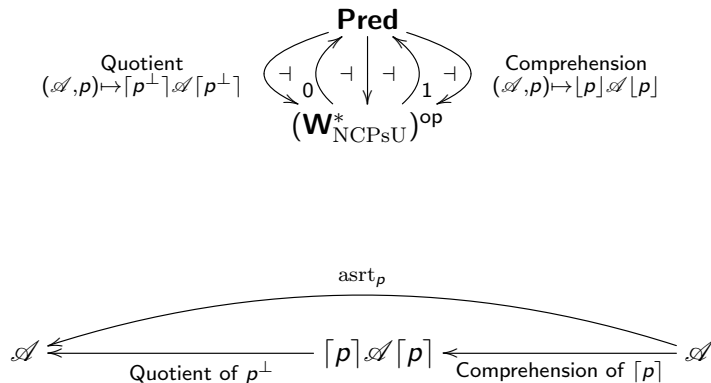


$$\mathcal{A} \xleftarrow{\text{Quotient of } p^\perp} [p] \mathcal{A} [p] \xleftarrow{\text{Comprehension of } [p]} \mathcal{A}$$

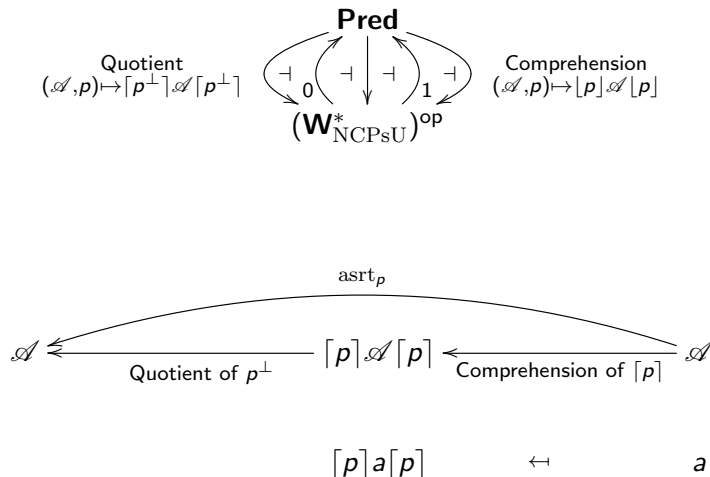
Quantum Example: Operator Algebras



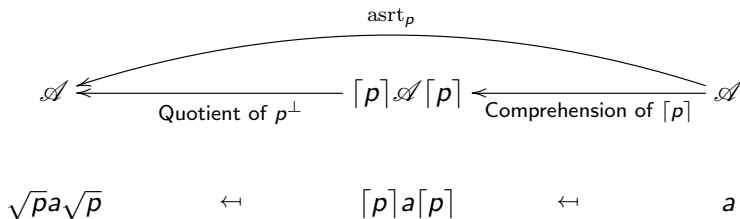
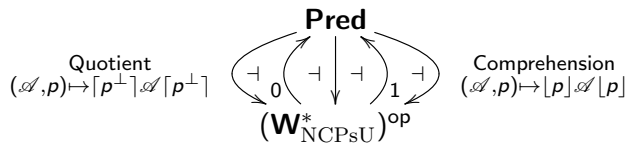
Quantum Example: Operator Algebras



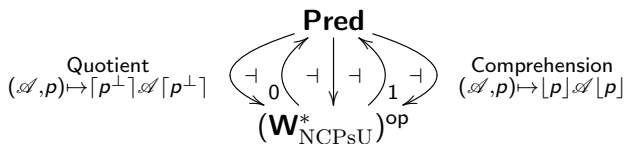
Quantum Example: Operator Algebras



Quantum Example: Operator Algebras

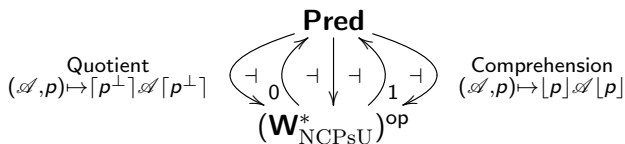


Quantum Example: Operator Algebras



$$\begin{array}{ccccc}
 \mathcal{A} & \xleftarrow{\text{Quotient of } p^\perp} & [p] \mathcal{A} [p] & \xleftarrow{u^* a u \leftarrow a} & [p] \mathcal{A} [p] & \xleftarrow{\text{Compreh. of } [p]} & \mathcal{A}
 \end{array}$$

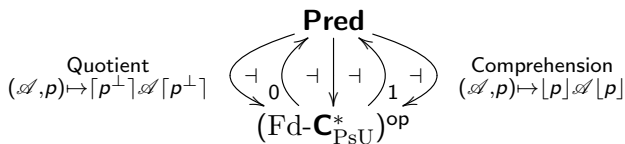
Quantum Example: Operator Algebras



$$\mathcal{A} \xleftarrow{\text{Quotient of } p^\perp} [p] \mathcal{A} [p] \xleftarrow{u^* a u \mapsto a} [p] \mathcal{A} [p] \xleftarrow{\text{Compreh. of } [p]} \mathcal{A}$$

$$\sqrt{p} u^* a u \sqrt{p} \mapsto a$$

Quantum Example: Operator Algebras



Categories with Quotient–Comprehension chain

1. **Vect**, **Hilb**
2. (*Boolean*) **Set**, **Top**, **Meas**, **CH**, $(\mathbf{CC}_{\text{MIU}}^*)^{\text{op}}$, $\mathbf{CRng}^{\text{op}}$,
any extensive category (with final object, such as a topos)
3. (*Probabilistic*) $\mathcal{KL}(\mathcal{D})$, $\mathcal{KL}(\mathcal{G})$, ...
4. (*Quantum*)

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5. For horrendous counterexamples: **OUS**^{op}

Questions?

