

Quotients, Comprehension, and Measurement

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Abstract—This paper establishes novel connections between on the one hand the notions of quotient and comprehension from (categorical) logic, and on the other hand the notion of measurement from quantum theory (and probability theory as well). Comprehension sends a predicate $\varphi(x)$ on a type σ to the type $\{x: \sigma \mid \varphi(x)\}$. Here we describe comprehension and quotient in a setting with partiality. This enables us to describe quotients for predicates, instead of relations (as usual). A quotient of a predicate φ on a type sigma σ then yields a new type σ/φ . The first leading observation of this paper is that such quotients and comprehension can be described via a chain of four consecutive adjunctions. This will be demonstrated in many examples: in vector spaces and Hilbert spaces; in partial, non-deterministic, and probabilistic computation; and in quantum computation (modeled by von Neumann algebras).

A second observation is: when comprehension and quotient are suitably connected in this chain of adjunctions, measurement maps can be defined (also known as instruments). This will be demonstrated in several examples. In the end we discuss how these observations lead towards a categorical axiomatization of quantum theory. We pinpoint the subtlety of such a categorical formulation: it is unclear yet to what extent these instruments can be determined uniquely.

I. INTRODUCTION

A topos is a well-established categorical notion for the study of constructive logic and set/type theory. Our long-term aim, to which this paper contributes, is to define a quantum-analogue of a topos, for which we already propose a name, namely: *telos* (Greek for effect, goal). The aim of “telos theory” is to axiomatize some of the main features of quantum theory. This paper builds on the theory of (measuring) instruments introduced in [1], but provides a completely new perspective, namely the relationship of measurement with quotients and comprehension. We claim that this relationship must be an essential part of the axiomatization that we are after. Since quotients and comprehension are important ingredients of (categorical) logic and type theory, we provide a basis for quantum measurement within a relatively familiar logical/categorical setting.

This notion of telos thus serves as a point on the horizon. Nevertheless, we claim that this paper brings us closer to a precise formulation: it introduces quotients and comprehension via a chain of adjunctions, and shows how such a chain, with an additional equation, namely (8) below, gives rise to the definition of a (measuring) instrument.

Roughly, the contributions in this paper consist of four parts.

- i) The paper describes the quotient–comprehension chain in many examples, including vector spaces and

Hilbert spaces, set-theoretic models for partial, non-deterministic and probabilistic computation, and also von Neumann algebras. The latter example captures the quantum world, and is mathematically highly non-trivial. In fact, the details of this example form a separate article [2], from which we pull the essentials.

- ii) The paper also describes these chain-of-adjunction examples in abstract categorical form, and identifies a condition under which quantum measuring instruments can be defined from such chains. This condition holds in several of the examples, and leads to the same instruments as identified in [1].
- iii) Given this abstract formulation, some basic theory is developed, involving partial maps, instruments and sequential products of predicates (“andthen”). Also this is only a part of a separate bigger project [3], [4].
- iv) Finally, some fundamental issues in the axiomatization of quantum theory are discussed in terms of ‘property’ versus ‘structure’. Typically in category theory everything is defined in terms of universal properties and is thus determined up-to-isomorphism. It seems that the instruments that we need here are at this point merely ‘structure’; not uniquely determined.

The latter point is currently our main obstacle on the road towards telos theory. The question remains: is the lack of universality a ‘bug’ or a ‘feature’? Unfortunately, at this stage we are unable to provide an answer to this cliff-hanger.

The paper is organized as follows. We introduce quotient–comprehension chains by example of vector spaces and Hilbert spaces in Section II. We continue with quotient–comprehension chains in a set-theoretic setting (in Section III), a probabilistic setting (in Section IV), and in a quantum setting (in Section V). We give a categorical framework for these chains in Section VI which we explore in Section VII defining instrument maps and deriving some of their properties in this abstract setting.

Early categorical work in quantum foundations is in [5] and [6]. The latter reference builds on Hilbert spaces, which do not satisfy our axioms (they are not an effectus, see Definition 8. Hilbert spaces do play a role in as far they give rise to von Neumann algebras. In follow-up of [6] work ‘CPM-style’ constructions of Selinger [7] (see also [8]) applied to categories of such Hilbert spaces come closer to what we are doing, since they can be equipped with the C^* -algebraic structure. The precise relation still needs to be clarified.

Notice that we have omitted the coprojection $\kappa_1: Y \rightarrow Y + 1$. A more precise formulation would be $f(x) = \kappa_1(y) \Rightarrow Q(y)$.

Each poset $\square(X) = \mathcal{P}(X)$ has a greatest element $1 = X \subseteq X$ and a least element $0 = \emptyset \subseteq X$. Moreover, $\square(f)(1) = 1$. Hence the conditions of Lemma 2 are satisfied, so that the forgetful functor $\int \square \rightarrow \mathbf{Sets}_\circ$ has both a left and a right adjoint. But there is more.

Proposition 4: In the set-theoretic case we have, like in (1):

$$\begin{array}{ccc} \text{Quotient} & \int \square & \text{Comprehension} \\ (P \subseteq X) \mapsto \neg P & \begin{array}{c} \curvearrowright \quad \curvearrowleft \\ \downarrow \quad \downarrow \\ \text{Sets}_\circ \end{array} & (P \subseteq X) \mapsto P \end{array} \quad (3)$$

We skip the proof of this result, but we will sketch the proof of a more interesting variation below. We do note that there is a clear similarity with the earlier vector space and Hilbert space examples: in a quotient V/P , for a linear subspace $P \subseteq V$, all elements from P are identified in V/P . Similarly, in the above set-theoretic case, a subset $P \subseteq X$ yields as quotient the complement $\neg P = \{x \mid x \notin P\}$. It is the part of X that remains when all elements from P are removed (or, identified with the base point, in a setting with partial functions). Thus, the quotient of $P \subseteq X$ is the comprehension of the negation of P .

For our next example we write \mathcal{P}_* for the *nonempty* powerset monad on \mathbf{Sets} , $\mathcal{Kl}(\mathcal{P}_*)$ for its Kleisli category, and $\mathcal{Kl}(\mathcal{P}_*)_\circ$ for the Kleisli category of the lift monad on $\mathcal{Kl}(\mathcal{P}_*)$. Thus, maps $X \rightarrow Y$ in $\mathcal{Kl}(\mathcal{P}_*)_\circ$ are functions $X \rightarrow \mathcal{P}_*(Y + 1)$. They capture non-deterministic computation, with multiple successor states and possibly also non-termination.

There is again a predicate functor $\square: \mathcal{Kl}(\mathcal{P}_*)_\circ \rightarrow \mathbf{PoSets}^{\text{op}}$ with $\square(X) = \mathcal{P}(X)$. For $f: X \rightarrow \mathcal{P}_*(Y + 1)$ we define:

$$\square(f)(Q) = \{x \in X \mid \forall y \in Y. y \in f(x) \Rightarrow Q(y)\}.$$

Proposition 5: Also for non-deterministic computation via the non-empty powerset monad \mathcal{P}_* we have:

$$\begin{array}{ccc} \text{Quotient} & \int \square & \text{Comprehension} \\ (P \subseteq X) \mapsto \neg P & \begin{array}{c} \curvearrowright \quad \curvearrowleft \\ \downarrow \quad \downarrow \\ \mathcal{Kl}(\mathcal{P}_*)_\circ \end{array} & (P \subseteq X) \mapsto P \end{array} \quad (4)$$

Proof The truth functor $1(X) = (X \subseteq X)$ and falsum functor $0(X) = (\emptyset \subseteq X)$ are obtained via Lemma 2.

The comprehension adjunction is easy: for a map $f: 1X \rightarrow (Y, Q)$ in $\int \square$ we have $f: X \rightarrow \mathcal{P}_*(Y + 1)$ satisfying $1X \subseteq \square(f)(Q)$. This means that for each $x \in X$ and $y \in Y$ we have: $y \in f(x) \Rightarrow Q(y)$. Thus we can factor f as $\bar{f}: X \rightarrow \mathcal{P}_*(Q + 1)$, giving us a map $\bar{f}: X \rightarrow Q$ in $\mathcal{Kl}(\mathcal{P}_*)_\circ$.

The quotient adjunction involves correspondences:

$$\begin{array}{ccc} (P \subseteq X) & \xrightarrow{f} & 0Y \\ \hline \neg P & \xrightarrow{g} & Y \end{array} \quad \begin{array}{l} \text{in } \int \square \\ \text{in } \mathcal{Kl}(\mathcal{P}_*)_\circ \end{array}$$

We spell out the transpose operations.

- Given a map $f: (P \subseteq X) \rightarrow (\emptyset \subseteq Y)$ in $\int \square$, we have $P \subseteq \square_f(\emptyset) = \{x \mid f(x) = \{*\}\}$. We can define $\bar{f}: \neg P \rightarrow \mathcal{P}_*(Y + 1)$ simply as $\bar{f}(x) = f(x)$.
- For $g: \neg P \rightarrow \mathcal{P}_*(Y + 1)$ we get $\bar{g}: X \rightarrow \mathcal{P}_*(Y + 1)$ as:

$$\bar{g}(x) = \begin{cases} \{*\} & \text{if } x \in P \\ g(x) & \text{if } x \in \neg P \end{cases}$$

This \bar{g} is a map $(P \subseteq X) \rightarrow (\emptyset \subseteq Y)$ in $\int \square$ since $\square_{\bar{g}}(\emptyset) = \{x \mid \bar{g}(x) = \{*\}\} \supseteq P$.

For $x \in X$, $\bar{\bar{f}}(x) = f(x)$, and $\bar{\bar{g}}(x) = g(x)$, for $x \in \neg P$. ■

IV. A PROBABILISTIC EXAMPLE

In this section we show how the quotient–comprehension chains of adjunctions also exist in probabilistic computation, via the (finite, discrete probability) distribution monad \mathcal{D} on \mathbf{Sets} . This monad sends a set X to the set of distributions:

$$\begin{aligned} \mathcal{D}(X) &= \{r_1|x_1\rangle + \dots + r_n|x_n\rangle \mid r_i \in [0, 1], x_i \in X, \sum_i r_i = 1\} \\ &\cong \{\varphi: X \rightarrow [0, 1] \mid \text{supp}(\varphi) \text{ is finite, and } \sum_x \varphi(x) = 1\}, \end{aligned}$$

where $\text{supp}(\varphi) = \{x \mid \varphi(x) \neq 0\}$. The ‘ket’ notation $|x\rangle$ is just syntactic sugar, used to distinguish an element $x \in X$ from its occurrence in a formal convex sum in $\mathcal{D}(X)$. In the sequel we shall freely switch between the above two descriptions of distributions. The unit of the monad is $\eta(x) = 1|x\rangle$, and the multiplication is $\mu(\Phi)(x) = \sum_\varphi \Phi(\varphi) \cdot \varphi(x)$.

We are primarily interested in the Kleisli category $\mathcal{Kl}(\mathcal{D})$ of the distribution monad. This category has coproducts, like in \mathbf{Sets} , and the singleton set $1 = \{*\}$ as final object, because $\mathcal{D}(1) \cong 1$. Hence we can consider the Kleisli category $\mathcal{Kl}(\mathcal{D})_\circ$ of the lift monad $(-)+1$ on $\mathcal{Kl}(\mathcal{D})$. Its objects are sets, and its maps $X \rightarrow Y$ are functions $X \rightarrow \mathcal{D}(Y + 1)$. Elements of $\mathcal{D}(Y + 1)$ are often called *subdistributions* on Y .

Like before we define a ‘predicate’ functor $\square: \mathcal{Kl}(\mathcal{D})_\circ \rightarrow \mathbf{PoSets}^{\text{op}}$. For a set X , we take $\square(X) = [0, 1]^X$, the set of ‘fuzzy’ predicates $X \rightarrow [0, 1]$ on X . They form a poset, by using pointwise the order on $[0, 1]$. This poset $[0, 1]^X$ contains a top (1) and bottom (0) element, namely the constant functions $x \mapsto 1$ and $x \mapsto 0$ respectively. For a predicate $p \in [0, 1]^X$ we write $p^\perp \in [0, 1]^X$ for the orthocomplement, given by $p^\perp(x) = 1 - p(x)$. Notice that $p^{\perp\perp} = p$, $1^\perp = 0$ and $0^\perp = 1$. Together with its partial sum operation, the set of fuzzy predicates $[0, 1]^X$ forms a what is called an *effect module*, that is, an effect algebra with a $[0, 1]$ -action (see [1] for details).

A predicate $p \in [0, 1]^X$ is called **sharp** if $p^2 = p$. This means that $p(x) \in \{0, 1\}$, so that p is a Boolean predicate in $\{0, 1\}^X$. Equivalently, p is sharp if $p \wedge p^\perp = 0$. For each predicate $p \in [0, 1]^X$ there is a least sharp predicate $\lceil p \rceil$ with $p \leq \lceil p \rceil$, and a greatest sharp predicate $\lfloor p \rfloor \leq p$, namely:

$$\lceil p \rceil(x) = \begin{cases} 0 & \text{if } p(x) = 0 \\ 1 & \text{otherwise.} \end{cases} \quad \lfloor p \rfloor(x) = \begin{cases} 1 & \text{if } p(x) = 1 \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that these least/greatest sharp predicates are each others De Morgan duals, that is, $[p^\perp] = [p]^\perp$. If p is a sharp, then $[p] = p = [p]$.

For a function $f: X \rightarrow \mathcal{D}(Y+1)$ we define $\square(f): [0,1]^Y \rightarrow [0,1]^X$ as:

$$\square(f)(q)(x) = \sum_{y \in Y} f(x)(y) \cdot q(y) + f(x)(*)$$

Since $f(x) \in \mathcal{D}(Y+1)$ is a distribution, we have $\sum_{y \in Y} f(x)(y) + f(x)(*) = 1$, so that $\square(f)(1) = 1$. Hence Lemma 2 applies, so that we have a functor $\int \square \rightarrow \mathcal{Kl}(\mathcal{D})_\circ$ with falsum 0 as left adjoint, and truth 1 as right adjoint. Recall that a map $(X, p) \rightarrow (Y, q)$ in $\int \square$ is a function $f: X \rightarrow \mathcal{D}(Y+1)$ with $p(x) \leq \square(f)(q)(x)$ for all $x \in X$.

Proposition 6: The distribution monad \mathcal{D} on **Sets**, used to model probabilistic computation, gives rise to:

$$\begin{array}{ccc} & \int \square & \\ \text{Quotient} & \begin{array}{c} \begin{array}{ccc} \begin{array}{c} \downarrow \text{0} \\ \text{---} \end{array} & \begin{array}{c} \downarrow \text{1} \\ \text{---} \end{array} & \begin{array}{c} \downarrow \text{1} \\ \text{---} \end{array} \\ \text{---} & \text{---} & \text{---} \\ \begin{array}{c} \downarrow \text{1} \\ \text{---} \end{array} & \begin{array}{c} \downarrow \text{0} \\ \text{---} \end{array} & \begin{array}{c} \downarrow \text{1} \\ \text{---} \end{array} \\ \text{---} & \text{---} & \text{---} \end{array} & \text{Comprehension} \\ (p \in [0,1]^X) \mapsto X/p & & (p \in [0,1]^X) \mapsto \{X|p\} \end{array} \quad (5) \end{array}$$

where $\{X|p\} = \{x \in X \mid p(x) = 1\}$, and $X/p = \{X|[p^\perp]\} = \{x \mid p(x) \neq 1\}$.

Proof For a map $f: 1Y \rightarrow (X, p)$ in $\int \square$ we have $f: Y \rightarrow \mathcal{D}(X+1)$ satisfying $1 \leq \square(f)(p)$. This means, for each $y \in Y$,

$$1 = (\sum_x f(y)(x) \cdot p(x)) + f(y)(*)$$

Since $\sum_x f(y)(x) + f(y)(*) = 1$, this can only happen if $f(y)(x) \neq 0 \Rightarrow p(x) = 1$. But then we can factor f as $\bar{f}: Y \rightarrow \{X|p\}$ in $\mathcal{Kl}(\mathcal{D})_\circ$, where:

$$\bar{f}(y) = \sum_{x, f(y)(x) \neq 0} f(y)(x)|x\rangle + f(y)(*)|*\rangle$$

In the other direction, given $g: Y \rightarrow \mathcal{D}(\{X|p\} + 1)$ we define $\bar{g}: Y \rightarrow \mathcal{D}(X+1)$ as $\bar{g}(y) = \sum_{x, p(x)=1} g(y)(x)|x\rangle + g(y)(*)|*\rangle$. Then, for each $y \in Y$,

$$\begin{aligned} \square(\bar{g})(p)(y) &= \sum_{x, p(x)=1} \bar{g}(y)(x) \cdot p(x) + \bar{g}(y)(*) \\ &= \sum_{x, p(x)=1} g(y)(x) + g(y)(*) \\ &= 1. \end{aligned}$$

Moreover, for $y \in Y$,

$$\begin{aligned} \bar{\bar{f}}(y) &= \sum_{x, p(x)=1} \bar{f}(y)(x)|x\rangle + \bar{f}(y)(*)|*\rangle \\ &= \sum_x f(y)(x)|x\rangle + \bar{f}(y)(*)|*\rangle \\ &\quad \text{since } p(x) \neq 1 \Rightarrow f(y)(x) = 0 \\ &= f(y) \\ \bar{\bar{g}}(y) &= \sum_{x, \bar{g}(y)(x) \neq 0} \bar{g}(y)(x)|x\rangle + \bar{g}(y)(*)|*\rangle \\ &= \sum_{x \in \{X|p\}} g(y)(x)|x\rangle + g(y)(*)|*\rangle \\ &= g(y). \end{aligned}$$

The quotient adjunction involves correspondences:

$$\frac{(X, p) \xrightarrow{f} 0Y}{X/p \xrightarrow{g} Y}$$

This works as follows.

- Given $f: (X, p) \rightarrow 0Y$ in $\int \square$, then $f: X \rightarrow \mathcal{D}(Y+1)$ satisfies $p \leq \square_f(0)$. This means that for each $x \in X$,

$$p(x) \leq \sum_y f(x)(y) \cdot 0(y) + f(x)(*) = f(x)(*)$$

We then define $\bar{f}: X/p \rightarrow \mathcal{D}(Y+1)$ as:

$$\bar{f}(x) = \sum_y \frac{f(x)(y)}{p^\perp(x)} |y\rangle + \frac{f(x)(*) - p(x)}{p^\perp(x)} |*\rangle$$

This is well-defined, since $p^\perp(x) \neq 0$ for $x \in X/p$.

- In the other direction, given $g: X/p \rightarrow \mathcal{D}(Y+1)$ we define $\bar{g}: X \rightarrow \mathcal{D}(Y+1)$ as:

$$\begin{aligned} \bar{g}(x) &= \sum_y p^\perp(x) \cdot g(x)(y)|y\rangle \\ &\quad + (p(x) + p^\perp(x) \cdot g(x)(*))|*\rangle. \end{aligned}$$

Notice that this extension of g outside the subset $\{X|[p^\perp]\} \hookrightarrow X$ is well-defined, since if $x \notin \{X|[p^\perp]\}$, then $p(x) = 1$, so $p^\perp(x) = 0$, which justifies writing $p^\perp(x) \cdot g(x)(y)$. In that case, when $p(x) = 1$, we get $\bar{g}(x) = 1|*\rangle$.

This \bar{g} is a morphism $(X, p) \rightarrow 0Y$ in $\int \square$, since $p \leq \square_{\bar{g}}(0)$, that is $p(x) \leq \bar{g}(x)(*)$. This follows since $p^\perp(x) \geq 0$ and $g(x)(*) \geq 0$ in:

$$\bar{g}(x)(*) = p(x) + p^\perp(x) \cdot g(x)(*) \geq p(x). \quad \blacksquare$$

The counit map $\pi_p: \{X|p\} \rightarrow \mathcal{D}(X+1)$ and the unit $\xi_p: X \rightarrow \mathcal{D}(X/p+1)$ are given by:

$$\pi_p(x)(u) = \begin{cases} 1 & \text{if } u = x \\ 0 & \text{otherwise} \end{cases} \quad \xi_p(x) = p^\perp(x)|x\rangle + p(x)|*\rangle$$

For the quotient $X/p^\perp = \{X|[p]\} = \{x \mid p(x) \neq 0\}$ we can form in the category $\mathcal{Kl}(\mathcal{D})_\circ$, the composites:

$$\begin{array}{ccc} X & \xrightarrow{\xi_{p^\perp}} & X/p^\perp & \xrightarrow{\pi_{[p]}} & X \\ & \searrow \text{asrt}_p & \downarrow \pi_{[p]} & \searrow & \downarrow \xi_{p^\perp} \\ & & X & & X/p^\perp \end{array}$$

On the left the ‘assert p ’ function $\text{asrt}_p: X \rightarrow \mathcal{D}(X+1)$ is:

$$\text{asrt}_p(x) = (\pi_{[p]} \circ \xi_{p^\perp})(x) = p(x)|x\rangle + p^\perp(x)|*\rangle$$

This map forms the basis for (measuring) instruments (and is essentially as in [1]). The other composite yields, for $x \in X/p^\perp$ and thus $p(x) \neq 0$,

$$(\xi_{p^\perp} \circ \pi_{[p]})(x) = p(x)|x\rangle + p^\perp(x)|*\rangle$$

Hence this composite is the identity if p is sharp.

The other composite yields, for $a \in [p]\mathcal{A}[p]$

$$(\pi_{[p]} \circ \xi_{p^\perp})(a) = [p]a[p].$$

Hence this composite is the identity iff p is a projection.

Proofsketch of Proposition 7 (Comprehension)

We must show that given a von Neumann algebra \mathcal{A} , an effect $p \in \mathcal{A}$, and a map $f: \mathcal{A} \rightarrow \mathcal{B}$ in \mathbf{vN}_\odot with $f(p) = f(1)$ there is a unique map $g: [p]\mathcal{A}[p] \rightarrow \mathcal{B}$ in \mathbf{vN}_\odot with $g([p]b[p]) = f(b)$. Put $g(b) = f(b)$; the difficulty is to show that $f([p]b[p]) = f(b)$. By a variant of Cauchy–Schwarz inequality for the completely positive map f (see [11], exercise 3.4) we can reduce this problem to proving that $f([p]) = f(1)$, that is, $f([p^\perp]) = 0$. Since $[p^\perp]$ is the supremum of $p^\perp \leq (p^\perp)^{1/2} \leq (p^\perp)^{1/4} \leq \dots$ and f is normal, $f([p^\perp])$ is the supremum of $f(p^\perp) \leq f((p^\perp)^{1/2}) \leq f((p^\perp)^{1/4}) \leq \dots$, all of which turn out to be zero by Cauchy–Schwarz since $f(p) = f(1)$. Thus $f([p^\perp]) = 0$, and we are done. Again, for more details, see [2].

(Quotient) We must show that given a von Neumann algebra \mathcal{A} , an effect $p \in \mathcal{A}$, and a map $f: \mathcal{B} \rightarrow \mathcal{A}$ in \mathbf{vN}_\odot with $f(1) \leq p^\perp$, there is a unique $g: \mathcal{B} \rightarrow [p^\perp]\mathcal{A}[p^\perp]$ in \mathbf{vN}_\odot such that $\sqrt{p^\perp}g(b)\sqrt{p^\perp} = f(b)$. If $\sqrt{p^\perp}$ is invertible, then we may define $g(b) = (\sqrt{p^\perp})^{-1}f(b)(\sqrt{p^\perp})^{-1}$, and this works. The proof is also straightforward if $\sqrt{p^\perp}$ is pseudoinvertible (=has norm-closed range). The trouble is that in general $\sqrt{p^\perp}$ is not (pseudo)invertible. However, there is a sequence s_n (which converges ultraweakly to the (pseudo)inverse if it exists and) for which $g(b) = \text{uwlim}_n s_n f(b) s_n$ exists and satisfies the requirements. For further details, see [2]. ■

VI. TOWARDS AN AXIOMATIZATION OF MEASUREMENT

We use the term ‘effectus’, like in [12] with a small twist (see Appendix A), for a category satisfying some basic assumptions identified in [1]. These assumptions already allow us to do a certain amount of logic, as will be explained next.

Definition 8: A category \mathbf{C} is called an **effectus** provided

- i) \mathbf{C} has finite coproducts $(+, 0)$ and a final object 1 ;
- ii) diagrams of the following form are pullbacks and

$$\begin{array}{ccc} A + X & \xrightarrow{\text{id}+f} & A + Y & & A & \xlongequal{\quad} & A \\ g+\text{id} \downarrow & & \downarrow g+\text{id} & & \kappa_1 \downarrow & & \downarrow \kappa_1 \\ B + X & \xrightarrow{\text{id}+f} & B + Y & & A + X & \xrightarrow{\text{id}+f} & A + Y \end{array}$$

- iii) the following maps are jointly monic,

$$(X + X) + 1 \xrightarrow[\triangleright_2, \kappa_2]{\triangleright_1, \kappa_2} X + 1$$

where $\triangleright_i: X + X \rightarrow X + 1$ are defined by $\triangleright_1 = [\kappa_1, \kappa_2 \circ !_X]$ and $\triangleright_2 = [\kappa_2 \circ !_X, \kappa_1]$.

A **state** on an object X is a map $\omega: 1 \rightarrow X$. A **predicate** on X is a map $p: X \rightarrow 1 + 1$. A map $f: X \rightarrow Y + 1$ is a **partial map** from X to Y , denoted by $f: X \dashrightarrow Y$. Recall

from Definition 3 that \mathbf{C}_\odot is the category of partial maps, i.e., the Kleisli category of the lift monad, whose composition is denoted by \circ . Explicitly, $g \circ f = [g, \kappa_2] \circ f$.

We here introduce some more notations and terminology for \mathbf{C}_\odot . We denote the canonical functor $\mathbf{C} \rightarrow \mathbf{C}_\odot$ by $(\hat{\quad})$; namely $\hat{X} = X$ and $\hat{f} = \kappa_1 \circ f$. Then we may write $\text{id}_X: X \dashrightarrow X$ for the identity on X in \mathbf{C}_\odot . We say a partial map $f: X \dashrightarrow Y$ is **total** if there exists a ‘total’ map $g: X \rightarrow Y$ such that $f = \hat{g}$. The category \mathbf{C}_\odot inherits coproducts from \mathbf{C} , which we denote by $\hat{+}$. Since $(\hat{\quad}): \mathbf{C} \rightarrow \mathbf{C}_\odot$ preserves the coproducts on the nose, $X \hat{+} Y = X + Y$ and $\hat{\kappa}_i$ are the coprojections, but note that $f \hat{+} g$ differs from $f + g$. The category \mathbf{C}_\odot has **zero maps** given by $0_{XY} := \kappa_2 \circ !_X: X \rightarrow Y + 1$, which satisfy $0_{YW} \circ f = 0_{XW} = g \circ 0_{XZ}$ for $f: X \dashrightarrow Y$ and $g: Z \dashrightarrow W$. The zero maps allow us to define **partial projections** (or quasi-projections) $\triangleright_i: X_1 + \dots + X_n \dashrightarrow X_i$ by $\triangleright_i \circ \hat{\kappa}_i = \hat{\text{id}}$ and $\triangleright_i \circ \hat{\kappa}_j = 0$ ($j \neq i$). This is consistent with \triangleright_1 and \triangleright_2 of Definition 8.

Example 9: The categories \mathbf{vN}^{op} , **Sets** and $\mathcal{Kl}(\mathcal{D})$ are effectuses. The category $\mathcal{Kl}(\mathcal{P}_*)$ is not as it does not satisfy the joint monicity requirement.

On the partial maps we can define a partial sum as follows.

Definition 10: Given $f, g: X \dashrightarrow Y$. We say f and g are **summable** (in symbols $f \perp g$), whenever there is a ‘bound’ $b: X \dashrightarrow Y + Y$ such that $\triangleright_1 \circ b = f$ and $\triangleright_2 \circ b = g$. In that case, the **sum** is defined as $f \oplus g = \nabla \circ b$. In the next section we observe that $\mathbf{C}_\odot(X, Y)$ is a partial commutative monoid (PCM for short) with the zero map 0_{XY} as a neutral element. The standard order $f \leq g \iff \exists h. f \oplus h = g$ turns out to be a partial order with a bottom 0_{XY} , unlike general PCMs (which are just preorders).

As predicates are partial maps to 1 , they come with a partial sum as well. We call the zero map $0_X := 0_{X1}: X \rightarrow 1 + 1$ the **falsum predicate**. For a predicate $p: X \rightarrow 1 + 1$, define its **orthocomplement** as $p^\perp := [\kappa_2, \kappa_1] \circ p$. Then $1_X := 0_X^\perp = \kappa_1 \circ !_X$ is called the **truth predicate**. With these structures the homsets $\mathbf{C}_\odot(X, 1) = \mathbf{C}(X, 1 + 1)$ are effect algebras [13], and posets with a bottom 0 and a top 1 . In particular $p \oplus p^\perp = 1$ holds. We say p is **sharp** whenever $p \wedge p^\perp = 0$. If there is a least sharp predicate above p , we denote it by $[p]$. In that case, write $[p] = [p^\perp]^\perp$, which is the greatest sharp predicate below p . For a partial map $f: X \dashrightarrow Y$, define its **domain predicate** as $\text{Dp}(f) := 1_Y \circ f = (!_Y + \text{id}) \circ f \in \mathbf{C}_\odot(X, 1)$.

Define the functor $\square: \mathbf{C}_\odot \rightarrow \mathbf{PoSets}^{\text{op}}$. by $\square(X) = \mathbf{C}_\odot(X, 1) = \mathbf{C}(X, 1 + 1)$ and $\square(f)(p) = [p, \kappa_1] \circ f$.

On top of an effectus we consider a chain of adjunctions, of the sort that we have seen in several examples. This is all unproblematic. The controversial requirement is the equality (8) below.

Definition 11: An **effectus with quotient–comprehension chain** is an effectus \mathbf{C} such that:

- i) For every predicate p , there is a least sharp predicate above it, denoted by $[p]$.
- ii) The forgetful functor $\int \square \rightarrow \mathbf{C}_\odot$ has as second left

adjoint Q and second right adjoint C :

$$\begin{array}{ccc}
 \text{Quotient} & \int \square & \text{Comprehension} \\
 (p, X) \mapsto X/p & \begin{array}{c} \circlearrowleft \\ \downarrow \\ \circlearrowright \end{array} & (p, X) \mapsto \{X|p\} \\
 & \mathbf{C}_\circ &
 \end{array} \quad (7)$$

We call these adjunctions the **quotient–comprehension chain** on \mathbf{C} .

We write the unit on the left and counit on the right as

$$\xi_p: X \rightarrow X/p \quad \pi_p: \{X|p\} \rightarrow X$$

iii) For any predicate p , we have $\{X|[p]\} = X/p^\perp$ and $\xi_{[p]^\perp} \circ \pi_{[p]} = \text{id}$, that is:

$$\begin{array}{ccc}
 \{X|[p]\} & \xlongequal{\quad} & X/[p]^\perp \\
 \searrow \pi_{[p]} & & \nearrow \xi_{[p]^\perp} \\
 & X &
 \end{array} \quad (8)$$

Define $\text{asrt}_p = \pi_{[p]} \circ \xi_{p^\perp}$. That is:

$$\begin{array}{ccc}
 X & \xrightarrow{\text{asrt}_p} & X \\
 \searrow \xi_{p^\perp} & & \nearrow \pi_{[p]} \\
 & X/p^\perp = \{X|[p]\} &
 \end{array} \quad (9)$$

Example 12: The categories \mathbf{vN}^{op} , **Sets** and $\mathcal{Kl}(\mathcal{D})$ are effectuses with quotient–comprehension chain.

The equation (8) is problematic since left and right adjoints are determined only up-to-isomorphism. What the requirement really says is that there is a particular choice of quotient and comprehension functor such that the triangle (8) commutes. This choice means that we are dealing with ‘structure’, and not with a ‘property’. We will come back to this issue in Subsection VII-A.

VII. ELEMENTARY PROPERTIES OF AN EFFECTUS WITH QUOTIENT–COMPREHENSION CHAIN

This section will describe some basic results about effectuses with quotient–comprehension chain. One of the difficulties in [1] was that (measuring) instrument maps had to be assumed part of the structure. We will show that in an effectus with quotient–comprehension chain there are canonical instrument maps, and that these satisfy two of the properties of instruments maps in [1] (namely (1) and (3) of [1, Assumption 2]). Nevertheless these canonical instruments do depend on the choice of quotient and comprehension, which we will demonstrate in the first subsection. It remains an important research problem for us to find additional axioms (if any) that remove this dependence.

In the second subsection we have collected some results about effectuses in general (see [1], [3] for more details).

We end this section with the definition of a *sequential product* on predicates, and give some of its properties. As an application of the sequential product we show that partial maps factor via total maps.

A. Dependence of the Assert Maps on the Quotient and Comprehension Functors

In this subsection we will show that the asrt_p map from (9) may depend on the choice of quotient and comprehension functor (see Lemma 14 below). However, this can only happen if p is not sharp:

Lemma 13: Let \mathbf{C} be an effectus with two quotient–comprehension chains giving us assert maps asrt_p and asrt'_p , respectively. Then $\text{asrt}_q = \text{asrt}'_q$ for sharp q .

Proof Let (Q, C) and (Q', C') be two quotient–comprehension chains for \mathbf{C} . By uniqueness of adjoints, there are natural isomorphisms $\alpha: Q \Rightarrow Q'$ and $\beta: C \Rightarrow C'$ with $\alpha \circ \xi = \xi'$ and $\pi = \pi' \circ \beta$. Let q be a sharp predicate. Note that $[q] = q$. By (8) the diagram below on the left commutes and so $\beta_q = \alpha_{q^\perp}$. Hence the diagram below on the right commutes and so $\text{asrt}_q = \text{asrt}'_q$.

$$\begin{array}{ccc}
 C(X, q) & \xlongequal{\quad} & Q(X, q^\perp) \\
 \beta_q \downarrow & \begin{array}{c} \nearrow \pi_q \\ \searrow \pi'_q \end{array} & \begin{array}{c} \nearrow \xi_{q^\perp} \\ \searrow \xi_{q^\perp} \end{array} \\
 C'(X, q) & \xlongequal{\quad} & Q'(X, q^\perp)
 \end{array} \quad \begin{array}{ccc}
 Q(X, q^\perp) & \xlongequal{\quad} & C(X, q) \\
 \xi_{q^\perp} \nearrow & & \searrow \pi_q \\
 X & \xrightarrow{\text{asrt}_q} & X \\
 \alpha_{q^\perp} \searrow & & \nearrow \pi'_q \\
 Q'(X, q^\perp) & \xlongequal{\quad} & C'(X, q)
 \end{array}$$

However, if p is not sharp, then asrt_p is not independent of the chosen quotient–comprehension chain, as the next result shows.

Lemma 14: Given an effectus with quotient–comprehension chain. Suppose for any $(X, p) \in \int \square$, there is an isomorphism $\alpha_{(X, p)}: X/p \rightarrow X/p$ with $\alpha_{(X, q)} = \text{id}$ for sharp q . Then: $\alpha_{(X, p)} \circ \xi_p$ is the unit of the quotient of another quotient–comprehension chain.

In particular, in the case of von Neumann algebras, we can find for any object \mathcal{A} , unitary $u \in \mathcal{A}$ and non-sharp predicate p a quotient–comprehension chain such that $\text{asrt}_p(a) = \sqrt{p}u^*au\sqrt{p}$. The precise structure of these unitaries is a topic of research in itself (see also [14]), going beyond the current paper.

B. Basic Effectus Theory

Throughout this subsection, \mathbf{C} denotes an effectus (see Definition 8). When we referring to items, simply as (iii), we mean: Definition 8 (iii). In [1], it is already shown that coprojections κ_i in \mathbf{C} are monic [1, Lemma 10], which implies that the canonical functor $(\widehat{\quad}): \mathbf{C} \rightarrow \mathbf{C}_\circ$ is faithful. Therefore we can identify \mathbf{C} with the subcategory of \mathbf{C}_\circ with total maps.

The following result constitutes a part of the *state-and-effect triangle*.

Proposition 15 ([1, Proposition 12]): Each homset $\mathbf{C}(X, 1 + 1)$ is an effect algebra, and the hom-functor $\mathbf{C}(-, 1 + 1)$ gives rise to a contravariant functor $\mathbf{C} \rightarrow \mathbf{EA}^{\text{op}}$ into the category of effect algebras.

In [1] the partial sum \oplus on homsets $\mathbf{C}(X, 1 + 1)$ of predicates is defined via bounds. This can be generalized to homsets $\mathbf{C}(X, Y + 1) = \mathbf{C}_\circ(X, Y)$ as shown in the previous

section. The homsets $\mathbf{C}_\circ(X, Y)$ are in general not effect algebras, but instead we have the following result.

Proposition 16: The category \mathbf{C}_\circ is enriched over partial commutative monoids (PCMs), *i.e.*, every homset $\mathbf{C}_\circ(X, Y)$ is a PCM and the composition \circ is a PCM-bihomomorphism.

Proof We refer to the proof of [1, Proposition 12] (*i.e.*, the previous proposition), which is easily generalized. ■

Moreover the enrichment satisfies some special properties.

Proposition 17: Let $f, g: X \rightarrow Y$ be maps in \mathbf{C}_\circ .

- (Compatible sum axiom) If there exists $b: X \rightarrow Y + Y$ such that $\triangleright_1 \circ b = f$ and $\triangleright_2 \circ b = g$, then f and g are summable.
- (Untying axiom) If f and g are summable, then $\widehat{\kappa}_1 \circ f, \widehat{\kappa}_2 \circ g: X \rightarrow Y + Y$ are summable too.

Proof The compatible sum axiom follows by definition, and it is straightforward to check the untying axiom. ■

This easy fact is notable, since these axioms are variants of the axioms of Arbib and Manes' *partially additive category* (PAC) [15], and therefore \mathbf{C}_\circ possesses similar properties to a PAC. The difference is that a PAC involves *countable* sums, but ours involves only *finite* sums.

The domain predicates of partial maps are a powerful tool. We can easily see that $\text{Dp}: \mathbf{C}_\circ(X, Y) \rightarrow \mathbf{C}_\circ(X, 1)$ is a PCM-homomorphism, as \mathbf{C}_\circ is enriched over PCMs. It in fact has even better properties.

Proposition 18: Let $f, g: X \rightarrow Y$ be partial maps.

- i) $f \perp g$ if and only if $\text{Dp}(f) \perp \text{Dp}(g)$. In that case $\text{Dp}(f \otimes g) = \text{Dp}(f) \otimes \text{Dp}(g)$.
- ii) $f = 0_{XY}$ if and only if $\text{Dp}(f) = 0_X$.
- iii) f is total if and only if $\text{Dp}(f) = 1_X$.

Proof See Appendix B. ■

In fact, the properties of \mathbf{C}_\circ we described so far are strong enough to characterize an effectus as a subcategory with “total” maps. The details will be elaborated in a forthcoming paper [3]. Finally, we show several facts on \mathbf{C}_\circ for later use.

Proposition 19: Let $p: X \rightarrow 1$ be a predicate, and let $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ be partial maps.

- i) Coprojections $\widehat{\kappa}_i$ in \mathbf{C}_\circ are (split) monic.
- ii) If f is monic in \mathbf{C}_\circ , then $f: X \rightarrow Y + 1$ is monic in \mathbf{C} too.
- iii) $\text{Dp}(p) = p$.
- iv) $\text{Dp}(g \circ f) = \text{Dp}(g) \circ f$.
- v) $\text{Dp}(g \circ f) \leq \text{Dp}(f)$. If g is total then $\text{Dp}(g \circ f) = \text{Dp}(f)$. In particular $\text{Dp}(\widehat{\kappa}_i \circ f) = \text{Dp}(f)$.

Proof

- i) Notice that $\triangleright_i \circ \widehat{\kappa}_i = \widehat{\text{id}}$.
- ii) Note that $f \circ h = f \circ \widehat{h}$. If f is monic in \mathbf{C}_\circ and $f \circ h = f \circ k$, then $f \circ \widehat{h} = f \circ \widehat{k}$ and $\widehat{h} = \widehat{k}$. It follows that $h = k$ since $(\widehat{\quad}): \mathbf{C} \rightarrow \mathbf{C}_\circ$ is faithful.
- iii) Because $1_1 = \kappa_1 = \widehat{\text{id}}_1$.
- iv) By definition, $\text{Dp}(g \circ f) = 1_Z \circ g \circ f = \text{Dp}(g) \circ f$.

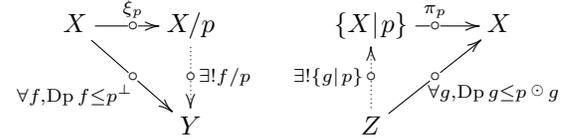
- v) $\text{Dp}(g \circ f) = \text{Dp}(g) \circ f \leq 1_X \circ f = \text{Dp}(f)$. By Proposition 18, we have equality when g is total. ■

C. Properties of Effectuses with Quotient–Comprehension Chain

Assumption ii in the definition of effectus with quotient–comprehension chain is by definition of $\int \square$ equivalent to the following statement.

Proposition 20: Given a predicate $p: X \rightarrow 2 = 1 + 1$.

- i) There is a X/p and $\xi_p: X \rightarrow X/p$ such that $\text{Dp} \xi_p \leq p^\perp$ and for every $f: X \rightarrow Y$ with $\text{Dp} f \leq p^\perp$ there is a unique $f/p: X/p \rightarrow Y$ with $f/p \circ \xi_p = f$.
- ii) There is a $\{X|p\}$ and $\pi_p: X \rightarrow \{X|p\}$ such that $\text{Dp} \pi_p \leq p \circ \pi_p$ and for every $g: Z \rightarrow X$ with $\text{Dp} g \leq p \circ g$ there is a unique $\{g|p\}: Z \rightarrow \{X|p\}$ with $\pi_p \circ \{g|p\} = g$.



We derive some properties without using the controversial assumption iii.

Lemma 21: Given a predicate $p: X \rightarrow 2$.

- i) In \mathbf{C}_\circ the map ξ_p is epic.
- ii) We have $\text{Dp} \xi_p = p^\perp$ and whenever $f: X \rightarrow Y$ satisfies $\text{Dp} f = p^\perp$ then f/p is total, *i.e.* $\text{Dp} f/p = 1$.
- iii) Given $g: Z \rightarrow X$ with $p \circ g = 1$. Then $\text{Dp} g \leq p \circ g$ and both g and $\{g|p\}$ are total.
- iv) In \mathbf{C}_\circ the map π_p is monic.

Proof One at the time.

- i) Given $\alpha, \beta: X/p \rightarrow Z$ with $\alpha \circ \xi_p = \beta \circ \xi_p$. Note $\text{Dp}(\alpha \circ \xi_p) \leq \text{Dp} \xi_p \leq p^\perp$. Thus $\alpha = (\alpha \circ \xi_p)/p = \beta$, by uniqueness of ξ_p .
- ii) We have $p^\perp = \text{Dp} p^\perp = \text{Dp}(p^\perp/p \circ \xi_p) \leq \text{Dp} \xi_p$, since $p^\perp \leq p^\perp$. Thus $\text{Dp} \xi_p = p^\perp$. Now $1 \circ \xi_p = p^\perp = \text{Dp} f = \text{Dp}(f/p \circ \xi_p) = \text{Dp}(f/p) \circ \xi_p$. As ξ_p is epic in \mathbf{C}_\circ , we see $\text{Dp} f/p = 1$.
- iii) Clearly $\text{Dp} g \leq 1 = \text{Dp}(p \circ g) \leq \text{Dp} g$. Thus g is total. Furthermore $1 = \text{Dp} g = \text{Dp}(\pi_p \circ \{g|p\}) \leq \text{Dp}\{g|p\}$.
- iv) Given $\alpha, \beta: Z \rightarrow \{X|p\}$ with $\pi_p \circ \alpha = \pi_p \circ \beta$. Observe $\text{Dp}(\pi_p \circ \alpha) = \text{Dp}(\pi_p) \circ \alpha = p \circ \pi_p \circ \alpha$, thus $\alpha = \{\pi_p \circ \alpha|p\} = \beta$. ■

Also using assumption iii, we may deduce:

Lemma 22: Given a predicate $p: X \rightarrow 2$.

- i) The map $\pi_{[p]}$ is total.
- ii) We have $\text{Dp} \text{asrt}_p = p$

Proof We prove one at the time.

- i) We have $\text{Dp} \pi_{[p]} \geq \text{Dp}(\xi_{[p]^\perp} \circ \pi_{[p]}) = \text{Dp} \kappa_1 = 1$.
- ii) Using totality of $\pi_{[p]}$, we derive $\text{Dp} \text{asrt}_p = \text{Dp}(\pi_{[p]} \circ \xi_{p^\perp}) = \text{Dp} \xi_{p^\perp} = p$. ■

Proposition 23: $\xi_p \hat{+} \xi_q$ has the universal property of $\xi_{[p,q]}$ for any predicates $p: X \rightarrow 2$ and $q: Y \rightarrow 2$.

Proof We leave this to the reader. (Hint: since the quotient functor is a left adjoint it preserves coproducts.) ■

The following is less obvious.

Proposition 24: $\pi_p \hat{+} \pi_q$ has the universal property of $\pi_{[p,q]}$ for sharp predicates $p: X \rightarrow 2$ and $q: Y \rightarrow 2$.

Proof To begin, note that $\text{Dp}(\pi_p \hat{+} \pi_q) = [\text{Dp}\pi_p, \text{Dp}\pi_q] \leq [p \circ \pi_p, q \circ \pi_q] = [p, q] \circ (\pi_p \hat{+} \pi_q)$.

Let $f: Z \rightarrow X + Y$ be a map with $\text{Dp}(f) \leq [p, q] \circ f$. We must show that there is unique $h: Z \rightarrow \{X|p\} + \{Y|q\}$ such that $f = (\pi_p \hat{+} \pi_q) \circ h$.

(Existence) Define $f_1 = \triangleright_1 \circ f$ and $f_2 = \triangleright_2 \circ f$. Since

$$(\widehat{\kappa}_1 \circ \triangleright_1) \circledast (\widehat{\kappa}_2 \circ \triangleright_2) = \text{id}: X + Y \rightarrow X + Y,$$

we have $f = (\widehat{\kappa}_1 \circ f_1) \circledast (\widehat{\kappa}_2 \circ f_2)$.

We claim that $\text{Dp} f_1 \leq p \circ f_1$. Indeed, $(\text{Dp} f_1) \circledast (\text{Dp} f_2) = \text{Dp} f \leq [p, q] \circ f = (p \circ f_1) \circledast (q \circ f_2) \leq (p \circ f_1) \circledast (\text{Dp} f_2)$. Now use that \circledast is cancellative. Similarly, $\text{Dp} f_2 \leq q \circ f_2$.

Therefore one obtains a map $\{f_1|p\}: Z \rightarrow \{X|p\}$ such that $f_1 = \pi_p \circ \{f_1|p\}$, and $\{f_2|q\}$ as well. Note that $\text{Dp}(\widehat{\kappa}_1 \circ \{f_1|p\}) = \text{Dp}\{f_1|p\} \stackrel{*}{=} \text{Dp}(\pi_p \circ \{f_1|p\}) = \text{Dp} f_1$ (using sharpness of p for $\stackrel{*}{=}$), and $\text{Dp}(\widehat{\kappa}_2 \circ \{f_2|q\}) = \text{Dp} f_2$ similarly. Hence $\widehat{\kappa}_1 \circ \{f_1|p\} \perp \widehat{\kappa}_2 \circ \{f_2|q\}$ because $\text{Dp} f_1 \perp \text{Dp} f_2$, and we may define $h: Z \rightarrow \{X|p\} + \{Y|q\}$ by

$$h = (\widehat{\kappa}_1 \circ \{f_1|p\}) \circledast (\widehat{\kappa}_2 \circ \{f_2|q\}), \quad (10)$$

so $(\pi_p \hat{+} \pi_q) \circ h = (\widehat{\kappa}_1 \circ \pi_p \circ \{f_1|p\}) \circledast (\widehat{\kappa}_2 \circ \pi_q \circ \{f_2|q\}) = (\widehat{\kappa}_1 \circ f_1) \circledast (\widehat{\kappa}_2 \circ f_2) = f$.

(Uniqueness) Let a map $h': Z \rightarrow \{X|p\} + \{Y|q\}$ such $f = (\pi_p \hat{+} \pi_q) \circ h'$ be given. We must show that $h' = h$.

Note that $f_1 = \triangleright_1 \circ f = \triangleright_1 \circ (\pi_p \hat{+} \pi_q) \circ h' = \pi_p \circ \triangleright_1 \circ h'$, so $\triangleright_1 \circ h' = \{f_1|p\}$. Similarly, $\triangleright_2 \circ h' = \{f_2|q\}$. Then $h' = (\widehat{\kappa}_1 \circ \triangleright_1 \circ h') \circledast (\widehat{\kappa}_2 \circ \triangleright_2 \circ h') = (\widehat{\kappa}_1 \circ \{f_1|p\}) \circledast (\widehat{\kappa}_2 \circ \{f_2|q\})$. Thus $h' = h$ by Equation (10). ■

Remark 25: Note that we have

$$\text{asrt}_p \hat{+} \text{asrt}_q = (\pi_{[p]} \hat{+} \pi_{[q]}) \circledast (\xi_{p^\perp} \hat{+} \xi_{q^\perp})$$

for predicates $p: X \rightarrow 2$ and $q: Y \rightarrow 2$, and also

$$\text{asrt}_{[p,q]} = \pi_{[[p],[q]]} \circledast \xi_{[p^\perp, q^\perp]},$$

and note that $\pi_{[[p],[q]]}$ and $\pi_{[p]} \hat{+} \pi_{[q]}$ satisfy the same universal property by Proposition 24, as do $\xi_{[p^\perp, q^\perp]}$ and $\xi_{p^\perp} \hat{+} \xi_{q^\perp}$ by Proposition 23. Yet from this alone we cannot conclude that $\text{asrt}_p \hat{+} \text{asrt}_q = \text{asrt}_{[p,q]}$. This a consequence of Lemma 14.

D. Instrument Maps

Definition 26: Given an object X and $n \in \mathbb{N}$, an n -test p on X is a n -tuple of predicates $p_1, \dots, p_n: X \rightarrow 2$ with $p_1 \circledast \dots \circledast p_n = 1$.

Lemma 27: Given an n -test $p_1, \dots, p_n: X \rightarrow 2$ the sum

$$(\widehat{\kappa}_1 \circ \text{asrt}_{p_1}) \circledast \dots \circledast (\widehat{\kappa}_n \circ \text{asrt}_{p_n}): X \rightarrow n \cdot X$$

exists and is total.

Proof Since $\text{Dp}(\widehat{\kappa}_i \circ \text{asrt}_{p_i}) = p_i$ and $p_1 \circledast \dots \circledast p_n = 1$ the claim follows immediately from Proposition 18, parts i and iii. ■

Definition 28: Given an object X and an n -test $p_1, \dots, p_n: X \rightarrow 2$ the **(measuring) instrument** for p is the unique map $\text{instr}_p: X \rightarrow n \cdot X$ with

$$\widehat{\text{instr}}_p = (\widehat{\kappa}_1 \circ \text{asrt}_{p_1}) \circledast \dots \circledast (\widehat{\kappa}_n \circ \text{asrt}_{p_n}).$$

Example 29: In the effectus \mathbf{vN}^{op} an n -test on a von Neumann algebra \mathcal{A} is given by tuple p_1, \dots, p_n of effects with $p_1 + \dots + p_n = 1$, and its instrument $\text{instr}_p: \mathcal{A}^{\oplus n} \rightarrow \mathcal{A}$ is given by

$$\text{instr}_p(a_1, \dots, a_n) = \sqrt{p_1} a_1 \sqrt{p_1} + \dots + \sqrt{p_n} a_n \sqrt{p_n}.$$

This equation is sometimes called the (generalized) Lüders rule, for unsharp predicates (effects). It occurs for instance in [16, Eq.(1.3)] (and also in [1]). Here it arises from our quotient-comprehension chain, using equation (iii).

Proposition 30 (Properties of instruments):

i) Let X be an object, p an n -test on X , and $\varphi: n \rightarrow m$ an injection. Then we have an m -test $\varphi \cdot p$ on X given by $(\varphi \cdot p)_j = p_i$ if $\varphi(i) = j$ and $q_j = 0$ otherwise; and we have a map $\varphi \cdot X: n \cdot X \rightarrow m \cdot X$ given by $(\varphi \cdot X) \circ \kappa_i = \kappa_{\varphi(i)}$; and these are related by

$$\text{instr}_{\varphi \cdot p} = (\varphi \cdot X) \circ \text{instr}_p.$$

iii) Let X be an object, and p be an $n+m$ -test on X . Then we have an $n+1$ -test p' on X given by $p'_i = p_i$ for $i < n$ and $p'_n = p_n \circledast \dots \circledast p_{n-1}$; and the following diagram commutes.

$$\begin{array}{ccc} & n \cdot X + m \cdot X & \\ \swarrow & \text{instr}_p & \searrow \\ X & & n \cdot X + 1 \\ \swarrow & \text{instr}_{p' \circ n} \cdot X + X & \searrow \\ & & \text{id}! \end{array}$$

Proof (Sketch)

i) We can show that $\triangleright_i \circ \text{instr}_{\varphi \cdot p} = \triangleright_i \circ (\varphi \cdot X) \circ \text{instr}_p$ for each $i \in \{1, \dots, m\}$. Partial projections \triangleright_i are jointly monic in \mathbf{C}_\circledast (see Appendix A), and also in \mathbf{C} for the same reasons as Proposition 19.ii. Therefore we obtain $\text{instr}_{\varphi \cdot p} = (\varphi \cdot X) \circ \text{instr}_p$.

iii) Similarly we can show that $[\triangleright_i, \kappa_2] \circ (\text{id} + !) \circ \text{instr}_p = [\triangleright_i, \kappa_2] \circ (\text{id} + !) \circ \text{instr}_{p'}$, i.e., $\triangleright_i \circ ((\text{id} + !) \circ \text{instr}_p) = \triangleright_i \circ ((\text{id} + !) \circ \text{instr}_{p'})$ for each $i \in \{1, \dots, n\}$. Then $(\text{id} + !) \circ \text{instr}_p = (\text{id} + !) \circ \text{instr}_{p'}$ by the joint monicity of partial projections. ■

Remark 31: The statements i) and iii) of Proposition 30 are (1) and (3) from the four requirements of [1, Assumption 2]. The other two requirements are the following.

- ii) Let X and Y be objects. Let p be an n -test on X , and q an n -test on Y . Then we have an n -test $[p, q]$ on $X + Y$ given by $[p, q]_i = [p_i, q_i]$; and this commutes:

$$\begin{array}{ccc} X + Y & \xrightarrow{\text{instr}_{[p,q]}} & n \cdot (X + Y) \\ & \searrow \text{instr}_p + \text{instr}_q & \downarrow [\kappa_n + \kappa_n]_n \\ & & n \cdot X + n \cdot Y \end{array}$$

- iv) Given an object X , and a map $q: X \rightarrow n \cdot X$ with $\nabla \circ q = \text{id}$ (where $\nabla \circ \kappa_i = \text{id}$ for all i), we have

$$q = \text{instr}_p,$$

where p is the n -test ‘induced by’ q , that is, $p_i = \triangleright_i \circ (n \cdot!) \circ q$.

We can translate them into the conditions on assert maps asrt_p as follows.

- ii’) For predicates $p: X \rightarrow 2$ and $q: Y \rightarrow 2$, we have $\text{asrt}_{[p,q]} = \text{asrt}_p \hat{+} \text{asrt}_q$.
iv’) If $f: X \rightarrow X$ satisfies $f \leq \widehat{\text{id}}_X$, then $\text{asrt}_{\text{Dp}(f)} = f$.

For the assert maps asrt_p and for the corresponding instruments instr_p (via Definition 28), we have ii) \iff ii’), and iv) \iff iv’).

For our examples \mathbf{vN}^{op} , **Sets** and $\mathcal{Kl}(\mathcal{D})$ of effectuses with (the standard choice of) quotient–comprehension chain, the instruments instr_p do satisfy ii) and iv), since the instruments we obtain coincide the chosen ones in [1]. In the current axiomatization, however, ii) is problematic as seen in Remark 25, and it seems iv) is more difficult.

E. Sequential Product

Definition 32: Given two predicates p and q on the same object, we define their **sequential product** by $p * q = q \circ \text{asrt}_p$.

This sequential product $p * q$ should be read as ‘ p and then q ’. In a probabilistic setting $*$ is commutative, but not in the quantum world. Via the map asrt_p in the definition of $p * q$ the side-effect of measuring the predicate p is taken into account. In [1] this sequential product $p * q$ is described as a modal operator $\langle p^? \rangle(q)$, understood as: test p , and then q .

Example 33: In \mathbf{vN}^{op} the sequential product is given by $a * b = \sqrt{ab} \sqrt{a}$. In the literature, this is called the sequential product as well. See for instance [17], [10], [18], [19].

In our axiomatic setting we can derive some basic properties of the sequential product.

Proposition 34: Given predicates $p, q, r: X \rightarrow 2$, the following is true.

- i) We have $1 * p = p * 1 = p$.
- ii) Whenever $p \perp q$, then $r * p \perp r * q$ and $r * (p \otimes q) = (r * p) \otimes (r * q)$.
- iii) If p is sharp, then $p * p = p$.
- iv) If p and q are sharp and $p \perp q$, then $p * q = q * p = 0$.
- v) If $p \leq q$ and q is sharp, then $q * p = p$.

Proof One by one.

- i) We have $p * 1 = 1 \circ \text{asrt}_p = \text{Dp asrt}_p = p$. and $1 * p = p \circ \text{asrt}_1 p \circ \widehat{\text{id}} = p$.

- ii) By left additivity of \otimes , we see $r * (p \otimes q) = (p \otimes q) \circ \text{asrt}_r = (p \circ \text{asrt}_r) \otimes (q \circ \text{asrt}_r) = (r * p) \otimes (r * q)$.
- iii) We compute $p * p = p \circ \text{asrt}_p = p \circ \pi_p \circ \xi_{p^\perp} = \text{Dp}(\pi_p) \circ \xi_{p^\perp} = 1 \circ \xi_{p^\perp} = \text{Dp} \xi_{p^\perp} = p$.
- iv) Write $r = (p \otimes q)^\perp$. By the previous $p = p * 1 = p * (p \otimes q \otimes r) = (p * p) \otimes (p * q) \otimes (p * r) = p \otimes (p * q) \otimes (p * r)$. By cancellation $0 = (p * q) \otimes (p * r)$. By positivity $p * q = p * r = 0$. Similarly, by multiplying on the right: $q * p = r * p = 0$.
- v) We derive $q * p = p \circ \text{asrt}_q = p \circ \pi_q \circ \xi_{q^\perp} = p/q^\perp \circ \xi_{q^\perp} \circ \pi_q \circ \xi_{q^\perp} = p/q^\perp \circ \xi_{q^\perp} = p$. \blacksquare

The last three properties are noteworthy: they are familiar properties of projections in linear algebra and were not derivable in the setting of [1]. We continue with another remarkable property of the sequential product.

Proposition 35: Given predicates $p, q: X \rightarrow 2$ with $p \leq q$. Then there is a predicate $r \leq [q]$ such that $p = q * r$.

Proof Recall there is a p/q^\perp such that $(p/q^\perp) \circ \xi_{q^\perp} = p$. Note $X/q^\perp = \{X/[q]\} = X/[q]^\perp$. Define $r = (p/q^\perp) \circ \xi_{[q]^\perp}$. Note $r = (p/q^\perp) \circ \xi_{[q]^\perp} \leq \text{Dp} \xi_{[q]^\perp} = [q]$. We check $q * r = r \circ \text{asrt}_q = (p/q^\perp) \circ \xi_{[q]^\perp} \circ \pi_{[q]} \circ \xi_{q^\perp} = (p/q^\perp) \circ \xi_{q^\perp} = p$. \blacksquare

F. Normalization for Partial Maps

Given a total map $h: X \rightarrow Y$ and any predicate p on X , the map $f \circ \text{asrt}_p$ is partial with $\text{Dp}(f \circ \text{asrt}_p) = p$. We show that every partial map arises in this fashion.

Proposition 36: Given a partial map $f: X \rightarrow Y$ and a state $\omega: 1 \rightarrow Y$, there is a total map $g: X \rightarrow Y$ such that $g \circ \text{asrt}_{\text{Dp} f} = f$.

Proof Write $p = \text{Dp} f$. Define $g = [f/p^\perp, \widehat{\omega}] \circ \widehat{\xi_{[p]^\perp}}$, which is total because maps f/p^\perp , $\widehat{\omega}$ and $\widehat{\xi_{[p]^\perp}}$ are total.

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \xi_{p^\perp} \downarrow & \nearrow f/p^\perp & \uparrow [f/p^\perp, \widehat{\omega}] \\ X/p^\perp & \xrightarrow{\text{asrt}_p} & (X/p^\perp) + 1 \\ \pi_{[p]} \downarrow & \nearrow g & \uparrow \widehat{\xi_{[p]^\perp}} \\ X & & X \end{array}$$

We write $\widehat{\pi_{[p]}}$ for the map with $\pi_{[p]} = (\widehat{\pi_{[p]}})^\perp$ (since $\pi_{[p]}$ is total). Using $h \circ \widehat{k} = h \circ k$, one has $\widehat{\xi_{[p]^\perp}} \circ \widehat{\pi_{[p]}} = \widehat{\xi_{[p]^\perp} \circ \pi_{[p]}} = \widehat{\widehat{\text{id}}} = \widehat{\kappa_1}$, and therefore $\widehat{\kappa_1} = (\widehat{\xi_{[p]^\perp} \circ \widehat{\pi_{[p]}})^\perp = \widehat{\xi_{[p]^\perp}} \circ \pi_{[p]}$. Then we obtain

$$\begin{aligned} g \circ \text{asrt}_p &= [f/p^\perp, \widehat{\omega}] \circ \widehat{\xi_{[p]^\perp}} \circ \pi_{[p]} \circ \xi_{p^\perp} \\ &= [f/p^\perp, \widehat{\omega}] \circ \widehat{\kappa_1} \circ \xi_{p^\perp} \\ &= (f/p^\perp) \circ \xi_{p^\perp} = f, \end{aligned}$$

as desired. \blacksquare

In the von Neumann algebra case this gives us the following non-trivial result.

Corollary 37: Let $f: \mathcal{A} \rightarrow \mathcal{B}$ be a completely positive normal subunital linear map between non-trivial von Neumann algebras. It models a possibly non-terminating quantum operation. Then there is a *unital* completely positive normal linear map $g: \mathcal{A} \rightarrow \mathcal{B}$ such that $f(a) = \sqrt{f(1)}g(a)\sqrt{f(1)}$; that is: the quantum operation factors into a terminating operation g and a canonical assertion measurement of $f(1)$.

VIII. CONCLUSIONS

Having connected quotient–comprehension chains with measurement in set-theoretic, probabilistic and quantum examples we believe we have taken the first steps towards an abstract theory of measurement, with it already proving a non-trivial factorization theorem for von Neumann algebras (Cor. 37). Whether it is possible to make the (measuring) instruments independent of the choice of quotient and comprehension functor, or more broadly, if measurement is ‘structure’ or ‘property’, remains an open problem for future research. This non-trivial problem will have to be clarified in a categorical axiomatization of quantum theory, leading to what we have already called a ‘telos’. Whatever such a telos will be, we think that the quotient–comprehension chain of adjunctions uncovered here should be a central part of it.

REFERENCES

- [1] B. Jacobs, “New directions in categorical logic, for classical, probabilistic and quantum logic,” 2014, see arxiv.org/abs/1205.3940v3.
- [2] B. Westerbaan and A. Westerbaan, “A universal property of sequential measurement,” 2015, preprint is available at <http://westerbaan.name/~bas/math/univ-prop-seq-prod.pdf>.
- [3] K. Cho, “The lift monad in categorical quantum foundations,” 2015, in preparation.
- [4] B. Westerbaan and A. Westerbaan, “Measurement in an effectus,” 2015, in preparation.
- [5] J. Butterfield and C. Isham, “A topos perspective on the Kochen–Specker theorem: I. quantum states as generalized valuations,” *Int. Journ. Theor. Physics*, vol. 37(11), pp. 2669–2733, 1998.
- [6] S. Abramsky and B. Coecke, “A categorical semantics of quantum protocols,” in *Logic in Computer Science*, IEEE. Computer Science Press, 2004, pp. 415–425.
- [7] P. Selinger, “Dagger compact closed categories and completely positive maps (extended abstract),” in *Proceedings of the 3rd International Workshop on Quantum Programming Languages (QPL 2005)*, ser. Elect. Notes in Theor. Comp. Sci., P. Selinger, Ed., no. 170. Elsevier, Amsterdam, 2007, pp. 139–163.
- [8] C. Heunen, A. Kissinger, and P. Selinger, “Completely positive projections and biproducts,” pp. 71–83, 2014.
- [9] B. Jacobs, *Categorical Logic and Type Theory*. Amsterdam: North Holland, 1999.
- [10] S. Gudder and G. Nagy, “Sequential quantum measurements,” *Journal of Mathematical Physics*, vol. 42, pp. 5212–5222, 2001.
- [11] V. Paulsen, *Completely bounded maps and operator algebras*. Cambridge University Press, 2002, vol. 78.
- [12] B. Jacobs, B. Westerbaan, and A. Westerbaan, “States of convex sets,” in *FoSSaCS 2015*, ser. Lect. Notes Comp. Sci. Springer, Berlin, 2015.
- [13] A. Dvurečenskij and S. Pulmannová, *New Trends in Quantum Structures*. Dordrecht: Kluwer Acad. Publ., 2000.
- [14] S. Abramsky, “Coalgebras, Chu spaces, and representations of physical systems,” in *Logic in Computer Science*, IEEE. Computer Science Press, 2010, pp. 411–420.
- [15] M. A. Arbib and E. G. Manes, “Partially additive categories and flow-diagram semantics,” *Journal of Algebra*, vol. 62, no. 1, pp. 203–227, 1980.
- [16] P. Busch and J. Singh, “Lüders theorem for unsharp quantum measurements,” *Phys. Letters A*, vol. 249, pp. 10–12, 1998.

- [17] S. Gudder and F. Latremoliere, “Characterization of the sequential product on quantum effects,” *arXiv preprint arXiv:0803.3867*, 2008.
- [18] S. Gudder and R. Greechie, “Sequential products on effect algebras,” *Reports on Mathematical Physics*, vol. 49, no. 1, pp. 87–111, 2002.
- [19] —, “Uniqueness and order in sequential effect algebras,” *International Journal of Theoretical Physics*, vol. 44, no. 7, pp. 755–770, 2005.

APPENDIX

A. Remark on the Joint Monicity Requirement

Several slightly different formulations of the joint monicity requirement appear in [1], [12]. In this section we discuss their relation.

The joint monicity requirements of Definition 8 and [1, Assumption 1] can be translated to the joint monicity in \mathbf{C}_\odot as follows. Definition 8 requires the joint monicity of the two partial projections $\triangleright_1, \triangleright_2: X + X \dashrightarrow X$, while [1, Assumption 1] requires the joint monicity of the n partial projections $\triangleright_i: n \cdot X \dashrightarrow X$ for each n . In the light of the following fact, the two requirements are equivalent.

Proposition 38: Let \mathbf{A} be a category with finite coproducts $(+, 0)$ and zero maps $0_{XY}: X \rightarrow Y$, in which we can define partial projections \triangleright_i . The following are equivalent.

- i) $\triangleright_1, \triangleright_2: X + X \rightarrow X$ are jointly monic for all $X \in \mathbf{A}$.
- ii) $\triangleright_1: X + Y \rightarrow X$ and $\triangleright_2: X + Y \rightarrow Y$ are jointly monic for all $X, Y \in \mathbf{A}$.
- iii) n maps $\triangleright_i: X_1 + \dots + X_n \rightarrow X_i$ are jointly monic for all $X_1, \dots, X_n \in \mathbf{A}$, for each n .

Proof iii \implies ii \implies i are trivial implications, and ii \implies iii follows by induction on n . We show i \implies ii.

Let $f, g: Z \rightarrow X + Y$ be arrows with $\triangleright_1 \circ f = \triangleright_1 \circ g$ and $\triangleright_2 \circ f = \triangleright_2 \circ g$. Note that the following diagram commutes.

$$\begin{array}{ccccc} X & \xleftarrow{\triangleright_1} & X + Y & \xrightarrow{\triangleright_2} & Y \\ \kappa_1 \downarrow & & \downarrow \kappa_1 + \kappa_2 & & \downarrow \kappa_2 \\ X + Y & \xleftarrow{\triangleright_1} & (X + Y) + (X + Y) & \xrightarrow{\triangleright_2} & X + Y \end{array}$$

For $i \in \{1, 2\}$, therefore, we have $\triangleright_i \circ (\kappa_1 + \kappa_2) \circ f = \triangleright_i \circ (\kappa_1 + \kappa_2) \circ g$, and by the joint-monocity of $\triangleright_1, \triangleright_2: (X + Y) + (X + Y) \rightarrow X + Y$, we obtain $(\kappa_1 + \kappa_2) \circ f = (\kappa_1 + \kappa_2) \circ g$. It follows that $f = g$, since $\kappa_1 + \kappa_2$ is a split mono as $\nabla \circ (\kappa_1 + \kappa_2) = \text{id}$. ■

However, the joint monicity requirement of [12, Definition 12], which states that $[\kappa_1, \kappa_2, \kappa_2], [\kappa_2, \kappa_1, \kappa_2]: X + X + X \rightarrow X + X$ are jointly monic in \mathbf{C} , is not equivalent to the others. The category of (not necessarily cancellative) convex sets does not satisfy their requirement as observed in [12], but the category does satisfy the requirement of Definition 8.

B. Proof of Proposition 18

First we show a lemma on pullbacks in an effectus.

Lemma 39: Let \mathbf{C} be an effectus. In \mathbf{C} , the following square is a pullback.

$$\begin{array}{ccc} A & \xrightarrow{g} & B \\ \kappa_1 \downarrow & & \downarrow \kappa_1 \\ A + X & \xrightarrow{g+f} & B + Y \end{array}$$

In \mathbf{C}_\circ , the following square is a pullback,

$$\begin{array}{ccc} X + A & \xrightarrow{\widehat{f} + \widehat{\text{id}}} & Y + A \\ \triangleright_1 \downarrow \circlearrowleft & & \downarrow \circlearrowleft \triangleright_1 \\ X & \xrightarrow{\widehat{f}} & Y \end{array}$$

where $f: X \rightarrow Y$ is a ‘‘total’’ map in \mathbf{C} .

Proof The first one is showed in [1, Lemma 10]. Note that the second square is a pullback in \mathbf{C}_\circ if and only if the following square is pullback in \mathbf{C} .

$$\begin{array}{ccc} (X + A) + 1 & \xrightarrow{[\widehat{f} + \widehat{\text{id}}, \kappa_2]} & (Y + A) + 1 \\ [\triangleright_1, \kappa_2] \downarrow & & \downarrow [\triangleright_1, \kappa_2] \\ X + 1 & \xrightarrow{[\widehat{f}, \kappa_2]} & Y + 1 \end{array}$$

Up to isomorphism, the square coincides with the following one, which is a pullback by definition.

$$\begin{array}{ccc} X + (A + 1) & \xrightarrow{f + \text{id}} & Y + (A + 1) \\ \text{id} + ! \downarrow & & \downarrow \text{id} + ! \\ X + 1 & \xrightarrow{f + \text{id}} & Y + 1 \end{array}$$

Now we prove Proposition 18.

Proof of Proposition 18 The ‘only if’ direction is easy in the light of the fact that Dp is a PCM-homomorphism. We prove ‘if’ directions.

i) Let $b: X \rightarrow 1 + 1$ be a bound for $\text{Dp}(f)$ and $\text{Dp}(g)$. Note that $1_Y = !_Y$ is a total map. Then we use a pullback of Lemma 39 and obtain a mediating map $c: X \rightarrow Y + 1$ as in the following diagram.

$$\begin{array}{ccccc} X & & & & \\ & \searrow & & & \searrow \\ & & c & & \\ & & \circlearrowleft & & \\ & & \downarrow & & \\ & & Y + 1 & \xrightarrow{1 + \widehat{\text{id}}} & 1 + 1 \\ & \searrow & \triangleright_1 \downarrow \circlearrowleft & & \downarrow \circlearrowleft \triangleright_1 \\ & & f & & \\ & & \circlearrowleft & & \\ & & Y & \xrightarrow{1} & 1 \end{array}$$

By using another pullback (which is indeed a pullback by the symmetry) we obtain $d: X \rightarrow Y + Y$ as in:

$$\begin{array}{ccccc} X & & & & \\ & \searrow & & & \searrow \\ & & d & & \\ & & \circlearrowleft & & \\ & & \downarrow & & \\ & & Y + Y & \xrightarrow{\widehat{\text{id}} + 1} & Y + 1 \\ & \searrow & \triangleright_2 \downarrow \circlearrowleft & & \downarrow \circlearrowleft \triangleright_2 \\ & & g & & \\ & & \circlearrowleft & & \\ & & Y & \xrightarrow{1} & 1 \end{array}$$

Then it is straightforward to check d is a bound for f and g .

ii) Assume that $\text{Dp}(f) = 0_X$, i.e., $(!_Y + \text{id}) \circ f = 1_Y \circ f = \kappa_2 \circ !_X$. By a pullback from the definition of a effectus (with the symmetry) we obtain the following diagram, which shows $f = \kappa_2 \circ !_X = 0_{XY}$.

$$\begin{array}{ccccc} X & & & & \\ & \searrow & & & \searrow \\ & & ! & & \\ & & \circlearrowleft & & \\ & & \downarrow & & \\ & & 1 & \xrightarrow{=} & 1 \\ & \searrow & \kappa_2 \downarrow & & \downarrow \kappa_2 \\ & & f & & \\ & & \circlearrowleft & & \\ & & Y + 1 & \xrightarrow{! + \text{id}} & 1 + 1 \end{array}$$

iii) Assume that $\text{Dp}(f) = 1_X$, i.e., $(!_Y + \text{id}) \circ f = \kappa_1 \circ !_X$. By a pullback from Lemma 39 we obtain $g: X \rightarrow Y$ with $f = \kappa_1 \circ g$ as in:

$$\begin{array}{ccccc} X & & & & \\ & \searrow & & & \searrow \\ & & g & & \\ & & \circlearrowleft & & \\ & & \downarrow & & \\ & & Y & \xrightarrow{!} & 1 \\ & \searrow & \kappa_1 \downarrow & & \downarrow \kappa_1 \\ & & f & & \\ & & \circlearrowleft & & \\ & & Y + 1 & \xrightarrow{! + \text{id}} & 1 + 1 \end{array}$$

Namely f is total. ■