# The Category of Von Neumann Algebras 

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## Chapter 1

## Introduction

What does this Ph.D. thesis offer? Proof, perhaps, to the manuscript committee of passable academic work; an advertisement, as it may be, of my school's perspective to colleagues; a display, even, of intellectual achievement to friends and family. But I believe such narrow and selfish goals alone barely serve to keep a writer's spirits energised - and are definitely detrimental to that of the readers. That is why I have foolhardily challenged myself not just to drily list contributions, but to write this thesis as the introduction, that I would have liked to read when I started research for this thesis back in May 2014.

The topic is von Neumann algebras, the category they form, and how they may be used to model aspects of quantum computation. Let us just say for now that a von Neumann algebra is a special type of complex vector space endowed with a multiplication operation among some other additional structure. An important example is the complex vector space $M_{2}$ of $2 \times 2$ complex matrices, because it models (the predicates on) a qubit; but all $N \times N$-complex matrices form a von Neumann algebra $M_{N}$ as well. Using von Neumann algebras (and their little cousins, $C^{*}$-algebras) to describe quantum data types seems to be quite a recent idea (see e.g. 17, 25,57, and (3) for an overview) and has two distinct features. Firstly, classical data types are neatly incorporated: $\mathbb{C}^{2} \equiv$ $\mathbb{C} \oplus \mathbb{C}$ models a bit, and the direct sum $M_{2} \oplus M_{3}$ models the union type of a qubit and a qutrit. Secondly, von Neumann algebras allow for infinite data types as well such as $\mathscr{B}\left(\ell^{2}(\mathbb{Z})\right)$, which represents a "quantum integer." ${ }^{*}$ It should be

[^0]said that this last feature is both a boon and a bane: it brings with it all the inherent intricacies of dealing with infinite dimensions; and it is no wonder that most authors choose to restrict themselves to finite dimensions, especially since this seems to be enough to describe quantum algorithms, see e.g. 52.

II In this thesis, however, we do face infinite dimensions, because the two main results demand it:

1. For the first result, that von Neumann algebras form a model of Selinger and Valiron's quantum lambda calculus, as Cho and I explained in 9 and for which I'll provide the foundation here, we need to interpret function types, some of which are essentially infinite dimensional.
2. The second result, an axiomatisation of the map $a \mapsto \sqrt{p} a \sqrt{p}: \mathscr{A} \rightarrow \mathscr{A}$ representing measurement of an element $p \in[0,1]_{\mathscr{A}}$ of a von Neumann algebra $\mathscr{A}$ was tailored by B.E. Westerbaan (my twin brother) and myself to work for both finite and infinite dimensional $\mathscr{A}$.

These results are part of a line of research that tries to find patterns in the category of von Neumann algebras, that may also be cut from other categories modelling computation-ideally in order to arrive at categorical axioms for (probabilistic) computation in general. When I joined the fray the notion of effectus 26 had already been established by Jacobs, and the two results above offer potential additional axioms. The work in this area has largely been a collaborative effort, primarily between Jacobs, Cho, my twin brother, and myself, and many of their insights have ended up in this thesis.

Of this I'd say no more than that my work appears conversely, and proportionally, in their writings too, except that the close cooperation with my brother begs further explanation. Our efforts on certain topics have been like interleaving of the pages of two phone books: separating them would be nigh impossible, especially the work on the axiomatisation of $a \mapsto \sqrt{p} a \sqrt{p}$ and Paschke dilations. So that's why we decided to write our theses as two volumes of the same work; preliminaries on von Neumann algebras, and the axiomatisation of $a \mapsto \sqrt{p} a \sqrt{p}$ appear in this thesis, while the work on dilations, and effectus theory appear in my brother's thesis, 74 .

III The two results mentioned above only make up about a third of this thesis; the rest of it is devoted to the introduction to the theory of von Neumann algebras needed to understand these results. My aim is that anyone with, say, a bachelor's degree in mathematics (c.q. basic knowledge of linear algebra, analysis 60],
action [23], and quantitive semantics 53].
topology [76] and set theory [14]) should at least be able to follow the lines of reasoning with only minimal recourse to external sources. But I hope that they will gain some deeper understanding of the material as well. To this end, and because I wanted to gain some of this insight for myself too, I've not just mixed and matched results from the literature, but I tailored a thorough treatise of everything that's needed, including proofs. Whenever possible, I've taken shortcuts (e.g. avoiding for example the theory of Banach algebras and locally convex spaces entirely) to prevent the mental tax the added concepts (and pages) would have brought. For the same reasons I've refrained from putting everything in its proper abstract (and categorical 46) context trusting that it'll shine through of its own accord. I've however not been able to restrain myself in making perhaps frivolous variations on the existing theory whenever not strictly necessary, taking for example Kadison's characterisation 42] of von Neumann algebras as my definition, and developing the elementary theory for it; in my defence I'll just say this adds to the original element that is expected of a thesis.

Advertisements Due to space-time constraints this thesis is based only on a selection $6,8,9,71,72$ of the works I produced under supervision of Jacobs, and while $7,38,73$ are incorporated in by brother's thesis, this means [36, 37] are unfortunately left out. If you like this thesis, then you might also want to take a look at these $17,24,44,45,58,59$ recent works on von Neumann algebras, and $C^{*}$-algebras. If you're curious about effectus theory and related matters, please have a look at $4,7,26,40$. But if you'd like more pictures instead, I'd suggest 11 .

Writing style I've replaced page numbers by paragraph numbers such as $V$ for this paragraph. The numbers after 134 refer to paragraphs in my twin brother's thesis 74 . Definitions are set like that (i.e. in blue), and can be found in the index. Proofs of certain facts that are easily obtained on the back of an envelope, and would clutter this manuscript, have been left out. Instead these facts have been phrased as exercises as a challenge to the reader.

Acknowledgements The work in this thesis specifically has benefited greatly from discussions with John van de Wetering, Robert Furber, Kenta Cho, and Bas Westerbaan, but I've also had the pleasure of discussing a variety of other topics with Aleks Kissinger, Andrew Polonsky, Bert Lindenhovius, Frank Roumen, Hans Maassen, Henk Barendregt, Joshua Moerman, Martti Karvonen, Robbert Krebbers, Robin Adams, Robin Kaarsgaard, Sam Staton, Sander Uijlen, Sebastiaan Joosten, and many others. I'm especially honoured to have been received in Edinburgh by Chris Heunen and in Oberwolfach by Jianchao Wu. I'm very
grateful to Arnoud van Rooij, Bas Westerbaan and John van de Wetering for proofreading large parts of this manuscript, without whose efforts even more shameful errors would have remained. I should of course not forget to mention the contribution of friends (both close and distant), family, and colleagues - too numerous to name - of keeping me sane these past years.

This is the second dissertation topic I've worked on; my first attempt under different supervision was unfortunately cut short after $1^{1 / 2}$ years. When Bart Jacobs graciously offered me a second chance, I initially had my reservations, but accepted on account of the challenging topic. Little did I know that behind the ambition and suit one finds a man of singular moral fibre, embodying what was said about von Neumann himself: "[he] had to understand and accept much that most of us do not want to accept and do not even wish to understand. ${ }^{\text {W }}$

VII Funding was received from the European Research Council under grant agreement № 320571.

[^1]
## Chapter 2

## C*-algebras

We redevelop the essentials of the theory of (unital) $C^{*}$-algebras in this chapter.
Since we are ultimately interested in von Neumann algebras (a special type of $C^{*}$-algebras) we will evade delicate topics such as tensor products (of $C^{*}$ algebras), quotients, approximate identities, and $C^{*}$-algebras without a unit. The zenith of this chapter is Gelfand's representation theorem (see 27 XXVII ), the fact that every commutative (unital) $C^{*}$-algebra is isomorphic to the $C^{*}$ algebra $C(X)$ of continuous functions on some compact Hausdorff space $X$ it yields a duality between the category CH of compact Hausdorff spaces (and continuous maps) and the category $\mathbf{c C}_{\text {MIU }}^{*}$ of commutative $C^{*}$-algebras (and unital $*$-homomorphisms, the appropriate structure preserving maps), see 29 .

As the road to Gelfand's representation theorem is a bit winding - involving intricate relations between technical concepts - we have put emphasis on the invertible and positive elements so that the important theorems about them may serve as landmarks along the way:

1. first we show that the norm on a $C^{*}$-algebra is determined by the invertible elements (via the spectral radius), see 16 II .
2. then we construct a square root of a positive element in 23 VII ,
3. and finally we show that an element of a commutative $C^{*}$-algebra is not invertible iff it is mapped to 0 by some multiplicative state, see 27 XV .

At every step along the way the positive and invertible elements (and the norm,
multiplicative states, multiplication and other structure on a $C^{*}$-algebra) are bound more tightly together until Gelfand's representation theorem emerges.

To make this chapter more accessible we have removed much material from the ordinary development of $C^{*}$-algebras such as the more general theory of Banach algebras (and its pathology). This forces us to take a slightly different path than is usual in the literature (see e.g. 16 VIIII ).

After Gelfand's representation theorem we deal with two smaller topics: that a $C^{*}$-algebra may be represented as a concrete $C^{*}$-algebra of bounded operators on a Hilbert space (see 30 Vl ), and that the $N \times N$-matrices with entries drawn from a $C^{*}$-algebra $\mathscr{A}$ form a $C^{*}$-algebra $M_{N}(\mathscr{A})$ (see 331). We end with an overture to von Neumann algebras - the topic of the next chapter.

### 2.1 Definition and Examples

3 Definition A $C^{*}$-algebra is a complex vector space $\mathscr{A}$ endowed with

1. a binary operation, called multiplication (and denoted as such), which is associative, and linear in both coordinates;
2. an element 1, called unit, such that $1 \cdot a=a=a \cdot 1$ for all $a \in \mathscr{A}$;
3. a unary operation $(\cdot)^{*}$, called involution such that $\left(a^{*}\right)^{*}=a,(a b)^{*}=$ $b^{*} a^{*},(\lambda a)^{*}=\bar{\lambda} a^{*}$, and $(a+b)^{*}=a^{*}+b^{*}$ for all $a, b \in \mathscr{A}$ and $\lambda \in \mathbb{C}$;
4. a complete norm $\|\cdot\|$ such that $\|a b\| \leqslant\|a\|\|b\|$ for all $a, b \in \mathscr{A}$, and

$$
\left\|a^{*} a\right\|=\|a\|^{2}
$$

holds; this equality is called the $C^{*}$-identity.
The $C^{*}$-algebra $\mathscr{A}$ is called commutative if $a b=b a$ for all $a, b \in \mathscr{A}$.
II Warning In the literature it is usually not required that a $C^{*}$-algebra possess a unit; but when it does it is called a unital $C^{*}$-algebra.

III Example The vector space $\mathbb{C}$ of complex numbers forms a commutative $C^{*}$ algebra in which multiplication and 1 have their usual meaning. Involution is given by conjugation $\left(z^{*}=\bar{z}\right)$, and norm by modulus $(\|z\|=|z|)$.
IV Example A $C^{*}$-subalgebra of a $C^{*}$-algebra $\mathscr{A}$ is a subset $\mathscr{B}$ of $\mathscr{A}$, which is a linear subspace of $\mathscr{A}$, contains the unit, 1 , is closed under multiplication and involution, and is closed with respect to the norm of $\mathscr{A} ;$ such a $C^{*}$-subalgebra of $\mathscr{A}$ is itself a $C^{*}$-algebra when endowed with the operations and norm of $\mathscr{A}$.

Example One can form products (in the categorical sense, see 10 VII ) of $C^{*}$ algebras as follows. Let $\mathscr{A}_{i}$ be a $C^{*}$-algebra for every element $i$ of some index set $I$. The direct sum of the family $\left(\mathscr{A}_{i}\right)_{i}$ is the $C^{*}$-algebra denoted by $\bigoplus_{i \in I} \mathscr{A}_{i}$ on the set

$$
\left\{a \in \prod_{i \in I} \mathscr{A}_{i}: \sup _{i \in I}\|a(i)\|<\infty\right\}
$$

whose operations are defined coordinatewise, and whose norm is a supremum norm given by $\|a\|=\sup _{i}\|a(i)\|$. If each $\mathscr{A}_{i}$ is commutative, then $\bigoplus_{i \in I} \mathscr{A}_{i}$ is commutative.

In particular, taking $\mathscr{A}_{i} \equiv \mathbb{C}$, we see that the vector space $\ell^{\infty}(X)$ of bounded complex-valued functions on a set $X$ forms a commutative $C^{*}$-algebra with pointwise operations and supremum norm.

Example The bounded continuous functions on a topological space $X$ form a commutative $C^{*}$-subalgebra $B C(X)$ of $\ell^{\infty}(X)$ (see above). In particular, since a continuous function on a compact Hausdorff space is automatically bounded, we see that the continuous functions on a compact Hausdorff space $X$ form a commutative $C^{*}$-algebra $C(X)$ with pointwise operations and sup-norm. We'll see that every commutative $C^{*}$-algebra is isomorphic to a $C(X)$ in 27 XXVII .

Example An example of a non-commutative $C^{*}$-algebra is the vector space $M_{n}$ of $n \times n$-matrices $(n>1)$ over $\mathbb{C}$ with the usual (matrix) multiplication, the identity matrix as unit, and conjugate transpose as involution (so $\left(A^{*}\right)_{i j}=\overline{A_{j i}}$ ). The norm $\|A\|$ of a matrix $A$ in $M_{n}$ is less obvious, being the operator norm (cf. 4 II) of the associated linear map $v \mapsto A v, \mathbb{C}^{n} \rightarrow \mathbb{C}^{n}$, that is, $\|A\|$ is the least number $r \geqslant 0$ with $\|A v\|_{2} \leqslant r\|v\|_{2}$ for all $v \in \mathbb{C}^{n}\left(\right.$ where $\|w\|_{2}=\left(\sum_{i}\left|w_{i}\right|^{2}\right)^{1 / 2}$ denotes the 2-norm of $w \in \mathbb{C}^{n}$ ).

It is not entirely obvious that $\left\|A^{*} A\right\|=\|A\|^{2}$ holds and that $M_{n}$ is complete. We will prove these facts in the more general setting of bounded operators between Hilbert spaces, see 51. Suffice it to say, $\mathbb{C}^{n}$ is a Hilbert space with $\langle v, w\rangle=\sum_{i} \bar{v}_{i} w_{i}$ as inner product, each matrix gives a (bounded) linear map $v \mapsto A v, \mathbb{C}^{n} \rightarrow \mathbb{C}^{n}$, and the conjugate transpose $A^{*}$ is adjoint to $A$ in the sense that $\langle v, A w\rangle=\left\langle A^{*} v, w\right\rangle$ for all $v, w \in \mathbb{C}^{n}$.

Remark Combining V and VII we see that $\bigoplus_{k} M_{n_{k}}$ is a finite-dimensional VIII $C^{*}$-algebra for any tuple $n_{1}, \ldots, n_{K}$ of natural numbers. In fact, any finitedimensional $C^{*}$-algebra is of this form as we'll see in 84 II *

[^2]
### 2.1.1 Operators

4 Example Let us now turn to perhaps the most important and difficult example: we'll show that the vector space $\mathscr{B}(\mathscr{H})$ of bounded operators on a Hilbert space $\mathscr{H}$ forms a $C^{*}$-algebra when endowed with the operator norm. Multiplication is given by composition, involution by taking the adjoint (see VIII), and unit by the identity operator. A concrete $C^{*}$-algebra or a $C^{*}$-algebra of bounded operators refers to a $C^{*}$-subalgebra of $\mathscr{B}(\mathscr{H})$. We will eventually see that every $C^{*}$-algebra is isomorphic to a $C^{*}$-algebra of bounded operators in 30XIV.
II Definition Let $\mathscr{X}$ and $\mathscr{Y}$ be normed vector spaces. We say that $r \in[0, \infty)$ is a bound for a linear map (=operator) $T: \mathscr{X} \rightarrow \mathscr{Y}$ when $\|T x\| \leqslant r\|x\|$ for all $x \in \mathscr{X}$, and we say that $T$ is bounded when there is such a bound. In that case $T$ has a least bound, which is called the operator norm of $T$, and is denoted by $\|T\|$. The vector space of bounded operators from $\mathscr{X}$ to $\mathscr{Y}$ is denoted by $\mathscr{B}(\mathscr{X}, \mathscr{Y})$, and the vector space of bounded operators from $\mathscr{X}$ to itself is denoted by $\mathscr{B}(\mathscr{X})$.
III Exercise Let $\mathscr{X}, \mathscr{Y}$ and $\mathscr{Z}$ be normed complex vector spaces.

1. Show that the operator norm on $\mathscr{B}(\mathscr{X}, \mathscr{Y})$ is, indeed, a norm.
2. Let $T: \mathscr{X} \rightarrow \mathscr{Y}$ and $S: \mathscr{Y} \rightarrow \mathscr{Z}$ be bounded operators. Show that $S T$ is bounded by $\|S\|\|T\|$, so that $\|S T\| \leqslant\|S\|\|T\|$.
3. Show that the identity operator id: $\mathscr{X} \rightarrow \mathscr{X}$ is bounded by 1 .

IV Exercise Let $T: \mathscr{X} \rightarrow \mathscr{Y}$ be a bounded operator between normed vector spaces, and let $r \in[0, \infty)$. Show that

$$
r\|T\|=\sup _{x \in(\mathscr{X})_{r}}\|T x\|
$$

where $(\mathscr{X})_{r}=\{x \in \mathscr{X}:\|x\| \leqslant r\}$. (The set $(\mathscr{X})_{1}$ is called the unit ball of $\left.\mathscr{X}.\right)$
V Lemma The operator norm on $\mathscr{B}(\mathscr{X}, \mathscr{Y})$ is complete when $\mathscr{Y}$ is a complete normed vector space.
VI Proof Let $\left(T_{n}\right)_{n}$ be a Cauchy sequence in $\mathscr{B}(\mathscr{X}, \mathscr{Y})$. We must show that $\left(T_{n}\right)_{n}$ converges to some bounded operator $T: \mathscr{X} \rightarrow \mathscr{Y}$. Let $x \in \mathscr{X}$ be given. Since

$$
\left\|T_{n} x-T_{m} x\right\|=\left\|\left(T_{n}-T_{m}\right) x\right\| \leqslant\left\|T_{n}-T_{m}\right\|\|x\|
$$

and $\left\|T_{n}-T_{m}\right\| \rightarrow 0$ as $n, m \rightarrow \infty$ (because $\left(T_{k}\right)_{k}$ is Cauchy), we see that $\left\|T_{n} x-T_{m} x\right\| \rightarrow 0$ as $n, m \rightarrow \infty$, and so $\left(T_{n} x\right)_{n}$ is a Cauchy sequence in $\mathscr{Y}$. Since $\mathscr{Y}$ is complete, $\left(T_{n} x\right)_{n}$ converges, and we may define $T x:=\lim _{n} T_{n} x$, giving a map $T: \mathscr{X} \rightarrow \mathscr{Y}$, which is easily seen to be linear (by continuity of addition and scalar multiplication).

It remains to be shown that $T$ is bounded, and that $\left(T_{n}\right)_{n}$ converges to $T$ with respect to the operator norm. Let $\varepsilon>0$ be given, and pick $N$ such that $\left\|T_{n}-T_{m}\right\| \leqslant \frac{1}{2} \varepsilon$ for all $n, m \geqslant N$. Then for every $x \in \mathscr{X}$ we can find $M \geqslant N$ with $\left\|T x-T_{m} x\right\| \leqslant \frac{1}{2} \varepsilon\|x\|$ for all $m \geqslant M$, and so, for $n \geqslant N, m \geqslant M$,

$$
\left\|\left(T-T_{n}\right) x\right\| \leqslant\left\|T x-T_{m} x\right\|+\left\|T_{m} x-T_{n} x\right\| \leqslant \varepsilon\|x\|
$$

giving that $T-T_{n}$ is bounded and $\left\|T-T_{n}\right\| \leqslant \varepsilon$ for all $n \geqslant N$. Whence $T$ is bounded too, and $\left(T_{n}\right)_{n}$ converges to $T$.
From $\operatorname{III}$ and V it is clear that the complex vector space of bounded operators $\mathscr{B}(\mathscr{X})$ on a complete normed vector space $\mathscr{X}$ with composition as multiplication and the identity operator as unit satisfies all the requirements to be $C^{*}$-algebra that do not involve the involution, $(\cdot)^{*}$ (that is, $\mathscr{B}(\mathscr{X})$ is a Banach algebra). To get an involution, we need the additional structure provided by a Hilbert space as follows.
Definition An inner product on a complex vector space $V$ is a map $\langle\cdot, \cdot\rangle: V \times$ $V \rightarrow \mathbb{C}$ such that, for all $x, y \in V,\langle x, \cdot\rangle: V \rightarrow V$ is linear; $\langle x, x\rangle \geqslant 0$; and $\langle x, y\rangle=\overline{\langle y, x\rangle}$. We say that the inner product is definite when $\langle x, x\rangle=0 \Longrightarrow$ $x=0$ for $x \in V$. A pre-Hilbert space $\mathscr{H}$ is a complex vector space endowed with a definite inner product. We'll shortly see that every such $\mathscr{H}$ carries a norm given by $\|x\|:=\langle x, x\rangle^{1 / 2}$; if $\mathscr{H}$ is complete with respect to this norm, we say that $\mathscr{H}$ is a Hilbert space.

Let $\mathscr{H}$ and $\mathscr{K}$ be pre-Hilbert spaces. We say that a operator $T: \mathscr{H} \rightarrow \mathscr{K}$ is adjoint to a operator $S: \mathscr{K} \rightarrow \mathscr{H}$ when

$$
\langle T x, y\rangle=\langle x, S y\rangle \quad \text { for all } x \in \mathscr{H} \text { and } y \in \mathscr{K} .
$$

In that case, we call $T$ adjoinable. We'll see (in $\mathbb{\text { p }}$ that such adjoinable $T$ is adjoint to exactly one $S$, which we denote by $T^{*}$.
Example We endow $\mathbb{C}^{N}$ (where $N$ is a natural number) with the inner product IX given by $\langle x, y\rangle=\sum_{i} \bar{x}_{i} y_{i}$, making it a Hilbert space.

The space $c_{00}$ of sequences $x_{1}, x_{2}, \ldots$ for which $x_{n}$ is non-zero for finitely many $n$ 's is an example of a pre-Hilbert which is not complete when endowed with $\langle x, y\rangle=\sum_{n=0}^{\infty} \bar{x}_{n} y_{n}$ as inner product.

For an example of an infinite-dimensional Hilbert space, we'll have to wait until 6 II where we'll show that the sequences $x_{1}, x_{2}, \ldots$ with $\sum_{n}\left|x_{n}\right|^{2}<\infty$ form a Hilbert space $\ell^{2}$ with $\langle x, y\rangle=\sum_{n=0}^{\infty} \bar{x}_{n} y_{n}$ as its inner product, because at this point it is not even clear that this sum converges.

X Exercise Let $x$ and $x^{\prime}$ be elements of a pre-Hilbert space $\mathscr{H}$ with $\langle y, x\rangle=\left\langle y, x^{\prime}\right\rangle$ for all $y \in \mathscr{H}$. Show that $x=x^{\prime}$ (by taking $y=x-x^{\prime}$ ). Conclude that every operator between pre-Hilbert spaces has at most one adjoint.
XI Remark Note that we did not require that an adjoinable operator $T: \mathscr{H} \rightarrow \mathscr{K}$ between pre-Hilbert spaces be bounded, and in fact, it might not be. Take for example the operator $T: c_{00} \rightarrow c_{00}$ given by $(T x)_{n}=n x_{n}$, which is adjoint to itself, and not bounded. On the other hand, if either $\mathscr{H}$ or $\mathscr{K}$ is complete, then both $T$ and $T^{*}$ are automatically bounded as we'll see in 35 VI .

XII Exercise Let $S$ and $T$ be adjoinable operators on a pre-Hilbert space.

1. Show that $T^{*}$ is adjoint to $T$ (and so $T^{* *}=T$ ).
2. Show that $(T+S)^{*}=T^{*}+S^{*}$ and $(\lambda S)^{*}=\bar{\lambda} S^{*}$ for every $\lambda \in \mathbb{C}$.
3. Show that $S T$ is adjoint to $T^{*} S^{*}$ (and so $\left.(S T)^{*}=T^{*} S^{*}\right)$.

We will, of course, show that every bounded operator on a Hilbert space is adjoinable, see 5 XI. But let us first show that $\|\cdot\|$ defined in VIII is a norm, which boils down to the following fact about $2 \times 2$-matrices.
XIII Lemma For a positive matrix $A \equiv\left(\begin{array}{c}p \\ c \\ c \\ q\end{array}\right)$ (i.e. $(\bar{u} \bar{v}) A\binom{u}{v} \geqslant 0$ for all $\left.u, v \in \mathbb{C}\right)$, we have $p, q \geqslant 0$, and $|c|^{2} \leqslant p q$.
XIV Proof Let $u, v \in \mathbb{C}$ be given. We have

$$
0 \leqslant(\bar{u} \bar{v}) A\binom{u}{v}=|u|^{2} p+\bar{u} v \bar{c}+u \bar{v} c+|v|^{2} q .
$$

By taking $u=1$ and $v=0$, we see that $p \geqslant 0$, and similarly $q \geqslant 0$.
The trick to see that $|c|^{2} \leqslant p q$ is to take $v=1$ and $u=t \bar{c}$ with $t \in \mathbb{R}$ :

$$
0 \leqslant p|c|^{2} t^{2}+2|c|^{2} t+q
$$

If $p=0$, then $-2|c|^{2} t \leqslant q$ for all $t \in \mathbb{R}$, which implies that $|c|^{2}=0=p q$.
Suppose that $p>0$. Then taking $t=-p^{-1}$ we see that

$$
0 \leqslant|c|^{2} p^{-1}-2|c|^{2} p^{-1}+q=-|c|^{2} p^{-1}+q
$$

Rewriting gives us $|c|^{2} \leqslant p q$.

Exercise Let $\langle\cdot, \cdot\rangle$ be an inner product on a vector space $V$. Show that the formula $\|x\|=\sqrt{\langle x, x\rangle}$ defines a seminorm on $V$, that is, $\|x\| \geqslant 0,\|\lambda x\|=|\lambda|\|x\|$, and-the triangle inequality- $\|x+y\| \leqslant\|x\|+\|y\|$ for all $\lambda \in \mathbb{C}$ and $x, y \in V$.

Moreover, prove that $\|\cdot\|$ is a norm when $\langle\cdot, \cdot\rangle$ is definite; and for $x, y \in V$ :

1. The Cauchy-Schwarz inequality: $|\langle x, y\rangle|^{2} \leqslant\langle x, x\rangle\langle y, y\rangle$;
2. Pythagoras' theorem: $\|x\|^{2}+\|y\|^{2}=\|x+y\|^{2}$ when $\langle x, y\rangle=0$;
3. The parallelogram law: $\|x\|^{2}+\|y\|^{2}=\frac{1}{2}\left(\|x+y\|^{2}+\|x-y\|^{2}\right)$;
4. The polarization identity: $\langle x, y\rangle=\frac{1}{4} \sum_{n=0}^{3} i^{n}\left\|i^{n} x+y\right\|^{2}$.
(Hint: prove the Cauchy-Schwarz inequality before the triangle inequality by applying XIII to the matrix $\left(\begin{array}{c}\langle x, x\rangle \\ \langle y, x\rangle\end{array}\left\langle\begin{array}{l}\langle x, y\rangle \\ y, y\rangle\end{array}\right)\right.$. Then prove $\|x+y\|^{2} \leqslant(\|x\|+\|y\|)^{2}$ using the inequalities $\langle x, y\rangle+\langle y, x\rangle \leqslant 2|\langle x, y\rangle| \leqslant 2\|x\|\|y\|$.)
Lemma For an adjoinable operator $T$ on a pre-Hilbert space $\mathscr{H}$

$$
\left\|T^{*} T\right\|=\|T\|^{2} \quad \text { and } \quad\left\|T^{*}\right\|=\|T\|
$$

Proof If $T=0$, then $T^{*}=0$, and the statements are surely true.
XVII
Suppose $T \neq 0$ (and so $T^{*} \neq 0$ ). Since $\|T x\|^{2}=\langle T x, T x\rangle=\left\langle x, T^{*} T x\right\rangle \leqslant$ $\|x\|\left\|T^{*} T x\right\| \leqslant\|x\|^{2}\left\|T^{*} T\right\|$ for every $x \in \mathscr{H}$ by Cauchy-Schwarz, we have $\|T\|^{2} \leqslant\left\|T^{*} T\right\|$. Since $\left\|T^{*} T\right\| \leqslant\left\|T^{*}\right\|\|T\|$ and $\|T\| \neq 0$, it follows that $\|T\| \leqslant$ $\left\|T^{*}\right\|$. Since by a similar reasoning $\left\|T^{*}\right\| \leqslant\|T\|$, we get $\|T\|=\left\|T^{*}\right\|$. But then $\|T\|^{2} \leqslant\left\|T^{*} T\right\| \leqslant\left\|T^{*}\right\|\|T\|=\|T\|^{2}$, and so $\|T\|^{2}=\left\|T^{*} T\right\|$.
Exercise Given a Hilbert space $\mathscr{H}$ show that the adjoinable operators form a XVIII closed subspace of $\mathscr{B}(\mathscr{H})$.
Exercise Let $x$ and $y$ be vectors from a Hilbert space $\mathscr{H}$.

1. Show that $|x\rangle\langle y|: z \mapsto\langle y, z\rangle x$ defines a bounded operator $\mathscr{H} \rightarrow \mathscr{H}$, and, moreover, that $\||x\rangle\langle y|\|=\| x\| \| y \|$.
2. Show that $|x\rangle\langle y|$ is adjoinable, and $(|x\rangle\langle y|)^{*}=|y\rangle\langle x|$.

At this point it is clear that the vector space of adjoinable operators on a Hilbert space forms a $C^{*}$-algebra. So to prove that $\mathscr{B}(\mathscr{H})$ is a $C^{*}$-algebra, it remains
to be shown that every bounded operator is adjoinable (which we'll do in XI). We first show that each bounded functional $f: \mathscr{H} \rightarrow \mathbb{C}$ has an adjoint, see IX, for which we need the (existence and) properties of "projections" on (closed) linear subspaces:

II Definition Let $x$ be an element of a pre-Hilbert space $\mathscr{H}$. We say that an element $y$ of a linear subspace $C$ of $\mathscr{H}$ is a projection of $x$ on $C$ if

$$
\|x-y\|=\min \left\{\left\|x-y^{\prime}\right\|: y^{\prime} \in C\right\} .
$$

(In other words, $y$ is one of the elements of $C$ closest to $x$.)
III Exercise We'll see in VII that on a closed linear subspace every vector has a projection. For arbitrary linear subspaces this isn't so: show that the only vectors in $\ell_{2}$ having a projection on the linear subspace $c_{00}$ (from 4 IX ) are the vectors in $c_{00}$ themselves.
IV Lemma Let $\mathscr{H}$ be a pre-Hilbert space, and let $x, e \in \mathscr{H}$ with $\|e\|=1$.
Then $y=\langle e, x\rangle e$ is the unique projection of $x$ on $e \mathbb{C}$.
$\checkmark$ Proof Let $y^{\prime} \in e \mathbb{C}$ with $y^{\prime} \neq y$ be given. To prove that $y$ is the unique projection of $x$ on $e \mathbb{C}$ it suffices to show that $\|x-y\|<\left\|x-y^{\prime}\right\|$. Since $y^{\prime} \neq y \equiv\langle e, x\rangle e$, there is $\lambda \in \mathbb{C}, \lambda \neq 0$ with $y^{\prime}=(\lambda+\langle e, x\rangle) e$.

Note that $\langle e, y\rangle=\langle e,\langle e, x\rangle e\rangle=\langle e, x\rangle\langle e, e\rangle=\langle e, x\rangle$, and so $\langle e, x-y\rangle=0$. Then $y^{\prime}-y \equiv \lambda e$ and $x-y$ are orthogonal too, and thus, by Pythagoras' theorem (see 4 XV , , we have $\left\|y^{\prime}-x\right\|^{2}=\left\|y^{\prime}-y\right\|^{2}+\|y-x\|^{2} \equiv|\lambda|^{2}+\|x-y\|^{2}>\|x-y\|^{2}$, because $\lambda \neq 0$. Hence $\left\|y^{\prime}-x\right\|>\|y-x\|$.
VI Exercise Let $y$ be a projection of an element $x$ of a pre-Hilbert space $\mathscr{H}$ on a linear subspace $C$. Show that $y$ is a projection of $x$ on $y \mathbb{C}$. Conclude that $y$ is the unique projection of $x$ on $C$, and that $\langle y, x-y\rangle=0$. Show that $y+c$ is the projection of $x+c$ on $C$ for every $c \in C$. Conclude that $\left\langle y^{\prime}, x-y\right\rangle \equiv$ $\left\langle y^{\prime},\left(x+y^{\prime}-y\right)-y^{\prime}\right\rangle=0$ for every $y^{\prime} \in C$.
VII Projection Theorem Let $C$ be a closed linear subspace of a Hilbert space $\mathscr{H}$. Each $x \in \mathscr{H}$ has a unique projection $y$ on $C$, and $\left\langle y^{\prime}, y\right\rangle=\left\langle y^{\prime}, x\right\rangle$ for $y^{\prime} \in C$.
VIII Proof We only need to show that there is a projection $y$ of $x$ on $C$, because VI gives us that such $y$ is unique and satisfies $\left\langle y^{\prime}, y\right\rangle=\left\langle y^{\prime}, x\right\rangle$ for all $y^{\prime} \in C$.

Write $r:=\inf \left\{\left\|x-y^{\prime}\right\|: y^{\prime} \in C\right\}$, and pick a sequence $y_{1}, y_{2}, \ldots \in C$ such that $\left\|x-y_{n}\right\| \rightarrow r$. We will show that $y_{1}, y_{2}, \ldots$ is Cauchy. Let $\varepsilon>0$ be given, and pick $N$ such that $\left\|y_{n}-x\right\|^{2} \leqslant r^{2}+\frac{1}{4} \varepsilon$ for all $n \geqslant N$. Let $n, m \geqslant N$ be given. Then since $\frac{1}{2}\left(y_{n}+y_{m}\right)$ is in $C$, we have $\left\|y_{n}+y_{m}-2 x\right\| \equiv 2\left\|\frac{1}{2}\left(y_{n}+y_{m}\right)-x\right\| \geqslant 2 r$,
and so by the parallelogram law (see 4XV),

$$
\begin{aligned}
\left\|y_{n}-y_{m}\right\|^{2} & \equiv\left\|\left(y_{n}-x\right)-\left(y_{m}-x\right)\right\|^{2} \\
& =2\left\|y_{n}-x\right\|^{2}+2\left\|y_{m}-x\right\|^{2}-\left\|y_{n}+y_{m}-2 x\right\|^{2} \\
& \leqslant 4 r^{2}+\varepsilon-4 r^{2} \leqslant \varepsilon .
\end{aligned}
$$

Hence $y_{1}, y_{2}, \ldots$ is Cauchy, and converges to some $y \in C$, because $\mathscr{H}$ is complete and $C$ is closed. It follows easily that $\|x-y\|=r$, and thus $y$ is the projection of $x$ on $C$.

Riesz' Representation Theorem Let $\mathscr{H}$ be a Hilbert space. For every bounded linear map $f: \mathscr{H} \rightarrow \mathbb{C}$ there is a unique vector $x \in \mathscr{H}$ with $\langle x, \cdot\rangle=f$.
Proof If $f=0$, then $x=0$ does the job. Suppose that $f \neq 0$. There is an $x^{\prime} \in \mathscr{H}$ with $f\left(x^{\prime}\right) \neq 0$. Note that $\operatorname{ker}(f)$ is closed, because $f$ is bounded. So by VII, we know that $x^{\prime}$ has a projection $y$ on $\operatorname{ker}(f)$, and $\left\langle x^{\prime}, z\right\rangle=\langle y, z\rangle$ for all $z \in \operatorname{ker}(f)$. Then for $x^{\prime \prime}:=f\left(x^{\prime}-y\right)^{-1}\left(x^{\prime}-y\right)$, we have $f\left(x^{\prime \prime}\right)=1$ and $\left\langle x^{\prime \prime}, y^{\prime}\right\rangle=0$ for all $y^{\prime} \in \operatorname{ker}(f)$. Given $z \in \mathscr{H}$, we have $f\left(z-f(z) x^{\prime \prime}\right)=0$, so $z-f(z) x^{\prime \prime} \in \operatorname{ker}(f)$, and thus $0=\left\langle x^{\prime \prime}, z-f(z) x^{\prime \prime}\right\rangle \equiv\left\langle x^{\prime \prime}, z\right\rangle-f(z)\left\|x^{\prime \prime}\right\|^{2}$. Hence writing $x:=x^{\prime \prime}\left\|x^{\prime \prime}\right\|^{-2}$ we have $f(z)=\langle x, z\rangle$ for all $z \in \mathscr{H}$.

Finally, uniqueness of $x$ follows from 4X
Exercise Prove that every bounded operator $T$ on a Hilbert space $\mathscr{H}$ is adjoinable, as follows. Let $x \in \mathscr{H}$ be given. Prove that $\langle x, T(\cdot)\rangle: \mathscr{H} \rightarrow \mathbb{C}$ is a bounded linear map. Let $S x$ be the unique vector with $\langle S x, \cdot\rangle=\langle x, T(\cdot)\rangle$, which exists by $\mathbb{X X}$. Show that $x \mapsto S x$ gives a bounded linear map $S$, which is adjoint to $T$.

Thus the bounded operators on a Hilbert space $\mathscr{H}$ form a $C^{*}$-algebra $\mathscr{B}(\mathscr{H})$ as described in 41. We will return to Hilbert spaces in 30XIV, where we show that every $C^{*}$-algebra is isomorphic to a $C^{*}$-subalgebra of a $\mathscr{B}(\mathscr{H})$.

Here is a non-trivial example of a Hilbert space that will be used later on.
Proposition Given a family $\left(\mathscr{H}_{i}\right)_{i \in I}$ of Hilbert spaces, the vector space

$$
\bigoplus_{i} \mathscr{H}_{i}=\left\{x \in \prod_{i} \mathscr{H}_{i}: \quad \sum_{i}\left\|x_{i}\right\|^{2}<\infty\right\} .
$$

is a Hilbert space when endowed with the inner product $\langle x, y\rangle=\sum_{i}\left\langle x_{i}, y_{i}\right\rangle$.
Proof To begin with we must show that $\sum_{i}\left\langle x_{i}, y_{i}\right\rangle$ converges for $x, y \in \bigoplus_{i} \mathscr{H}_{i}$.
Given $\varepsilon>0$ we must find a finite subset $G$ of $I$ such that $\left|\sum_{i \in F}\left\langle x_{i}, y_{i}\right\rangle\right| \leqslant \varepsilon$
for all finite $F \subseteq I \backslash G$. Since an obvious application of the Cauchy-Schwarz inequality gives us that for every finite subset $F$ of $I$

$$
\left|\sum_{i \in F}\left\langle x_{i}, y_{i}\right\rangle\right|^{2} \leqslant \sum_{i \in F}\left\|x_{i}\right\|^{2} \sum_{i \in F}\left\|y_{i}\right\|^{2}
$$

any $G \subseteq I$ with $\sum_{i \in I \backslash G}\left\|x_{i}\right\|^{2} \leqslant \sqrt{\varepsilon}$ and $\sum_{i \in I \backslash G}\left\|y_{i}\right\|^{2} \leqslant \sqrt{\varepsilon}$ will do.
It is easily seen that $\langle x, y\rangle:=\sum_{i}\left\langle x_{i}, y_{i}\right\rangle$ gives a definite inner product on $\bigoplus_{i} \mathscr{H}_{i}$; the remaining difficulty lies in showing that resulting norm is complete. To this end, let $x_{1}, x_{2}, \ldots$ be a Cauchy sequence in $\bigoplus_{i \in I} \mathscr{H}_{i}$; we must show that it converges to some $x_{\infty} \in \bigoplus_{i} \mathscr{H}_{i}$. We do the obvious thing: since for every $i \in I$ the sequence $\left(x_{1}\right)_{i},\left(x_{2}\right)_{i}, \ldots$ is Cauchy in $\mathscr{H}_{i}$ we may define $\left(x_{\infty}\right)_{i}:=\lim _{n}\left(x_{n}\right)_{i}$, and thereby get an element $x_{\infty}$ of $\prod_{i} \mathscr{H}_{i}$. Since for each finite subset $F$ of $I$ we have $\sum_{i \in F}\left\|\left(x_{\infty}\right)_{i}\right\|^{2}=\lim _{n} \sum_{i \in F}\left\|\left(x_{n}\right)_{i}\right\|^{2} \leqslant \lim _{n}\left\|x_{n}\right\|^{2}$, we have $\sum_{i \in I}\left\|\left(x_{\infty}\right)_{i}\right\|^{2} \leqslant \lim _{n}\left\|x_{n}\right\|^{2}<\infty$, and so $x_{\infty} \in \bigoplus_{i} \mathscr{H}_{i}$.

It remains to be shown that $x_{1}, x_{2}, \ldots$ converges to $x_{\infty}$ (not only coordinatewise but also) with respect to the inner product on $\bigoplus_{i} \mathscr{H}_{i}$. Given $\varepsilon>0$ pick $N$ such that $\left\|x_{n}-x_{m}\right\| \leqslant \frac{1}{2 \sqrt{2}} \varepsilon$ for all $n, m \geqslant N$. We claim that for such $n$ we have $\left\|x_{\infty}-x\right\| \leqslant \varepsilon$. Indeed, first note that since the sum

$$
\sum_{i \in I}\left\|\left(x_{\infty}\right)_{i}-\left(x_{n}\right)_{i}\right\|^{2} \equiv \sum_{i \in F}\left\|\left(x_{\infty}\right)_{i}-\left(x_{n}\right)_{i}\right\|^{2}+\sum_{i \in I \backslash F}\left\|\left(x_{\infty}\right)_{i}-\left(x_{n}\right)_{i}\right\|^{2}
$$

converges (to $\left\|x_{\infty}-x_{n}\right\|^{2}$ ), we can find a finite subset $F$ (depending on $n$ ) such that second term in the right-hand side above is smaller than $\frac{1}{2} \varepsilon^{2}$. To see that the first term is also below $\frac{1}{2} \varepsilon^{2}$, begin by noting that for every $m$,

$$
\left(\sum_{i \in F}\left\|\left(x_{\infty}\right)_{i}-\left(x_{n}\right)_{i}\right\|^{2}\right)^{1 / 2} \leqslant\left(\sum_{i \in F}\left\|\left(x_{\infty}\right)_{i}-\left(x_{m}\right)_{i}\right\|^{2}\right)^{1 / 2}+\left(\sum_{i \in F}\left\|\left(x_{m}\right)_{i}-\left(x_{n}\right)_{i}\right\|^{2}\right)^{1 / 2}
$$

Since $F$ is finite, and $\left(x_{m}\right)$ converges to $x_{\infty}$ coordinatewise we can find an $m$ large enough that the first term on the right-hand side above is below $\frac{1}{2 \sqrt{2}} \varepsilon$. If we choose $m \geqslant N$ we see that the second term is below $\frac{1}{2 \sqrt{2}} \varepsilon$ as well, and we conclude that $\left\|x_{\infty}-x_{n}\right\| \leqslant \varepsilon$.

### 2.2 The Basics

7 Now that we have seen the most important examples of $C^{*}$-algebras, we can begin developing the theory. We'll start easy with the self-adjoint elements:

Definition Given an element $a$ of a $C^{*}$-algebra $\mathscr{A}$,

1. we say that $a$ is self-adjoint if $a^{*}=a$, and
2. we write $a_{\mathbb{R}}:=\frac{1}{2}\left(a+a^{*}\right)$ and $a_{\mathbb{I}}:=\frac{1}{2 i}\left(a-a^{*}\right)$ for the real and imaginary part of $a$, respectively.

The set of self-adjoint elements of $\mathscr{A}$ is denoted by $\mathscr{A}_{\mathbb{R}}$.
Exercise Let $a$ be an element of a $C^{*}$-algebra $\mathscr{A}$.

1. Show that $a_{\mathbb{R}}$ and $a_{\mathbb{I}}$ are self-adjoint, and $a=a_{\mathbb{R}}+i a_{\mathbb{I}}$.
2. Show that if $a \equiv b+i c$ for self-adjoint elements $b, c$ of $\mathscr{A}$, then $b=a_{\mathbb{R}}$ and $c=a_{\mathbb{I}}$.
3. Show that $\left(a^{*}\right)_{\mathbb{R}}=a_{\mathbb{R}}$ and $\left(a^{*}\right)_{\mathbb{I}}=-a_{\mathbb{I}}$.
4. Show that $a$ is self-adjoint iff $a_{\mathbb{R}}=a$ iff $a_{\mathbb{I}}=0$.
5. Show that $a \mapsto a_{\mathbb{R}}$ and $a \mapsto a_{\mathbb{I}}$ give $\mathbb{R}$-linear maps $\mathscr{A} \rightarrow \mathscr{A}$.
6. Show that $a_{\mathbb{I}}=-(i a)_{\mathbb{R}}$ and $a_{\mathbb{R}}=(i a)_{\mathbb{I}}$.
7. Show that $a^{*} a$ is self-adjoint, and $a^{*} a=a_{\mathbb{R}}^{2}+a_{\mathbb{I}}^{2}+i\left(a_{\mathbb{R}} a_{\mathbb{I}}-a_{\mathbb{I}} a_{\mathbb{R}}\right)$.
8. Give an example of $\mathscr{A}$ and $a$ with $a_{\mathbb{R}} a_{\mathbb{I}} \neq a_{\mathbb{I}} a_{\mathbb{R}}$.
(This inequality is a source of many technical difficulties.)
9. Show that $a^{*} a+a a^{*}=2\left(a_{\mathbb{R}}^{2}+a_{\mathbb{I}}^{2}\right)$.
10. The product of self-adjoint elements $b, c$ need not be self-adjoint; show that, in fact, $b c$ is self-adjoint iff $b c=c b$.
11. Show that $\left\|a^{*}\right\|=\|a\|$. (Hint: $\|a\|^{2}=\left\|a^{*} a\right\| \leqslant\left\|a^{*}\right\|\|a\|$.)
12. Show that $\left\|a_{\mathbb{R}}\right\| \leqslant\|a\|$ and $\left\|a_{\mathbb{I}}\right\| \leqslant\|a\|$.
13. Show that $\left\|a^{2}\right\|=\|a\|^{2}$ when $a$ is self-adjoint.

However, show that $\left\|a^{2}\right\| \neq\|a\|^{2}$ might occur when $a$ is not self-adjoint. (Hint: $\left(\begin{array}{ll}0 & 1 \\ 0 & 0\end{array}\right)$.)

8 Notation Recall that (in this text) every $C^{*}$-algebra $\mathscr{A}$ has a unit, 1. Thus, for every scalar $\lambda \in \mathbb{C}$, we have an element $\lambda \cdot 1$ of $\mathscr{A}$, which we will simply denote by $\lambda$. This should hardly cause any confusion, for while an expression of an element of $\mathscr{A}$ such as $i+2+5 a$ (where $a \in \mathscr{A}$ ) may be interpreted in several ways, the result is always the same.
II Exercise There is a subtle point regarding the norm $\|\lambda\|$ of a scalar $\lambda \in \mathbb{C}$ inside a $C^{*}$-algebra $\mathscr{A}$ : we do not always have $\|\lambda\|=|\lambda|$ on the nose.

1. Indeed, show that $\|1\|=0 \neq 1$ when $\mathscr{A}=\{0\}$ is the trivial $C^{*}$-algebra.
2. Show that $\|\lambda\| \leqslant|\lambda|$ (in $\mathbb{C})$.
3. Show that $\|\lambda\|=|\lambda|$ when $\|\lambda\|$ and $|\lambda|$ are interpreted as elements of $\mathscr{A}$.

9 Let us now generalize the notion of a positive function in $C(X)$ to a positive element of a $C^{*}$-algebra. There are several descriptions of positive functions in $C(X)$ in terms of the $C^{*}$-algebra structure (see III) on which we can base such a generalization, and while we will eventually see that these all yield the same notion of positive element of a $C^{*}$-algebra (see 251) we base our definition of positive element $(X)$ on the description that is perhaps not most familiar, but does give us the richest structure at this stage.

II Exercise Let $X$ be a compact Hausdorff space. Show that for self-adjoint $f \in C(X)$, the following are equivalent.

1. $f(X) \subseteq[0, \infty)$;
2. $f \equiv g^{2}$ for some $g \in C(X)_{\mathbb{R}}$;
3. $f \equiv g^{*} g$ for some $g \in C(X)$;
4. $\|f-t\| \leqslant t$ for some $t \geqslant \frac{1}{2}\|f\|$.

III Exercise To see how condition 1 can be expressed in terms of the $C^{*}$-algebra structure of $C(X)$, prove that $\lambda \in f(X)$ iff $f-\lambda$ is not invertible.

Definition A self-adjoint element $a$ of a $C^{*}$-algebra $\mathscr{A}$ is called positive if $\|a-t\| \leqslant t$ for some $t \geqslant \frac{1}{2}\|a\|$. We write $a \leqslant b$ for $a, b \in \mathscr{A}$ when $b-a$ is positive, and we denote the set of positive elements of $\mathscr{A}$ by $\mathscr{A}_{+}$.
Remark One advantage of this definition over, say, taking the elements of the form $a^{*} a$ to be positive, is that it is immediately clear that an element $b$ of a $C^{*}$-subalgebra $\mathscr{B}$ of a $C^{*}$-algebra $\mathscr{A}$ is positive in $\mathscr{B}$ iff $b$ is positive in $\mathscr{A}$ that is, 'positive permanence' comes for free (cf. 11 XXIII). Another advantage is that it's also pretty easy to see that the sum of such positive elements is again positive, see VII.
Example We'll see in 25 V , that a bounded operator $T$ on a Hilbert space $\mathscr{H}$ is positive iff $\langle x, T x\rangle \geqslant 0$ for all $x \in \mathscr{H}$.

Lemma Let $a, b$ be positive elements of a $C^{*}$-algebra. Then $a+b$ is positive.
Proof Since $t \geqslant 0$, there is $t \geqslant \frac{1}{2}\|a\|$ with $\|a-t\| \leqslant t$. Similarly, there VIII is $s \geqslant \frac{1}{2}\|b\|$ with $\|b-s\| \leqslant s$. Then $\|a+b-(t+s)\| \leqslant\|a-t\|+\|b-s\| \leqslant t+s$ and $t+s \geqslant \frac{1}{2}(\|a\|+\|b\|) \geqslant \frac{1}{2}\|a+b\|$, so $a+b \geqslant 0$.
Exercise Given an element $a$ of a $C^{*}$-algebra $\mathscr{A}$ with $0 \leqslant a \leqslant 1$ (which is called an effect) show that the orthosupplement $a^{\perp}:=1-a$ is an effect too.

Exercise Let $\mathscr{A}$ be a $C^{*}$-algebra.

1. Show that $\mathscr{A}_{+}$is a cone: $0 \in \mathscr{A}_{+}, a+b \in \mathscr{A}_{+}$for all $a, b \in \mathscr{A}_{+}$, and $\lambda a \in \mathscr{A}_{+}$for all $a \in \mathscr{A}_{+}$and $\lambda \in[0, \infty)$. Conclude that $\leqslant$ is a preorder.
2. Show that 1 is positive, and $-\|a\| \leqslant a \leqslant\|a\|$ for every self-adjoint element $a$ of $\mathscr{A}$. (Thus 1 is an order unit of $\mathscr{A}_{\mathbb{R}}$.)
3. The behavior of positive elements may be surprising: give an example of positive elements $a$ and $b$ from a $C^{*}$-algebra such that $a b$ is not positive.
4. Given a self-adjoint element $a$ of $\mathscr{A}$ define

$$
\|a\|_{o}=\inf \{\lambda \in[0, \infty):-\lambda \leqslant a \leqslant \lambda\} .
$$

Show that $\|-\|_{o}$ is a seminorm on $\mathscr{A}_{\mathbb{R}}$, and that $\|a\|_{o} \leqslant\|a\|$ for all $a \in \mathscr{A}_{\mathbb{R}}$.
Prove that $0 \leqslant a \leqslant b$ implies that $\|a\|_{o} \leqslant\|b\|_{o}$ for $a, b \in \mathscr{A}_{\mathbb{R}}$.
5. There is not much more that can easily be proven about positive elements, at this point, but don't take my word for it: try to prove the following facts about a self-adjoint element $a$ of $\mathscr{A}$ directly.
(a) $a^{2}$ is positive;
(b) if $a$ is the limit of positive $a_{n} \in \mathscr{A}$, then $a$ is positive;
(c) if $a \geqslant-\frac{1}{n}$ for all $n \in \mathbb{N}$, then $a \geqslant 0$;
(d) $\|a\|=\|a\|_{o}$;
(e) $a=0$ when $0 \leqslant a \leqslant 0$.

We will prove these facts when we return to the positive elements in 17

10 Let us spend some words on the morphisms between $C^{*}$-algebras.
II Definition A linear map $f: \mathscr{A} \rightarrow \mathscr{B}$ between $C^{*}$-algebras is called

1. multiplicative if $f(a b)=f(a) f(b)$ for all $a, b \in \mathscr{A}$;
2. involution preserving if $f\left(a^{*}\right)=f(a)^{*}$ for all $a \in \mathscr{A}$;
3. unital if $f(1)=1$;
4. subunital if $f(1) \leqslant 1$;
5. positive if $f(a)$ is positive for every positive $a \in \mathscr{A}$, and
6. completely positive if $\sum_{i, j} b_{i}^{*} f\left(a_{i}^{*} a_{j}\right) b_{j}$ is positive for all $a_{1}, \ldots, a_{n} \in \mathscr{A}$, and $b_{1}, \ldots, b_{n} \in \mathscr{B}$ (see Remark 5.1 of (54).

III We use the bold letters as abbreviations, so for instance, $f$ is pu if it is positive and unital, and a miu-map is a multiplicative, involution preserving, unital linear map between $C^{*}$-algebras (which is usually called a unital $*$-homomorphism).

We'll denote the category of $C^{*}$-algebras and miu-maps by $\mathbf{C}_{\text {MIU }}^{*}$, and the subcategory of commutative $C^{*}$-algebras by $\mathbf{c C}_{\text {miU }}^{*}$. We'll use similar notation for the other classes of maps, but will, naturally, only mention $\mathbf{C}_{\mathrm{CPU}}^{*}$ after having established that cp-maps are closed under composition.

The advantages of completely positive maps become apparent only later on when we start dealing with matrices (see 34II) and the tensor product (see 115 II ).

Lemma (" $\mathbf{p} \Rightarrow \mathbf{i}$ ") A positive map $f: \mathscr{A} \rightarrow \mathscr{B}$ between $C^{*}$-algebras is involution preserving.
Proof Let $a \in \mathscr{A}$ be given. We must show that $f\left(a^{*}\right)=f(a)^{*}$.
But first we'll show that if $a$ is self adjoint, then so is $f(a)$. Indeed, since $\|a\|$ and $\|a\|-a$ are positive (see 9 X ), we see that $f(\|a\|)$ ) and $f(\|a\|-a)$ are positive, and so $f(a)=f(\|a\|)-f(\|a\|-a)$ being positive is self adjoint.

It follows that $f(a)_{\mathbb{R}}=f\left(a_{\mathbb{R}}\right)$ and $f(a)_{\mathbb{I}}=f\left(a_{\mathbb{I}}\right)$ (for $a \in \mathscr{A}$ ), because $f(a) \equiv f\left(a_{\mathbb{R}}\right)+i f\left(a_{\mathbb{I}}\right)$, and $f\left(a_{\mathbb{R}}\right)$ and $f\left(a_{\mathbb{I}}\right)$ are self adjoint (see 7 III$)$.

Hence $f\left(a^{*}\right) \equiv f\left(a_{\mathbb{R}}-i a_{\mathbb{I}}\right)=f(a)_{\mathbb{R}}-i f(a)_{\mathbb{I}} \equiv f(a)^{*}$.
Remark Other important relations between these types of morphisms can only be established later on once we have a firmer grasp on the positive elements. We will then see that every mi-map is completely positive (in 34 IV ), and that every completely positive map is positive (in 25 II ).

Exercise Show that the product $\bigoplus_{i \in I} \mathscr{A}_{i}$ of a family $\left(\mathscr{A}_{i}\right)_{j \in I}$ of $C^{*}$-algebras defined in 3 V is also the categorical product of these $C^{*}$-algebras in $\mathbf{C}_{\mathrm{MIU}}^{*}$ with as projections the maps $\pi_{j}: \bigoplus_{i \in I} \mathscr{A}_{i} \rightarrow \mathscr{A}_{j}$ given by $\pi_{j}(a)=a(j)$.

Show that the same description applies to $\mathbf{c C}_{\text {MIU }}^{*}$.
Of course, $\bigoplus_{i} \mathscr{A}_{i}$ also gives the product in $\mathbf{C}_{\mathrm{PU}}^{*}$, but we must postpone the proof to 18 I when we're able to prove that an element $a$ of $\bigoplus_{i} \mathscr{A}_{i}$ is positive provided that each $a(i)$ is positive.

Exercise Show that given miu-maps $f, g: \mathscr{A} \rightarrow \mathscr{B}$ between $C^{*}$-algebras the collection $\mathscr{E}:=\{a \in \mathscr{A}: f(a)=g(a)\}$ is a $C^{*}$-subalgebra of $\mathscr{A}$, and that the inclusion $e: \mathscr{E} \rightarrow \mathscr{A}$ is a positive miu-map that is in fact the equaliser of $f$ and $g$ in $\mathbf{C}_{\mathrm{MIU}}^{*}$ and $\mathbf{C}_{\mathrm{PU}}^{*}$. Show that the same description applies to $\mathbf{c C}_{\mathrm{MIU}}^{*}$ and $\mathbf{c C}_{\mathrm{PU}}^{*}$.
Remark The assumption here that $f$ and $g$ are miu-maps might be essential: it is not clear whether arbitrary pu-maps $f, g: \mathscr{A} \rightarrow \mathscr{B}$ have an equaliser in $\mathbf{C}_{\mathrm{PU}}^{*}$.

After having visited the positive elements, let us explore our second landmark, the invertible elements of a $C^{*}$-algebra, whose role is as important as it is technical. This paragraph culminates in what is essentially spectral permanence (XXIII): the fact that if an element $a$ of a $C^{*}$-subalgebra $\mathscr{B}$ of a $C^{*}$-algebra $\mathscr{A}$ is invertible in $\mathscr{A}$, then $a$ is already invertible in $\mathscr{B}$, see XVI.

Lemma Let $a$ be an element of a $C^{*}$-algebra $\mathscr{A}$ with $\|a\|<1$. Then $a^{\perp} \equiv 1-a$ has an inverse, namely $\left(a^{\perp}\right)^{-1}=\sum_{n=0}^{\infty} a^{n}$. Moreover, this series converges absolutely, that is, $\sum_{n=0}^{\infty}\left\|a^{n}\right\|<\infty$.

III Proof Note that $(1-\|a\|)\left(1+\|a\|+\|a\|^{2}+\cdots+\|a\|^{N}\right)=1-\|a\|^{N+1}$, and so

$$
\sum_{n=0}^{N}\|a\|^{n}=\frac{1-\|a\|^{N+1}}{1-\|a\|}
$$

for every $N$. Thus, since $\|a\|^{N}$ converges to 0 (becaus ${ }^{\dagger}\|a\|<1$ ), we get $\sum_{n=0}^{\infty}\|a\|^{n}=(1-\|a\|)^{-1}$.

IV Note that $a^{N}$ norm converges to 0 , because $\|a\|^{N}$ converges to 0 . Also (but slightly less obvious), $\sum_{n} a^{n}$ norm converges, because $\sum_{n}\|a\|^{n}$ converges.
$\checkmark$ Thus, taking the norm limit on both sides of $(1-a)\left(1+a+a^{2}+\cdots a^{N}\right)=1-a^{N+1}$, gives us $(1-a)\left(\sum_{n} a^{n}\right)=1$. Since we can derive $\left(\sum_{n} a^{n}\right)(1-a)=1$ in a similar manner, we see that $\sum_{n} a^{n}$ is the inverse of $1-a$.

VI Exercise Let $a$ be an element of a $C^{*}$-algebra $\mathscr{A}$.

1. Show that $a-\lambda$ is invertible for every $\lambda \in \mathbb{C}$ with $\|a\|<|\lambda|$.
2. Show that $a-b$ is invertible when $b \in \mathscr{A}$ is invertible and $\|a\|<\|b\|$.
3. Show that $U:=\{b \in \mathscr{A}: b$ is invertible $\}$ is an open subset of $\mathscr{A}$.

VII Lemma For a self-adjoint element $a$ of $\mathscr{A}$ the series $\sum_{n} a^{n}$ converges iff $\|a\|<1$; and in that case converges absolutely.
VIII Proof We have already seen in III that $\sum_{n} a^{n}$ converges absolutely when $\|a\|<1$. Now, if $\sum_{n} a^{n}$ converges, then $\left\|a^{2^{n}}\right\| \equiv\|a\|^{2^{n}}$ (being the norm of the difference between consecutive partial sums of $\sum_{n} a^{n}$ ) converges to 0 , which only happens when $\|a\|<1$.
IX Remark For non-self-adjoint elements $a$ of $\mathscr{A}$, the convergence of $\sum_{n} a^{n}$ is a more delicate matter. Take for example the matrix $A:=\left(\begin{array}{ll}0 & 2 \\ 0 & 0\end{array}\right)$ for which the series $\sum_{n} A^{n}$ converges (to $A$ ), while $\|A\|=2$ - the problem being that $\left\|A^{2}\right\|^{1 / 2}$ differs from $\|A\|$. In fact, we'll see from $13 \|$ (although we won't need it) that $\sum_{n} a^{n}$ converges absolutely when $1>\lim \sup _{n}\left\|a^{n}\right\|^{1 / n}$, and diverges when $1<\lim \sup _{n}\left\|a^{n}\right\|^{1 / n}$. This begs the question what happens when $1=$ $\lim \sup _{n}\left\|a^{n}\right\|^{1 / n}$ - which I do not know.

[^3]Lemma Let $\mathscr{A}$ be a $C^{*}$-algebra. The assignment $a \mapsto a^{-1}$ gives a continuous
X map (from the set $\{b \in \mathscr{A}: b$ is invertible $\}$ to $\mathscr{A}$.)
Proof (Based on Proposition 3.1.6 of 43].)
XI
First we establish continuity at 1 : let $a \in \mathscr{A}$ with $\|1-a\| \leqslant \frac{1}{2}$ be given; we claim that $a$ is invertible, and $\left\|1-a^{-1}\right\| \leqslant 2\|1-a\|$.

Indeed, since $\|1-a\| \leqslant \frac{1}{2}<1, a$ is invertible by $\|$ and $a^{-1}=\sum_{n=0}^{\infty}(1-a)^{n}$. Then $\left\|1-a^{-1}\right\|=\left\|\sum_{n=1}^{\infty}(1-a)^{n}\right\| \leqslant \sum_{n=1}^{\infty}\|1-a\|^{n}=\|1-a\|(1-\|1-a\|)^{-1}$. Thus, as $\|1-a\| \leqslant \frac{1}{2}$, we get $(1-\|1-a\|)^{-1} \leqslant 2$, and so $\left\|1-a^{-1}\right\| \leqslant 2\|1-a\|$. Let $a$ be an invertible element of $\mathscr{A}$, and let $b \in \mathscr{A}$ with $\|a-b\| \leqslant \frac{1}{2}\left\|a^{-1}\right\|$. We claim that $b$ is invertible, and $\left\|a^{-1}-b^{-1}\right\| \leqslant 2\|a-b\|\left\|a^{-1}\right\|^{2}$. Since $\|a-b\| \leqslant$ $\frac{1}{2}\left\|a^{-1}\right\|$ we have $\left\|1-a^{-1} b\right\| \leqslant\left\|a^{-1}\right\|\|a-b\| \leqslant \frac{1}{2}$. By XI $a^{-1} b$ is invertible, and $\left\|1-\left(a^{-1} b\right)^{-1}\right\| \leqslant 2\left\|1-a^{-1} b\right\| \leqslant 2\|a-b\|\left\|a^{-1}\right\|$. Hence $\left\|a^{-1}-b^{-1}\right\|=$ $\left\|\left(1-\left(a^{-1} b\right)^{-1}\right) a^{-1}\right\| \leqslant\left\|1-\left(a^{-1} b\right)^{-1}\right\|\left\|a^{-1}\right\| \leqslant 2\|a-b\|\left\|a^{-1}\right\|^{2}$.
Lemma For a self-adjoint element $a$ from a $C^{*}$-algebra, $a-i$ is invertible.
Proof (Based on Proposition 4.1.1(ii) of 43].)
The trick is to write $a-i \equiv(a+n i)-(n+1) i$ for sufficiently large $n$, because then by $\widehat{\mathrm{VI}} a-i$ is invertible provided that $n+1>\|a+n i\|$. Indeed, for $n$ such that $\|a\|<2 n+1$, we have $\|a+n i\|^{2}=\left\|(a+n i)^{*}(a+n i)\right\|=\left\|a^{2}+n^{2}\right\| \leqslant$ $\|a\|^{2}+n^{2}<2 n+1+n^{2}=(n+1)^{2}$, and so $\|a+n i\|<n+1$.
Exercise Let $a$ be a self-adjoint element of a $C^{*}$-algebra.

1. Show that $a-\lambda$ is invertible for all $\lambda \in \mathbb{C} \backslash \mathbb{R}$.
2. Show that $a^{2}-\lambda$ is invertible for all $\lambda \in \mathbb{C} \backslash[0, \infty)$.
(Hint: first prove that $a^{2}+1 \equiv(a+i)(a-i)$ is invertible.)
Conclude that $a^{n}-\lambda$ is invertible for all $\lambda \in \mathbb{C} \backslash[0, \infty)$ and even $n \in \mathbb{N}$.
3. Let $n \in \mathbb{N}$ be odd. Show that $a^{n}-\lambda$ is invertible for all $\lambda \in \mathbb{C} \backslash[0, \infty)$ if and only if $a-\lambda$ is invertible for all $\lambda \in \mathbb{C} \backslash[0, \infty)$.
(Hint: show that $a^{n}+1=\prod_{k=1}^{n} a+\zeta^{2 k+1}$ where $\zeta=e^{\frac{\pi i}{n}}$.)

Proposition Let $\mathscr{A}$ be a $C^{*}$-subalgebra of a $C^{*}$-algebra $\mathscr{B}$. Let $a$ be a self- XVI adjoint element of $\mathscr{A}$, which has an inverse, $a^{-1}$, in $\mathscr{B}$. Then $a^{-1} \in \mathscr{A}$.
Proof While we do not know yet that $a$ is invertible in $\mathscr{A}$, we do know that $a+i / n$ XVII has an inverse $(a+i / n)^{-1}$ in $\mathscr{A}$ by XV for each $n$ (using that $a$ is self-adjoint.) Since $a+i / n$ converges to $a$ in $\mathscr{B}$ as $n$ increases, we see that $(a+i / n)^{-1}$ converges
to $a^{-1}$ in $\mathscr{B}$ by . Thus, as all $(a+i / n)^{-1}$ are in $\mathscr{A}$, and $\mathscr{A}$ is closed in $\mathscr{B}$, we see that $a^{-1}$ is in $\mathscr{A}$.

XVIII Exercise Show that the assumption in XVI that $a$ is self-adjoint may be dropped. (Hint: consider $a^{*} a$, see Proposition VIII.1.14 of 13].)
XIX Definition The spectrum, $\operatorname{sp}(a)$, of an element $a$ of a $C^{*}$-algebra is the set of complex numbers $\lambda$ for which $a-\lambda$ is not invertible.
XX Exercise Verify the following examples.

1. The spectrum of a continuous function $f: X \rightarrow \mathbb{R}$ on a compact Hausdorff space $X$ being an element of the $C^{*}$-algebra $C(X)$ is the image of $f$, that is, $\operatorname{sp}(f)=\{f(x): x \in X\}$.
2. The spectrum of a square matrix $A$ from the $C^{*}$-algebra $M_{n}$ is the set of eigenvalues of $A$.

XXI Exercise Let $a$ be an element of a $C^{*}$-algebra $\mathscr{A}$.

1. Prove that $\operatorname{sp}(a) \subseteq \mathbb{R}$ when $a$ is self-adjoint (see XV).

The reverse implication does not hold: show that $\operatorname{sp}\left(\left(\begin{array}{ll}0 & 2 \\ 0 & 0\end{array}\right)\right)=\{0\}$.
2. Show that $\operatorname{sp}\left(a^{2}\right) \subseteq[0, \infty)$ when $a$ is self-adjoint (see XV).
3. Show that $|\lambda| \leqslant\|a\|$ for all $\lambda \in \operatorname{sp}(a)$ using VI,

In fact, we will see in 16 II, that $\|a\|=\sup \{|\lambda|: \lambda \in \operatorname{sp}(a)\}$.
4. Show that $\operatorname{sp}(a)$ is closed (using VII).

Conclude that $\operatorname{sp}(a)$ is compact.
5. Show that $\operatorname{sp}(a+z)=\{\lambda+z: \lambda \in \operatorname{sp}(a)\}$ for all $z \in \mathbb{C}$.
6. Prove that $\operatorname{sp}\left(a^{-1}\right)=\left\{\lambda^{-1}: \lambda \in \operatorname{sp}(a)\right\}$ if $a$ is invertible (and $0 \notin \operatorname{sp}(a)$ ).

XXII On first sight, the spectrum $\operatorname{sp}(a)$ of an element $a$ of a $C^{*}$-algebra $\mathscr{A}$ depends not only on $a$, but also on the surrounding $C^{*}$-algebra $\mathscr{A}$ for it determines for which $\lambda \in \mathbb{C}$ the operator $a-\lambda$ is invertible. Thus we should perhaps write $\operatorname{sp}_{\mathscr{A}}(a)$ instead of $\operatorname{sp}(a)$. However, such careful bookkeeping turns out be unnecessary by the following result.

Theorem (Spectral Permanence) Let $\mathscr{B}$ be a $C^{*}$-subalgebra of a $C^{*}$-algebra XXIII $\mathscr{A}$. Then $\operatorname{sp}_{\mathscr{A}}(a)=\operatorname{sp}_{\mathscr{B}}(a)$ for every element $a$ of $\mathscr{B}$.
Proof Let $a$ be an element of $\mathscr{B}$, and let $\lambda \in \mathbb{C}$. We must show that $a-\lambda$ XXIV is invertible in $\mathscr{A}$ iff $a-\lambda$ is invertible in $\mathscr{B}$. Surely, if $a-\lambda$ has an inverse $(a-\lambda)^{-1}$ in $\mathscr{B}$, then $(a-\lambda)^{-1}$ is also an inverse of $a-\lambda$ in $\mathscr{A}$, since $\mathscr{B} \subseteq \mathscr{A}$. The other, non-trivial, direction follows directly from XVI.

### 2.3 Positive Elements

### 2.3.1 Holomorphic Functions

The next order of business is to show that the spectrum $\operatorname{sp}(a)$ of an element $a$ of a $C^{*}$-algebra contains enough points, so to speak. One incarnation of this idea is that $\operatorname{sp}(a)$ is non-empty (see 16 V ), but we will need more, and prove that $\|a\|=|\lambda|$ for some $\lambda \in \operatorname{sp}(a)$ (provided that $a$ is self-adjoint). Somewhat bafflingly, the canonical and apparently easiest way to derive this fact is by considering the power series expansion of a cleverly chosen $\mathscr{A}$-valued function (see 16 II$)$. To this end, we'll first quickly redevelop some complex analysis for $\mathscr{A}$ valued functions (instead of $\mathbb{C}$-valued functions), which will only be needed to prove this fact.
Setting Fix a $C^{*}$-algebra $\mathscr{A}$ for the remainder of this paragraph. For brevity, we'll say that a function is a partially defined map $f: \mathbb{C} \rightarrow \mathscr{A}$ whose domain of definition $\operatorname{dom}(f)$ is an open subset of $\mathbb{C}$. Such a function is called holomorphic at a point $z \in \mathbb{C}$ if $f$ is defined on $z$, and

$$
\frac{f(x)-f(y)}{x-y}
$$

converges (with respect to the norm on $\mathscr{A}$ ) to some element $f^{\prime}(x)$ of $\mathscr{A}$ as $y \in \operatorname{dom}(f) \backslash\{x\}$ converges to $x$.

We say that $f$ is holomorphic if $f$ is holomorphic at $x$ for all $x \in \operatorname{dom}(f)$, and the function $z \mapsto f^{\prime}(z)$ with $\operatorname{dom}\left(f^{\prime}\right)=\operatorname{dom}(f)$ is called its derivative.
Exercise Verify the following examples of holomorphic functions.

12

II

III

1. If $f$ and $g$ are holomorphic functions with $\operatorname{dom}(f)=\operatorname{dom}(g)$, then $f+g$ and $f \cdot g$ are holomorphic, and $(f+g)^{\prime}=f^{\prime}+g^{\prime}$ and $(f \cdot g)^{\prime}=f^{\prime} g+g^{\prime} f$.
2. The function $f$ given by $f(z)=z$ and $\operatorname{dom}(f)=\mathbb{C}$ is holomorphic, and $f^{\prime}(z)=1$ for all $z \in \mathbb{C}$.
3. Let $a \in \mathscr{A}$. The constant function $f$ given by $f(z)=a$ for all $z \in \mathbb{C}$ is holomorphic, and $f^{\prime}(z)=0$ for all $z \in \mathscr{A}$.
4. Any polynomial, that is, function $f$ of the form $f(z) \equiv a_{n} z^{n}+\cdots+a_{1} z+a_{0}$ with $a_{i} \in \mathscr{A}$ is holomorphic with $f^{\prime}(z)=n a_{n} z^{n-1}+\cdots+2 a_{2} z+a_{1}$.

13 We now turn to perhaps the most important example of a holomorphic $\mathscr{A}$ valued function - or at the very least the very source from which (as we'll see) all holomorphic functions draw their interesting and pleasant properties: the holomorphic $\mathscr{A}$-valued function given by a power series $\sum_{n} a_{n} z^{n}$.
II Theorem Let $a_{0}, a_{1}, a_{2}, \ldots \in \mathscr{A}$ be given, and write $R:=\left(\limsup _{n}\left\|a_{n}\right\|^{1 / n}\right)^{-1}$. Then for every $z \in \mathbb{C}$,

1. $\sum_{n} a_{n} z^{n}$ converges absolutely when $|z|<R$, and
2. if $\sum_{n} a_{n} z^{n}$ converges, then $|z| \leqslant R$.
(The number $R \in[0, \infty]$ is called the radius of convergence of the series $\sum_{n} a_{n} z^{n}$.)
III Proof Suppose that $|z|<R$. To show that the series $\sum_{n} a_{n} z^{n}$ converges absolutely, we must show that $\sum_{n}\left\|a_{n}\right\||z|^{n} \equiv \sum_{n}\left(\left\|a_{n}\right\|^{1 / n}|z|\right)^{n}<\infty$. If $z=$ 0 , this is obvious, so we'll assume that $|z|>0$. Then, since $|z|<R$, we have $R^{-1}|z|<1$ (and $R^{-1}<\infty$ ). Note that there is $\varepsilon>0$ with $\left(R^{-1}+\varepsilon\right)|z|<1$. The point of this $\varepsilon$ is that $\limsup _{n}\left\|a_{n}\right\|^{1 / n}<R^{-1}+\varepsilon$, so that we can find $N$ with $\left\|a_{n}\right\|^{1 / n} \leqslant R^{-1}+\varepsilon$ for all $n \geqslant N$. Then $\left\|a_{n}\right\|^{1 / n}|z| \leqslant\left(R^{-1}+\varepsilon\right)|z|<1$ for all $n \geqslant N$, and so $\sum_{n}\left\|a_{n}\right\||z|^{n} \leqslant \sum_{n=0}^{N-1}\left\|a_{n}\right\||z|^{n}+\sum_{n=N}^{\infty}\left(\left(R^{-1}+\varepsilon\right)|z|\right)^{n}<\infty$ by convergence of the geometric series (c.f. 11 II ).

Suppose now instead that $\sum_{n} a_{n} z^{n}$ converges. Then $\left\|a_{n}\right\||z|^{n}$ converges to 0 . In particular, there is $N$ with $\left\|a_{n}\right\||z|^{n} \leqslant 1$ for all $n \geqslant N$. Then $\left\|a_{n}\right\|^{1 / n}|z| \leqslant 1$, and $\left\|a_{n}\right\|^{1 / n} \leqslant|z|^{-1}$ for all $n \geqslant N$, so that $R^{-1} \equiv \lim \sup _{n}\left\|a_{n}\right\|^{1 / n} \leqslant|z|^{-1}$, giving $|z| \leqslant R$.
IV Proposition The $\mathscr{A}$-valued function $f$ given by a series $\sum_{n} a_{n} z^{n}$ with radius of convergence $R:=\left(\lim \sup _{n}\left\|a_{n}\right\|^{1 / n}\right)^{-1}$ is holomorphic when defined on the disk $\operatorname{dom}(f)=\{z \in \mathbb{C}:|z|<R\}$, and $f^{\prime}(z)=\sum_{n=1}^{\infty} n a_{n} z^{n-1}$ for all $z \in \operatorname{dom}(f)$.
$\checkmark$ Proof If $R=0$, the statement is rather dull, but clearly true, so we assume that $R \neq 0$, that is, $\lim _{\sup _{n}}\left\|a_{n}\right\|^{1 / n}<\infty$.

Note that the radius of convergence of $\sum_{n=1}^{\infty} n a_{n} z^{n-1} \equiv \sum_{n=0}^{\infty}(n+1) a_{n+1} z^{n}$ is also $R$, because

$$
\left\|(n+1) a_{n+1}\right\|^{1 / n}=(n+1)^{1 / n}\left\|a_{n+1}\right\|^{\frac{1}{n+1}}\left(\left\|a_{n+1}\right\|^{\frac{1}{n+1}}\right)^{1 / n}
$$

and $R^{-1}=\lim \sup _{n}\left\|a_{n+1}\right\|^{\frac{1}{n+1}}$, and both $(n+1)^{1 / n}$ and $\left(\left\|a_{n+1}\right\|^{\frac{1}{n+1}}\right)^{1 / n}$ converge to 1 as $n \rightarrow \infty$ (using here that $\left.\lim \sup _{n}\left\|a_{n}\right\|^{1 / n}<\infty\right)$.

Hence $\sum_{n=1}^{\infty} n a_{n} z^{n-1}$ converges absolutely for every $z \in \mathbb{C}$ with $|z|<R$. Let $z \in \mathbb{C}$ with $|z|<R$ be given. We must show that $f$ is holomorphic at $z$ with $f^{\prime}(z)=\sum_{n} n a_{n} z^{n-1}$. For this it suffices to show that

$$
\begin{equation*}
\sum_{n=0}^{\infty}\left\|a_{n}\right\|\left|\frac{(z+h)^{n}-z^{n}}{h}-n z^{n-1}\right| \tag{2.1}
\end{equation*}
$$

converges to 0 as $h \in \mathbb{C}$ (with $h \neq 0$ and $|z+h|<R$ ) tends to 0 .
Pick $r>0$ with $|z|<r<R$. With the appropriate algebraic gymnastics (involving the identity $a^{n}-b^{n}=(a-b) \sum_{k=1}^{n} a^{n-k} b^{k-1}$ and the inequalities $|z+h| \leqslant r$ and $|z| \leqslant r$ ) we get, for every $n$ and $h \in \mathbb{C}$ with $h \neq 0$ and $|z+h|<r$,

$$
\begin{align*}
\left|\frac{(z+h)^{n}-z^{n}}{h}-n z^{n-1}\right| & =\left|\sum_{k=1}^{n}\left((z+h)^{n-k}-z^{n-k}\right) z^{k-1}\right|  \tag{2.2}\\
& \leqslant 2 n r^{n-1} \tag{2.3}
\end{align*}
$$

On the one hand, we see from (2.2) that any term - and thus any partial sum - of the series from (2.1) converges to 0 as $h$ tends to 0 . On the other hand, we see from (2.3) that the series from (2.1) is dominated by $2 \sum_{n}\left\|a_{n}\right\| n r^{n-1}$ (which converges because the radius of convergence of $\sum_{n} a_{n} n z^{n-1}$ is $R>r$ ), so that the tails of the series in (2.1) vanish uniformly in $h$. All in all, the sum of the infinite series from (2.1) converges to 0 as $h$ tends to 0 .
Exercise Let $\sum_{n} a_{n} z^{n}$ be a power series over $\mathscr{A}$ with radius of convergence $R>$ 0 such that $\sum_{n} a_{n} z^{n}=0$ for all $z$ from some disk around 0 with radius $r<R$. Show that $0=a_{0}=a_{1}=a_{2}=\cdots$.
(Hint: clearly $a_{0}=0$. Show that the derivative of the power series also vanishes on the disk around 0 with radius $r$.)

All holomorphic functions are power series in the sense that any $\mathscr{A}$-valued holo-
morphic function $f$ defined on 0 is given by some power series $\sum_{n} a_{n} z^{n}$ on the largest disk around 0 that fits in $\operatorname{dom}(f)$. This fact, which follows from 15 V
and 15 VII below, is all the more remarkable, because here the pointwise ("local") property of being holomorphic entails the uniform ("global") property of being equal to a power series (on some disk). The device that bridges this gap is integration of $\mathscr{A}$-valued holomorphic functions along line segments.

II Exercise We're going to define as quickly as possible an integral $\int f$ for every continuous map $f:[0,1] \rightarrow \mathscr{A}$. Any interval $I$ in $[0,1]$ is of one of the following forms

$$
[s, t] \quad[s, t) \quad(s, t] \quad(s, t)
$$

where $0 \leqslant s \leqslant t \leqslant 1$; we'll denote the length of an interval $I-$ being $t-s$ in the four cases above - by $|I|$. An $\mathscr{A}$-valued step function is a function $f:[0,1] \rightarrow \mathscr{A}$ of the form $f \equiv \sum_{n} a_{n} \mathbf{1}_{I_{n}}$ for some $a_{1}, \ldots, a_{N} \in \mathscr{A}$ and intervals $I_{1}, \ldots, I_{N}$ (where $\mathbf{1}_{I_{n}}$ is 1 is the indicator function of $I_{n}$ which is 1 on $I_{n}$ and 0 elsewhere); and the set of $\mathscr{A}$-valued step functions is denoted by $S_{\mathscr{A}}$, which is a subset of the space of all bounded functions $f:[0,1] \rightarrow \mathscr{A}$ which we'll denote by $B_{\mathscr{A}}$.

1. Show that there is a unique linear map $\int: S_{\mathscr{A}} \rightarrow \mathscr{A}$ with $\int a \mathbf{1}_{I}=|I| a$ for every interval $I$ in $[0,1]$ and $a \in \mathscr{A}$.
(Hint: the difficulty here is to show that no contradiction arises in the sense that $\sum_{n} a_{n}\left|I_{n}\right|=\sum_{m} a_{m}^{\prime}\left|I_{m}^{\prime}\right|$ when $\sum_{n} a_{n} \mathbf{1}_{I_{n}}=\sum_{m} a_{m}^{\prime} \mathbf{1}_{I_{n}^{\prime}}$ for intervals $I_{1}, \ldots, I_{N}, I_{1}^{\prime}, \ldots, I_{M}^{\prime}$ in $[0,1]$ and $a_{1}, \ldots, a_{N}, a_{1}^{\prime}, \ldots, a_{M}^{\prime} \in \mathscr{A}$.)
2. We endow $B_{\mathscr{A}}$ with the supremum norm, viz. $\|f\|=\sup _{t \in[0,1]}\|f(t)\|$ for all $f \in B_{\mathscr{A}}$.

Show that every $\mathscr{A}$-valued step function $f$ may be written as $f \equiv \sum_{n} a_{n} \mathbf{1}_{I_{n}}$ where $I_{1}, \ldots, I_{N}$ are disjoint and non-empty intervals in $[0,1]$.

Show that for such a representation $\|f\|=\sup _{n}\left\|a_{n}\right\|$, and $\sum_{n}\left|I_{n}\right| \leqslant 1$. Deduce that $\left\|\int f\right\| \leqslant \sum_{n}\left\|a_{n}\right\|\left|I_{n}\right| \leqslant\|f\|$.
Conclude that $\int: S_{\mathscr{A}} \rightarrow \mathscr{A}$ is a bounded linear map and can therefore be uniquely extended to a bounded linear map $\int: \bar{S}_{\mathscr{A}} \rightarrow \mathscr{A}$ on the closure $\bar{S}_{\mathscr{A}}$ of $S_{\mathscr{A}}$.
3. Show that every continuous function $f:[0,1] \rightarrow \mathscr{A}$ is the supremum norm limit of a sequence $g_{1}, g_{2}, \ldots$ of $\mathscr{A}$-valued step functions (i.e. $f \in \bar{S}_{\mathscr{A}}$ ).
4. Show that $\int a f=a \int f$ when $f:[0,1] \rightarrow \mathbb{C}$ is continuous and $a \in \mathscr{A}$.

Definition The integral of a holomorphic $\mathscr{A}$-valued function $f$ along a line segment $\left[w, w^{\prime}\right] \subseteq \operatorname{dom}(f)$ (where $w$ and $w^{\prime}$ are thus complex numbers) is now defined as

$$
\int_{w}^{w^{\prime}} f=\left(w^{\prime}-w\right) \int_{0}^{1} f\left(w+t\left(w^{\prime}-w\right)\right) d t
$$

We'll also need integration along a triangle $T$, which is for this purpose a triple of complex numbers $w_{0}, w_{1}, w_{2}$ (of which the order does matter) called the vertices of $T$. The boundary of such a triangle $T$ is $\partial T:=\left[w_{0}, w_{1}\right] \cup\left[w_{1}, w_{2}\right] \cup\left[w_{2}, w_{0}\right]$, and given any $\mathscr{A}$-valued holomorphic function $f$ with $\partial T \subseteq \operatorname{dom}(f)$ we define

$$
\int_{T} f=\int_{w_{0}}^{w_{1}} f+\int_{w_{1}}^{w_{2}} f+\int_{w_{2}}^{w_{0}} f
$$

We'll need some more terminology relating to our triangle $T$. Its closure, written $\operatorname{cl}(T)$, is the convex hull of $w_{0}, w_{1}, w_{2}$, and its interior is simply $\operatorname{in}(T)=$ $\operatorname{cl}(T) \backslash \partial T$. The length of $T$ is given by length $(T):=\left|w_{1}-w_{0}\right|+\left|w_{2}-w_{1}\right|+$ $\left|w_{0}-w_{2}\right|$.

The number of times the triangle $T$ winds around a point $z \in \mathbb{C} \backslash \partial T$ in the counterclockwise direction is called the winding numberis written $\mathrm{wn}_{\mathrm{T}}(z)$, is either 1 or -1 when $z \in \operatorname{in}(T)$ (depending on the order of the vertices), is 0 when $z \notin \operatorname{cl}(T)$, and undefined on $\partial T$. It is defined formally for $z \in \mathbb{C} \backslash \partial T$ by

$$
2 \pi \mathrm{wn}_{T}(z)=\measuredangle\left(w_{0}, z, w_{1}\right)+\measuredangle\left(w_{1}, z, w_{2}\right)+\measuredangle\left(w_{2}, z, w_{0}\right),
$$

and pops up in the value of the integral $\int_{T}\left(z-z_{0}\right)^{-1} d z$ later on, see VIII.
(Here $\measuredangle\left(w_{0}, z, w_{1}\right)$ denotes the number of radians in $(-\pi, \pi)$ needed to rotate the line through $z$ and $w_{0}$ counterclockwise around $z$ to hit $w_{1}$, that is, the angle of the corner on the right when travelling from $w_{0}$ to $w_{1}$ via $z$.)
Goursat's Theorem Let $f$ be a holomorphic function, and let $T$ be a triangle whose closure is entirely contained in $\operatorname{dom}(f)$. Then $\int_{T} f=0$.
Proof (Based on 49.) If two vertices of $T$ coincide the result is obviously true, so we may assume that they're all distinct, that is, $\operatorname{in}(T) \neq \varnothing$.

Note that if $f$ has an antiderivative, that is, $f \equiv g^{\prime}$ for some holomorphic function $g$, then one can show that $\int_{T} f=0$ (after deriving the fundamental theorem of calculus). Although it is true that every holomorphic function with simply connected domain has a antiderivative, this result is not yet available (and in fact usually depends on this very theorem). Instead we will approximate $f$ by an affine function (which does have an antiderivative) using the
derivative of $f$. But since such an approximation only concerns a single point, we first need to zoom in.
VI If we split $T$ into four similar triangles $T^{\mathrm{i}}, T^{\mathrm{ii}}, T^{\mathrm{iii}}, T^{\mathrm{iv}}$

we have $\int_{T} f=\sum_{n=\mathrm{i}}^{\mathrm{iv}} \int_{T^{n}} f$. There is $T^{\prime}$ among $T^{\mathrm{i}}, T^{\mathrm{ii}}, T^{\mathrm{iii}}, T^{\mathrm{iv}}$ with $\left\|\int_{T} f\right\| \leqslant$ $4\left\|\int_{T^{\prime}} f\right\|$. Clearly, length $(T)=2$ length $\left(T^{\prime}\right)$. Write $T_{0}:=T$ and $T_{1}:=T^{\prime}$.

From this it is clear how to get a sequence of similar triangles $T_{0}, T_{1}, T_{2}, \ldots$ with $\left\|\int_{T} f\right\| \leqslant 4^{n}\left\|\int_{T_{n}} f\right\|$, and length $(T)=2^{n}$ length $\left(T_{n}\right)$.
VII If we pick a point on the closure $\operatorname{cl}\left(T_{n}\right)$ of each triangle $T_{n}$ we get a Cauchy sequence that converges to some point $z_{0} \in \mathbb{C}$ which lies in $\bigcap_{n} \operatorname{cl}\left(T_{n}\right)$. We can approximate $f$ by an affine function at $z_{0}$ as follows. For $z \in \operatorname{dom}(f)$,

$$
f(z)=f\left(z_{0}\right)+f^{\prime}\left(z_{0}\right)\left(z-z_{0}\right)-r(z)\left(z-z_{0}\right)
$$

where $r: \operatorname{dom}(f) \rightarrow \mathbb{C}$ is given by $r(z)=f^{\prime}\left(z_{0}\right)-\left(f(z)-f\left(z_{0}\right)\right)\left(z-z_{0}\right)^{-1}$ for $z \neq z_{0}$ and $r\left(z_{0}\right)=0$. We see that $r(z)$ converges to 0 as $z \rightarrow z_{0}$.

Let $\varepsilon>0$ be given. There is $\delta>0$ such that $z \in \operatorname{dom}(f)$ and $\|r(z)\| \leqslant \varepsilon$ for all $z \in \mathbb{C}$ with $\left\|z-z_{0}\right\|<\delta$. There is $n$ such that the triangle $T_{n}$ is contained in the ball around $z_{0}$ of radius $\delta$. Note that $\int_{T_{n}} f\left(z_{0}\right)+f^{\prime}\left(z_{0}\right)\left(z-z_{0}\right) d z=0$ by the discussion in $\nabla$, because the integrated function is affine. Thus

$$
\int_{T_{n}} f=-\int_{T_{n}} r(z)\left(z-z_{0}\right) d z .
$$

Note that for $z \in T_{n}$, we have $\left\|z-z_{0}\right\| \leqslant$ length $\left(T_{n}\right)$, and $\|r(z)\| \leqslant \varepsilon$ (because $\left.\left\|z-z_{0}\right\|<\delta\right)$, and so $\left\|r(z)\left(z-z_{0}\right)\right\| \leqslant \varepsilon$ length $\left(T_{n}\right)$. Thus:

$$
\left\|\int_{T_{n}} f\right\|=\left\|\int_{T_{n}} r(z)\left(z-z_{0}\right) d z\right\| \leqslant \varepsilon \text { length }\left(T_{n}\right)^{2} .
$$

Using the inequalities from VI, we get

$$
\left\|\int_{T} f\right\| \leqslant 4^{n}\left\|\int_{T_{n}} f\right\| \leqslant \varepsilon 4^{n} \text { length }\left(T_{n}\right)^{2} \equiv \varepsilon \operatorname{length}(T)^{2}
$$

Since $\varepsilon>0$ was arbitrary, we see that $\int_{T} f=0$.

Exercise The assumption in Goursat's Theorem IVP that the holomorphic function $f$ is defined not only on the boundary $\partial T$ of the triangle $T$ but also on the interior $\operatorname{in}(T)$ is essential, for if only a single hole in $\operatorname{dom}(f)$ is allowed within $\operatorname{in}(T)$ the integral $\int_{T} f$ can become non-zero-which we will demonstrate here by computing $\int_{T}\left(z-z_{0}\right)^{-1} d z$.

1. Show that for a non-zero complex number $z$ we have

$$
z^{-1}=\frac{z_{\mathbb{R}}-i z_{\mathbb{I}}}{z_{\mathbb{R}}^{2}+z_{\mathbb{I}}^{2}}
$$

2. Given real numbers $a \neq 0$ and $b$, show that

$$
\begin{aligned}
\int_{a}^{a+i b} z^{-1} d z & =i \int_{0}^{t} \frac{a-i t}{a^{2}+t^{2}} d t \\
& =i \int_{0}^{b} \frac{a}{a^{2}+t^{2}} d t+\int_{0}^{b} \frac{t}{a^{2}+t^{2}} d t \\
& =i \arctan (b / a)+\log |a+i b|-\log |i a|
\end{aligned}
$$

and similarly, show that for real numbers $a$ and $b \neq 0$

$$
\int_{a+i b}^{i b} z^{-1} d z=i \arctan (a / b)+\log |i b|-\log |a+i b|
$$

3. Show that for complex numbers $w, w^{\prime}$ and $z_{0}$ with $z_{0} \notin\left[w, w^{\prime}\right]$

$$
\int_{w}^{w^{\prime}}\left(z-z_{0}\right)^{-1} d z=i \measuredangle\left(w, z_{0}, w^{\prime}\right)+\log \frac{\left|w^{\prime}-z_{0}\right|}{\left|w-z_{0}\right|}
$$

where $\measuredangle\left(w, z_{0}, w^{\prime}\right)$ denotes the number of radians in $(-\pi, \pi)$ needed to rotate the line through $z_{0}$ and $w$ counterclockwise around $z_{0}$ to hit $w^{\prime}$.
(Hint: using Goursat's Theorem, $\mathbb{I V}$, one may reduce the problem to integration along horizontal and vertical line segments.)
4. Given a triangle $T$ and $z_{0} \in \mathbb{C} \backslash \partial T$, show that

$$
\frac{1}{2 \pi i} \int_{T}\left(z-z_{0}\right)^{-1} d z=\operatorname{wn}_{T}\left(z_{0}\right)
$$

IX Thus integration of $z \mapsto\left(z-z_{0}\right)^{-1}$ along a triangle $T$ detects the number of times $T$ winds around $z_{0}$. There is nothing special about a triangle: a similar result-not needed here - holds for a broad class of curves (c.f. Thm 2.9 of 13]).

Integration along a curve can also be used to probe the value of a holomorphic function at a point $z_{0}$. On this occasion we restrict ourselves to regular $N$-gons.

15 Theorem (Cauchy's Integral Formula) Let $f$ be a holomorphic $\mathscr{A}$-valued function which is defined on the interior and boundary of some regular $N$-gon with center $c \in \mathbb{C}$, circumradius $r$ and vertices $w_{n}:=c+r \cos (2 \pi / n)+i r \sin (2 \pi / n)$. Then for any complex number $z_{0}$ in the interior of the $N$-gon we have

$$
f\left(z_{0}\right)=\frac{1}{2 \pi i} \sum_{n=0}^{N-1} \int_{w_{n}}^{w_{n+1}} \frac{f(z)}{z-z_{0}} d z
$$

II Proof Since $\sum_{n=0}^{N-1} \int_{w_{n}}^{w_{n+1}} \frac{f\left(z_{0}\right)}{z-z_{0}} d z=2 \pi i f\left(z_{0}\right)$ by 14 VIII it suffices to show that

$$
\begin{equation*}
\sum_{n=0}^{N-1} \int_{w_{n}}^{w_{n+1}} \frac{f(z)-f\left(z_{0}\right)}{z-z_{0}} d z=0 \tag{2.4}
\end{equation*}
$$

III Let $\varepsilon>0$ be given. Since $f$ is holomorphic at $z_{0}$ we can find $\delta>0$ with

$$
\left\|\frac{f(z)-f\left(z_{0}\right)}{z-z_{0}}\right\| \leqslant\left\|f^{\prime}\left(z_{0}\right)\right\|+37
$$

for all $z \in \operatorname{dom}(f)$ with $\left|z-z_{0}\right| \leqslant \delta$.
IV To use III we must restrict our attention to a smaller polygon. Let $T$ be a triangle that is entirely inside the $N$-gon such that $\mathrm{wn}_{T}\left(z_{0}\right)=-1$, length $(T) \leqslant$ $\varepsilon$, and $\left\|z_{0}-z\right\| \leqslant \delta$ for all $z \in \partial T$. By partitioning the area between $T$ and the $N$-gon in the obvious manner into triangles $T_{1}, \ldots, T_{M}$ (for which $\int_{T_{m}} f=0$ for all $m$ by 14 IV we see that

$$
\begin{equation*}
\sum_{n=0}^{N-1} \int_{w_{n}}^{w_{n+1}} \frac{f(z)-f\left(z_{0}\right)}{z-z_{0}} d z=\int_{T} \frac{f(z)-f\left(z_{0}\right)}{z-z_{0}} d z \tag{2.5}
\end{equation*}
$$

Hence by [III we have

$$
\begin{aligned}
\left\|\sum_{n=0}^{N-1} \int_{w_{n}}^{w_{n+1}} \frac{f(z)-f\left(z_{0}\right)}{z-z_{0}} d z\right\| & \leqslant \operatorname{length}(T) \cdot \sup _{z \in \partial T}\left\|\frac{f(z)-f\left(z_{0}\right)}{z-z_{0}}\right\| \\
& \leqslant\left\|f^{\prime}\left(z_{0}\right)\right\| \varepsilon+37 \varepsilon
\end{aligned}
$$

Since $\varepsilon>0$ was arbitrary, (2.4) follows from 2.5).
Proposition Let $f$ be a holomorphic $\mathscr{A}$-valued function defined on the boundary and interior of a regular $K$-gon with vertices $w_{0}, \ldots, w_{K-1}, w_{K}=w_{0}$ as in Then for every element $z$ of an open disk in the interior of the $K$-gon with center $w$,

$$
f(z)=\sum_{n=0}^{\infty}\left(\frac{1}{2 \pi i} \sum_{k=0}^{K-1} \int_{w_{k}}^{w_{k+1}} \frac{f(u)}{(u-w)^{n+1}} d u\right)(z-w)^{n} .
$$

Proof By 几and some easy algebra we have

$$
2 \pi i f(z)=\sum_{k=0}^{K-1} \int_{w_{k}}^{w_{k+1}} \frac{f(u)}{u-z} d u=\sum_{k=0}^{K-1} \int_{w_{k}}^{w_{k+1}} \frac{f(u)}{u-w} \frac{1}{1-\frac{z-w}{u-w}} d u
$$

Note that $|z-w|<|u-w|$ for all $u \in\left[w_{k}, w_{k+1}\right]$ and $k$, because the open disk with center $w$ from which $z$ came lies entirely in the $K$-gon. Hence, by 11 II,

$$
\begin{aligned}
2 \pi i f(z) & =\sum_{k=0}^{K-1} \int_{w_{k}}^{w_{k+1}} \frac{f(u)}{u-w} \sum_{n=0}^{\infty} \frac{(z-w)^{n}}{(u-w)^{n}} d u \\
& =\sum_{n=0}^{\infty} \sum_{k=0}^{K-1} \int_{w_{k}}^{w_{k+1}} \frac{f(u)}{(u-w)^{n+1}} d u(z-w)^{n},
\end{aligned}
$$

where the interchange of " $\sum$ " and " $\int$ " is allowed because the partial sum $\sum_{n=0}^{N} f(u) \frac{(z-w)^{n}}{(u-w)^{n+1}}$ converges uniformly in $u$ as $N \rightarrow \infty$.
Proposition Let $f$ be an $\mathscr{A}$-valued holomorphic function that can be written as a power series $f(z)=\sum_{n} a_{n}(z-w)^{n}$ where $a_{0}, a_{1}, \ldots \in \mathscr{A}$ for all $z$ from some disk in $\operatorname{dom}(f)$ around $w$ with radius $r>0$.

Then the formula $f(z)=\sum_{n} a_{n}(z-w)^{n}$ holds also for any $z$ from a larger disk with radius $R>r$ around $w$ that still fits in $\operatorname{dom}(f)$.
Proof Let $z$ with $|z-w|<R$ be given. By choosing $K$ large enough we can fit VIII the boundary of a regular $K$-gon centered around $w$ with vertices $w_{0}, \ldots, w_{K-1}, w_{K} \equiv$ $w_{0}$ inside the difference between the two disks, and we can moreover, by $\nabla$, choose the polygon in such a way that $f\left(z^{\prime}\right)=\sum_{n} b_{n}\left(z^{\prime}-w\right)^{n}$ for all $z^{\prime} \in \mathbb{C}$ with $\left|z^{\prime}-w\right| \leqslant|z-w|$ where $b_{n}=\sum_{k=0}^{K-1} \int_{w_{k}}^{w_{k+1}} \frac{f(u)}{(u-w)^{n+1}} d u$.

Thus to show that $f(z)=\sum_{n} a_{n}(z-w)^{n}$ it suffices to show that $a_{n}=b_{n}$ for all $n$. This in turn follows by 13 VI from the fact that $\sum_{n} a_{n}\left(z^{\prime}-w\right)^{n}=$ $\sum_{n} b_{n}\left(z^{\prime}-w\right)^{n}$ for all $z^{\prime} \in \mathbb{C}$ with $\left|z^{\prime}-w\right|<r$.

### 2.3.2 Spectral Radius

16 Our analysis of $\mathscr{A}$-valued holomorphic functions allows us to expose the following connection between the norm and the invertible elements in a $C^{*}$-algebra.

II Proposition For a self-adjoint element $a$ of a $C^{*}$-algebra $\mathscr{A}$, we have

$$
\|a\|=\sup \{|\lambda|: \lambda \in \operatorname{sp}(a)\} .
$$

(The quantity on the right hand-side above is called the spectral radius of $a$.)
III Proof Write $r=\sup \{|\lambda|: \lambda \in \operatorname{sp}(a) \backslash\{0\}\}$ where the supremum is computed in $[0, \infty]$ so that $\sup \varnothing=0$. Since $|\lambda| \leqslant\|a\|$ for all $\lambda \in \operatorname{sp}(a)$ (11VI) we see that $r \leqslant\|a\|$, and so we only need to show that $\|a\| \leqslant r$. Note that this is clearly true if $\|a\|=0$, so we may assume that $\|a\| \neq 0$.

The trick is to consider the power series expansion around 0 of the holomorphic function $f$ defined on $G:=\{z \in \mathbb{C}: 1-a z$ is invertible $\}$ by $f(z)=$ $z(1-a z)^{-1}$. More specifically, we are interested in the distance $R$ of 0 to the complement of $G$, viz. $R=\inf \{|\lambda|: \lambda \in \mathbb{C} \backslash G\}$ (where the infimum is computed in $[0, \infty]$ so that $\inf \varnothing=\infty)$ because since $0 \in G$ and $z \notin G \Longleftrightarrow z^{-1} \in \operatorname{sp}(a)$, we have $R=r^{-1}$ (using the convention $0^{-1}=\infty$ ).

Note that $f$ has the power series expansion $f(z)=\sum_{n} a^{n} z^{n+1}$ for all $z \in \mathbb{C}$ with $\|z\|<\|a\|^{-1}$, because for such $z$ we have $\sum_{n}(a z)^{n}=(1-a z)^{-1}$ by 11\|, and thus $f(z)=z(1-a z)^{-1}=z \sum_{n}(a z)^{n}=\sum_{n} a^{n} z^{n+1}$.

By 15 VII we know that $f(z)=\sum_{n} a^{n} z^{n+1}$ is valid not only for $z \in \mathbb{C}$ with $|z|<\|a\|^{-1}$, but for all $z$ with $|z|<R$. However, $R$ cannot be strictly larger than $\|a\|^{-1}$, because for every $z \in \mathbb{C}$ with $|z|>\|a\|^{-1}$ the series $\sum_{n}(a z)^{n}$ and thus $\sum_{n} a^{n} z^{n+1}$ diverges (see 11 VII - using here that $a$ is self-adjoint. Hence $R=\|a\|^{-1}$, and so $r=\|a\|$.
IV Remark For an arbitrary (possibly non-self-adjoint) element $a$ of a $C^{*}$-algebra $\mathscr{A}$ the formula in $\Pi$ might be incorrect, eg. $\left\|\left(\begin{array}{ll}0 & 1 \\ 0 & 0\end{array}\right)\right\|=1$ while $\operatorname{sp}\left(\left(\begin{array}{ll}0 & 1 \\ 0 & 0\end{array}\right)\right)=\{0\}$ cf. 11 IX . For such $a$ the formula $\sup \{|\lambda|: \lambda \in \operatorname{sp}(a)\}=\limsup _{n}\left\|a^{n}\right\|^{1 / n}$ can be derived (see e.g. Theorem 3.3 .3 of 43 ) - which we won't need here.
$\checkmark$ Exercise Given a self-adjoint element $a$ of a $C^{*}$-algebra show that $\operatorname{sp}(a) \neq \varnothing$.
VI Exercise Given a self-adjoint element $a$ of a $C^{*}$-algebra and $\lambda \in \mathbb{R}$ show that $\operatorname{sp}(a)=\{\lambda\}$ iff $a=\lambda$.
VII Exercise (Gelfand-Mazur's Theorem for $C^{*}$-algebras) Prove that if every non-zero element of a $C^{*}$-algebra $\mathscr{A}$ is invertible, then $\mathscr{A}=\mathbb{C}$ or $\mathscr{A}=\{0\}$.

Remark A logical next step towards Gelfand's representation theorem is to show that if $\lambda \in \operatorname{sp}(a)$ for some element $a$ of a commutative $C^{*}$-algebra $\mathscr{A}$, then there is a miu-map $f: \mathscr{A} \rightarrow \mathbb{C}$ with $f(a)=\lambda$. Here we have moved ourselves into a tight spot by evading Banach algebras, because the mentioned result is usually obtained by finding a maximal ideal $I$ of $\mathscr{A}$ (by Zorn's Lemma) that contains $\lambda-a$, and then forming the Banach algebra quotient $\mathscr{A} / I$. One then applies Gelfand-Mazur's Theorem for Banach algebras, to see that $\mathscr{A} / I=\mathbb{C}$, and thereby obtain a miu-map $f: \mathscr{A} \rightarrow \mathbb{C}$ with $f(a-\lambda)=0$. The problem here is that while $\mathscr{A} / I$ will turn out to be a $C^{*}$-algebra (indeed, be $\mathbb{C}$ ) the formation of the $C^{*}$-algebra quotient is non-trivial and depends on Gelfand's representation theorem (see e.g. §VIII. 4 of $[13]$ ) which is the very theorem we are working towards. The way out of this predicament is to avoid ideals and quotients of $C^{*}$ - and Banach algebras altogether, and instead work with order ideals (and what are essentially quotients of Riesz and order unit spaces). To this end, we develop the theory of the positive elements of a $C^{*}$-algebra farther than is usually done for Gelfand's representation theorem.

We return to the positive elements in a $C^{*}$-algebra (see 9 X ). We'll see that the connection we have established between the norm and invertible elements of a $C^{*}$-algebra via the spectral radius 16 II affects the positive elements as well, see V.
Exercise Show that $|\lambda-t| \leqslant t$ iff $\lambda \in[0,2 t]$, where $\lambda, t \in \mathbb{R}$.
Proposition For a self-adjoint element $a$ from a $C^{*}$-algebra, and $t \in[0, \infty]$,

$$
\|a-t\| \leqslant t \quad \Longleftrightarrow \quad \operatorname{sp}(a) \subseteq[0,2 t]
$$

Proof To begin, note that $\operatorname{sp}(a-t)=\operatorname{sp}(a)-t \subseteq \mathbb{R}$ by 11 XXI , because $a$ is self-adjoint. Thus $\|a-t\|=\sup \{|\lambda-t|: \lambda \in \operatorname{sp}(a)\}$ by $16 \pi \|$ Hence $\|a-t\| \leqslant t$ iff $|\lambda-t| \leqslant t$ for all $\lambda \in \operatorname{sp}(a)$ iff $\operatorname{sp}(a) \subseteq[0,2 t]$ (by (II).
Exercise Show (using III and 11 XXI ) that for any self-adjoint element $a$ of a $C^{*}$-algebra $\mathscr{A}$, the following are equivalent.

1. $\|a-t\| \leqslant t$ for some $t \geqslant \frac{1}{2}\|a\|$;
2. $\|a-t\| \leqslant t$ for all $t \geqslant \frac{1}{2}\|a\|$;
3. $\operatorname{sp}(a) \subseteq[0, \infty) ;$
4. $a$ is positive.
$16,17 .$.

We will complete this list in 251 .
VI Exercise Let $\mathscr{A}$ be a $C^{*}$-algebra.

1. Show that $0 \leqslant a \leqslant 0$ entails that $a=0$ for all $a \in \mathscr{A}$.
2. Show that $\mathscr{A}_{+}$is closed.
3. Let $a$ be a self-adjoint element of $\mathscr{A}$. Show that $-\lambda \leqslant a \leqslant \lambda$ iff $\|a\| \leqslant \lambda$, for $\lambda \in[0, \infty)$. Conclude that $\|a\|=\inf \{\lambda \in \mathbb{R}:-\lambda \leqslant a \leqslant \lambda\}$.
(In other words $\mathscr{A}_{\mathbb{R}}$ is a complete Archimedean order unit space, see Definition 1.12 of 1 -a type of structure first studied in 41].)
Show that $0 \leqslant a \leqslant b$ entails $\|a\| \leqslant\|b\|$ for $a, b \in \mathscr{A}_{\mathbb{R}}$.
4. Recall that $a b$ need not be positive if $a, b \geqslant 0$. However:

Show that $a^{2}$ is positive for every self-adjoint element $a$ of $\mathscr{A}$.
Show that $a^{n}$ is positive for even $n \in \mathbb{N}$ and $a \in \mathscr{A}_{\mathbb{R}}$.
Show that $a^{n}$ is positive iff $a$ is positive for $\operatorname{odd} n \in \mathbb{N}$ and $a \in \mathscr{A}_{\mathbb{R}}$.
Show that $a^{n}$ is positive for every positive $a$ from $\mathscr{A}$ and $n \in \mathbb{N}$.
5. Let $a$ be an invertible element of $\mathscr{A}$. Show that $a \geqslant 0$ iff $a^{-1} \geqslant 0$.
6. Show that a positive element $a$ of $\mathscr{A}$ is invertible iff $a \geqslant \frac{1}{n}$ for some $n>0$. (Hint: show that $\operatorname{sp}(a) \subseteq\left[\frac{1}{n}, \infty\right)$ when $a \geqslant \frac{1}{n}$.)

18 Exercise With our new-found knowledge about positive elements verify that the product $\bigoplus_{i} \mathscr{A}_{i}$ of $C^{*}$-algebras $\mathscr{A}_{i}$ from 3 V is indeed the product of these $C^{*}$-algebras $\mathscr{A}_{i}$ in the category $\mathbf{C}_{\mathrm{PU}}^{*}$.

1. Show that an element $a$ of $\bigoplus_{i \in I} \mathscr{A}_{i}$ is positive iff $a(i) \geqslant 0$ for all $i$.
2. Show that the projections $\pi_{j}: \bigoplus_{i \in I} \mathscr{A}_{i} \rightarrow \mathscr{A}_{j}$ (defined in 10 VIII ) are positive.
3. Show that for any $C^{*}$-algebra $\mathscr{B}$ and collection of pu-maps $f_{i}: \mathscr{B} \rightarrow \mathscr{A}_{i}$ there is a unique pu-map $\left\langle f_{i}\right\rangle_{i}: \mathscr{B} \rightarrow \bigoplus_{i} \mathscr{A}_{i}$ with $\pi_{j} \circ\left\langle f_{i}\right\rangle_{i}=f_{j}$ for all $j$.
4. Conclude that $\bigoplus_{i} \mathscr{A}_{i}$ is the (categorical) product of the $\mathscr{A}_{i} \mathrm{~s}$ in $\mathbf{C}_{\mathrm{PU}}^{*}$.
(We'll return to the product of $C^{*}$-algebras a final time in 34 VI .)
Lemma For elements $a$ and $b$ from a $C^{*}$-algebra, we have

$$
\operatorname{sp}(a b) \backslash\{0\}=\operatorname{sp}(b a) \backslash\{0\} .
$$

Proof Let $\lambda \in \mathbb{C}$ with $\lambda \neq 0$ be given. We must show that $\lambda-a b$ is invertible iff $\lambda-b a$ is invertible. Suppose that $\lambda-a b$ is invertible. Then using the equality $a(\lambda-b a)=(\lambda-a b) a$ one sees that $\left(1+b(\lambda-a b)^{-1} a\right)(\lambda-b a)=\lambda$. Since similarly $(\lambda-b a)\left(1+b(\lambda-a b)^{-1} a\right)=\lambda$, we see that $\lambda^{-1}(1+b(\lambda-a b) a)$ is the inverse of $\lambda-b a$.

Lemma We have $a^{*} a \leqslant 0 \Longrightarrow a=0$ for every element $a$ of a $C^{*}$-algebra.
Proof Suppose that $a^{*} a \leqslant 0$. Then $\operatorname{sp}\left(a^{*} a\right) \subseteq(-\infty, 0]$, almost by definition, and so $\operatorname{sp}\left(a a^{*}\right) \subseteq(-\infty, 0]$ by [] giving $a a^{*} \leqslant 0$. Thus $a^{*} a+a a^{*} \leqslant 0$.

But on the other hand, $a^{*} a+a a^{*}=2\left(a_{\mathbb{R}}^{2}+a_{\mathbb{I}}^{2}\right) \geqslant 0$, and so $a^{*} a+a a^{*}=0$. Then $0 \geqslant a^{*} a=-a a^{*} \geqslant 0$ gives $a^{*} a=0$, and $a=0$.

Observe that the norm and order on (the self-adjoint elements of a) $C^{*}$-algebra $\mathscr{A}$ completely determine one another (using the unit): on the one hand $\|a\|=$ $\inf \{\lambda \geqslant 0:-\lambda \leqslant a \leqslant \lambda\}$ by 17 VI , and on the other hand $a \geqslant 0$ iff $\|a-s\| \leqslant s$ for some $s \geqslant \frac{1}{2}\|a\|$ by definition 9 IV). This has some useful consequences.
Lemma A positive map $f: \mathscr{A} \rightarrow \mathscr{B}$ between $C^{*}$-algebras is bounded. More specifically, we have $\|f(a)\| \leqslant\|f(1)\|\|a\|$ for all self-adjoint $a \in \mathscr{A}_{\mathbb{R}}$, and we have $\|f(a)\| \leqslant 2\|f(1)\|\|a\|$ for arbitrary $a \in \mathscr{A}$.
Proof Given $a \in \mathscr{A}_{\mathbb{R}}$ we have $-\|a\| \leqslant a \leqslant\|a\|$, and $-\|a\| f(1) \leqslant f(a) \leqslant\|a\| f(1)$ (because $f$ is positive), and thus $\|f(a)\| \leqslant f(1)\|a\| \leqslant\|f(1)\|\|a\|$ by 17 VI .

For an arbitrary element $a \equiv a_{\mathbb{R}}+i a_{\mathbb{I}}$ of $\mathscr{A}$ we have $\|f(a)\| \leqslant\left\|f\left(a_{\mathbb{R}}\right)\right\|+$ $\left\|f\left(a_{\mathbb{I}}\right)\right\| \leqslant 2\|f(1)\|\|a\|$.
Remark It is a non-trivial theorem (known as Russo-Dye in [55), that the factor " 2 " in the statement above can be dropped, i.e. $\|f\|=\|f(1)\|$ (c.f. Corollary 1 of 61]). Fortunately for us, we only need this improved bound for completely positive maps for which it's much easier to obtain (see 34 XVII ).

Exercise Show that $\|\varrho\|=\varrho(1)$ for every mi-map $\varrho$ (using the $C^{*}$-identity).
Lemma Show that for a pu-map $f: \mathscr{A} \rightarrow \mathscr{B}$ between $C^{*}$-algebras, the following are equivalent.

1. $f$ is bipositive, that is, $f(a) \geqslant 0$ iff $a \geqslant 0$ for all $a \in \mathscr{A}$;
2. $f$ is an isometry on $\mathscr{A}_{\mathbb{R}}$, that is, $\|f(a)\|=\|a\|$ for all $\in \mathscr{A}_{\mathbb{R}}$;
3. $f$ is an isometry on $\mathscr{A}_{+}$.

VII Proof It is clear that 2 implies 3 .
VIII 12 Let $a \in \mathscr{A}_{\mathbb{R}}$ be given. Note that $-\lambda \leqslant a \leqslant \lambda$ iff $-\lambda \leqslant f(a) \leqslant \lambda$ for all $\lambda \geqslant 0$, because $f$ is bipositive and unital. In particular, since $-\|a\| \leqslant a \leqslant$ $\|a\|$, we have $-\|a\| \leqslant f(a) \leqslant\|a\|$, and so $\|f(a)\| \leqslant\|a\|$. On the other hand, $-\|f(a)\| \leqslant f(a) \leqslant\|f(a)\|$ implies $-\|f(a)\| \leqslant a \leqslant\|f(a)\|$, and so $\|a\| \leqslant\|f(a)\|$. Thus $\|a\|=\|f(a)\|$, and $f$ is an isometry on $\mathscr{A}_{\mathbb{R}}$.
IX 3 Let $a \in \mathscr{A}$ be given. We must show that $f(a) \geqslant 0$ iff $a \geqslant 0$. Since $f$ is involution preserving (10IV) $a$ is self-adjoint iff $f(a)$ is self-adjoint, and so we might as well assume that $a$ is self-adjoint to start with. Since $f$ is an isometry on $\mathscr{A}_{+},\|a\|-a$ is positive, and $f$ is unital, we have $\|\|a\|-a\|=$ $\|f(\|a\|-a)\|=\| \| a\|-f(a)\|$. Now, observe that $0 \leqslant a$ iff $\|\|a\|-a\| \leqslant\|a\|$, and that $\|\|a\|-f(a)\| \leqslant\|a\|$ iff $0 \leqslant f(a)$, by 17 VI , because $\frac{1}{2}\|a\| \leqslant\|a\|$ and $\frac{1}{2}\|f(a)\| \leqslant\|a\|$ (by II).
$\times$ Warning Such a map $f$ need not preserve the norm of arbitrary elements: the map $A \mapsto \frac{1}{2} A+\frac{1}{2} A^{T}: M_{2} \rightarrow M_{2}$ is bipositive and unital, but

$$
\left\|\left(\begin{array}{ll}
0 & 1 \\
0 & 0
\end{array}\right)\right\|=1 \neq \frac{1}{2}=\left\|\left(\begin{array}{cc}
0 & 1 / 2 \\
0 & 0
\end{array}\right)+\left(\begin{array}{cc}
0 & 0 \\
1 / 2 & 0
\end{array}\right)\right\| .
$$

(Even if $f$ is completely positive, 34 IV it might still only preserve the norm of self-adjoint elements cf. 21 IX )

21 We just saw in 20 VI that a map on a $C^{*}$-algebra $\mathscr{A}$ that preserves and reflects the order determines the norm of the self-adjoint - but not all - elements of $\mathscr{A}$. This theme, to what extend a linear map (or a collection of linear maps) on a $C^{*}$-algebra determines its structure, while tangential at the moment, will grow ever more important until it is essential for the theory of von Neumann algebras. That's why we introduce the four levels of discernment that a collection of maps on a $C^{*}$-algebra might have already here.

II Definition A collection $\Omega$ of maps on a $C^{*}$-algebra $\mathscr{A}$ will be called

1. order separating if an element $a$ of $\mathscr{A}$ is positive iff $0 \leqslant \omega(a)$ for all $\omega \in \Omega$;
2. separating if an element $a$ of $\mathscr{A}$ is zero iff $\omega(a)=0$ for all $\omega \in \Omega$;
3. faithful if an element $a$ of $\mathscr{A}_{+}$is zero iff $\omega(a)=0$ for all $\omega \in \Omega$; and
4. centre separating if $a \in \mathscr{A}_{+}$is zero iff $\omega\left(b^{*} a b\right)=0$ for all $\omega \in \Omega$ and $b \in \mathscr{A}$. (The "centre" in "centre separating" will be explained in 69IX.)

Examples We'll see later on that the following collections are order separating. III

1. The set of all pu-maps $\omega: \mathscr{A} \rightarrow \mathbb{C}$ on a $C^{*}$-algebra (see 22 VIII ).
2. The set of all miu-maps $\omega: \mathscr{A} \rightarrow \mathbb{C}$ on a commutative $C^{*}$-algebra (see 27 XVIIII ).
3. The set of functionals on $\mathscr{B}(\mathscr{H})$, where $\mathscr{H}$ is a Hilbert space, of the form $\langle x,(\cdot) x\rangle: \mathscr{B}(\mathscr{H}) \rightarrow \mathbb{C}$ where $x \in \mathscr{H}$ (see 25III).
We'll call these functionals vector functionals. (They are clearly bounded and involution preserving linear maps, and once we know that each positive element of a $C^{*}$-algebra is a square, in 23 VII , it'll be obvious that vector functionals are positive too.)

None of the four levels of separation coincides. This follows from the following IV examples, that we'll just mention here, but can't verify yet.

1. A single non-zero vector $x$ from a Hilbert space $\mathscr{H}$ gives a vector functional $\langle x,(\cdot) x\rangle$ on $\mathscr{B}(\mathscr{H})$ that is centre separating on its own, but is not faithful when $\mathscr{H}$ has dimension $\geqslant 2$.
2. Given an orthonormal basis $\mathscr{E}$ of a Hilbert space $\mathscr{H}$ the collection

$$
\{\langle e,(\cdot) e\rangle: e \in \mathscr{E}\}
$$

of vector functionals on $\mathscr{B}(\mathscr{H})$ is faithful, but not separating when $\mathscr{E}$ has more than one element.
3. Given Hilbert spaces $\mathscr{H}$ and $\mathscr{K}$ the set of vector functionals

$$
\{\langle x \otimes y,(\cdot) x \otimes y\rangle: x \in \mathscr{H}, y \in \mathscr{K}\}
$$

on $\mathscr{B}(\mathscr{H} \otimes \mathscr{K})$ is separating, but not order separating when both $\mathscr{H}$ and $\mathscr{K}$ are at least two dimensional.
$\checkmark$ Exercise One use for a separating collection $\Omega$ of involution preserving maps on a $C^{*}$-algebra $\mathscr{A}$ is checking whether an element $a \in \mathscr{A}$ is self-adjoint: show that $a \in \mathscr{A}$ is self-adjoint iff $\omega(a)$ is self-adjoint for all $\omega \in \Omega$.
VI An order separating collection senses the norm of a self-adjoint element:
VII Proposition For a collection $\Omega$ of pu-maps on a $C^{*}$-algebra $\mathscr{A}$ the following are equivalent.

1. $\Omega$ is order separating;
2. $\|a\|=\sup _{\omega \in \Omega}\|\omega(a)\|$ for all $a \in \mathscr{A}_{\mathbb{R}}$;
3. $\|a\|=\sup _{\omega \in \Omega}\|\omega(a)\|$ for all $a \in \mathscr{A}_{+}$.

VIII Proof Denoting the codomain of $\omega \in \Omega$ by $\mathscr{B}_{\omega}$ (so that $\omega: \mathscr{A} \rightarrow \mathscr{B}_{\omega}$ ), apply 20 VI to the pu-map $\langle\omega\rangle_{\omega \in \Omega}: \mathscr{A} \rightarrow \bigoplus_{\omega \in \Omega} \mathscr{B}_{\omega}$ (see 18I).
IX Warning The formula $\|a\|=\sup _{\omega \in \Omega}\|\omega(a)\|$ need not be correct for an arbitrary (not necessarily self-adjoint) element $a$. Indeed, consider the matrix $A:=\left(\begin{array}{ll}0 & 1 \\ 0 & 0\end{array}\right)$, and the collection $\Omega=\left\{\langle x,(\cdot) x\rangle: x \in \mathbb{C}^{2},\|x\|=1\right\}$, which will turn out to be order separating. We have $\|A\|=1$, while $|\langle x, \omega(A) x\rangle|=\left|x_{1}\right|\left|x_{2}\right|$ never exceeds $1 / 2$ for $x \equiv\left(x_{1}, x_{2}\right) \in \mathscr{H}$ with $1=\|x\|$.

X Exercise Show that any operator norm dense subset $\Omega^{\prime}$ of an order separating collection $\Omega$ of positive functionals on a $C^{*}$-algebra $\mathscr{A}$ is order separating too.

22 We'll use 21 VII to show that the pu-maps $\omega: \mathscr{A} \rightarrow \mathbb{C}$ on a $C^{*}$-algebra $\mathscr{A}$ (called states of $\mathscr{A}$ for short) are order separating by showing that for every self-adjoint element $a \in \mathscr{A}$ there is a state $\omega$ of $\mathscr{A}$ with $\omega(a)=\|a\|$ or $\omega(a)=-\|a\|$. To obtain such a state we first find its kernel, which leads us to the following definitions.

II Definition An order ideal of a $C^{*}$-algebra $\mathscr{A}$ is a linear subspace $I$ of $\mathscr{A}$ with $b \in I \Longrightarrow b^{*} \in I$ and $b \in I \cap \mathscr{A}_{+} \Longrightarrow[-b, b] \subseteq I$. It is called proper if $1 \notin I$, and maximal if it is maximal among all proper order ideals.

III Exercise Let $\mathscr{A}$ be a $C^{*}$-algebra.

1. Show that the kernel of a state is a maximal order ideal.
(Hint: the kernel of a state is already maximal as linear subspace.)
2. Let $I$ be a proper order ideal of $\mathscr{A}$. Show that there is a maximal order ideal $J$ of $\mathscr{A}$ with $I \subseteq J$. (Hint: Zorn's Lemma may be useful.)
3. Let $a \in \mathscr{A} \mathbb{R}$. Show that there is a least order ideal $(a)$ that contains $a$, and that given $b \in \mathscr{A} \mathbb{R}$ we have $b \in(a)$ iff there are $\lambda, \mu \in \mathbb{R}$ with $\lambda a \leqslant b \leqslant \mu a$.
Show that $(a)=\mathbb{C} a$ when $0 \nless a \nless 0$.
Show that $1 \in(a)$ if and only if $a$ is invertible and either $0 \leqslant a$ or $a \leqslant 0$.
4. Let $a$ be a self-adjoint element of $\mathscr{A}$ which is not invertible. Show that there is a maximal order ideal $J$ of $\mathscr{A}$ with $a \in J$.
5. Let $a$ be a self-adjoint element of $\mathscr{A}$. Show that $\|a\|-a$ or $\|a\|+a$ is not invertible (perhaps by considering the spectrum of $a$.)

Lemma For every maximal order ideal $I$ of a $C^{*}$-algebra $\mathscr{A}$, there is a state $\omega: \mathscr{A} \rightarrow \mathbb{C}$ with $\operatorname{ker}(\omega)=I$.
Proof Form the quotient vector space $\mathscr{A} / I$ with quotient map $q: \mathscr{A} \rightarrow \mathscr{A} / I$. Note that since $1 \notin I$ we have $q(1) \neq 0$ and so we may regard $\mathbb{C}$ to be a linear subspace of $\mathscr{A} / I$ via $\lambda \mapsto q(\lambda)$. We will, in fact, show that $\mathbb{C}=\mathscr{A} / I$.

But let us first put an order on $\mathscr{A} / I$ : we say that $\mathfrak{a} \in \mathscr{A} / I$ is positive if $\mathfrak{a} \equiv q(a)$ for some $a \in \mathscr{A}_{+}$, and write $\mathfrak{a} \leqslant \mathfrak{b}$ if $\mathfrak{b}-\mathfrak{a}$ is positive for $\mathfrak{a}, \mathfrak{b} \in \mathscr{A} / I$. Note that the definition of "order ideal" is such that if both $\mathfrak{a}$ and $-\mathfrak{a}$ are positive, then $\mathfrak{a}=0$. We leave it to the reader to verify that $\mathscr{A} / I$ becomes a partially ordered vector space with the order defined above. There is, however, one detail we'd like to draw attention to, namely that a scalar $\lambda$ is positive in $\mathscr{A} / I$ iff $\lambda$ is positive in $\mathbb{C}$. Indeed, if $\lambda \geqslant 0$ in $\mathbb{C}$, then $\lambda \geqslant 0$ in $\mathscr{A}$, and so $\lambda \geqslant 0$ in $\mathscr{A} / I$. On the other hand, if $\lambda \geqslant 0$ in $\mathscr{A} / I$, but $\lambda \leqslant 0$ in $\mathbb{C}$, then $\lambda \leqslant 0$ in $\mathscr{A} / I$, and so $\lambda=0$. This detail has the pleasant consequence that once we have shown that $\mathscr{A} / I=\mathbb{C}$, we automatically get that $q: \mathscr{A} \rightarrow \mathbb{C}$ is positive.

Let $a \in \mathscr{A}_{\mathbb{R}}$ be given. Define $\alpha:=\inf \{\lambda \in \mathbb{R}: q(a) \leqslant \lambda\}$. Note that $-\|a\| \leqslant$ $\alpha \leqslant\|a\|$. We will prove that $q(a)=\alpha$ by considering the order ideal

$$
\begin{aligned}
J:=\{b \in \mathscr{A}: & \exists \lambda, \mu \in \mathbb{R}\left[\lambda(\alpha-q(a)) \leqslant b_{\mathbb{R}} \leqslant \mu(\alpha-q(a))\right] \wedge \\
& \left.\exists \lambda, \mu \in \mathbb{R}\left[\lambda(\alpha-q(a)) \leqslant b_{\mathbb{I}} \leqslant \mu(\alpha-q(a))\right]\right\} .
\end{aligned}
$$

We claim that $1 \notin J$. Indeed, suppose not-towards a contradiction. Then there is $\mu \in \mathbb{R}$ with $1 \leqslant \mu(\alpha-q(a))$. What can we say about $\mu$ ? If $\mu<0$, then $0 \geqslant 1 / \mu \geqslant \alpha-q(a)$, so $\alpha-1 / \mu \leqslant q(a)$, but $q(a) \leqslant \alpha+\varepsilon$ for every $\varepsilon>0$, and so $\alpha-1 / \mu \leqslant q(a) \leqslant \alpha-1 / 2 \mu$, which is absurd. If $\mu=0$, then we get $1 \leqslant \mu(\alpha-q(a)) \equiv 0$, which is absurd. If $\mu>0$, then $1 / \mu \leqslant \alpha-q(a)$, or in other
words, $q(a) \leqslant \alpha-1 / \mu$, giving $\alpha \leqslant \alpha-1 / \mu$ by definition of $\alpha$, which is absurd. Hence $1 \notin J$.

But then since $I \subseteq J$, we get $I=J$, by maximality of $I$. Thus, as $\alpha-a \in J$, we have $\alpha-a \in I$, and so $q(a)=\alpha$, as desired.
VII Let $a \in \mathscr{A}$ be given. Then $a=a_{\mathbb{R}}+i a_{\mathbb{I}}$. By VI there are $\alpha, \beta \in \mathbb{R}$ with $q\left(a_{\mathbb{R}}\right)=\alpha$, and $q\left(a_{\mathbb{I}}\right)=\beta$. Thus $q(a)=\alpha+i \beta$. Hence $\mathscr{A} / I=\mathbb{C}$. Since the quotient map $q: \mathscr{A} \rightarrow \mathscr{A} / I \equiv \mathbb{C}$ is pu, and $\operatorname{ker}(q)=I$, we are done.
VIII Exercise Show using IV that given a self-adjoint element $a$ of a $C^{*}$-algebra $\mathscr{A}$ there is a state $\omega$ with $|\omega(a)|=\|a\|$. Conclude that the set of states of a $C^{*}$-algebra is order separating (see 21 II ).

### 2.3.3 The Square Root

23 The key that unlocks the remaining basic facts about the (positive) elements of a $C^{*}$-algebra is the existence of the square root $\sqrt{a}$ of a positive element $a$, and its properties. For technical reasons, we will assume $\|a\| \leqslant 1$, and construct $1-\sqrt{1-a}$ instead of $\sqrt{a}$.

II Lemma Let $a$ be an element of a $C^{*}$-algebra $\mathscr{A}$ with $0 \leqslant a \leqslant 1$. Then there is a unique element $b \in \mathscr{A}$ with, $0 \leqslant b \leqslant 1, a b=b a$, and $(1-b)^{2}=1-a$. To be more specific, $b$ is the norm limit of the sequence $b_{0} \leqslant b_{1} \leqslant \cdots$ given by $b_{0}=0$ and $b_{n+1}=\frac{1}{2}\left(a+b_{n}^{2}\right)$. Moreover, if $c \in \mathscr{A}$ commutes with $a$, then $c$ commutes with $b$, and if in addition $a \leqslant 1-c^{2}$ and $c^{*}=c$, we have $b \leqslant 1-c$.
III Proof When discussing $b_{n}$ it is convenient to write $b_{n} \equiv q_{n}(a)$ where $q_{0}, q_{1}, \ldots$ are the polynomials over $\mathbb{R}$ given by $q_{0}=0$ and $q_{n+1}=\frac{1}{2}\left(x+q_{n}^{2}\right)$. For example, we have $b_{n} \geqslant 0$, because all coefficients of $q_{n}$ are all positive, and $a, a^{2}, a^{3}, \ldots$ are positive by 17 VI . With a similar argument we can see that $b_{0} \leqslant b_{1} \leqslant b_{2} \leqslant \cdots$. Indeed, the coefficients of $q_{n+1}-q_{n}$ are positive, by induction, because

$$
\begin{aligned}
q_{n+2}-q_{n+1} & =\frac{1}{2}\left(x+q_{n+1}^{2}\right)-\frac{1}{2}\left(x+q_{n}^{2}\right) \\
& =\frac{1}{2}\left(q_{n+1}^{2}-q_{n}^{2}\right) \\
& =\frac{1}{2}\left(q_{n+1}+q_{n}\right)\left(q_{n+1}-q_{n}\right) \\
& =\left(q_{n}+\frac{1}{2}\left(q_{n+1}-q_{n}\right)\right)\left(q_{n+1}-q_{n}\right),
\end{aligned}
$$

has positive coefficients if $q_{n+1}-q_{n}$ has positive coefficients, and $q_{1}-q_{0} \equiv \frac{1}{2} x$ clearly has positive coefficients. Hence $b_{n+1}-b_{n}=q_{n+1}(a)-q_{n}(a)$ is positive.
(Note that we have carefully avoided using the fact here that the product of positive commuting elements is positive, which is not available to us until V.)

Let us now show that $b_{0} \leqslant b_{1} \leqslant \cdots$ converges. Let $n \geqslant N$ from $\mathbb{N}$ be given. Since the coefficients of $q_{n}-q_{N}$ are positive, and $\|a\| \leqslant 1$, the triangle inequality gives us $\left\|b_{n}-b_{N}\right\| \equiv\left\|\left(q_{n}-q_{N}\right)(a)\right\| \leqslant q_{n}(1)-q_{N}(1)$, and so it suffices to show that the ascending sequence $q_{0}(1) \leqslant q_{1}(1) \leqslant \cdots$ of real numbers converges, c.q. is bounded. Indeed, we have $q_{n}(1) \leqslant 1$, by induction, because $q_{n+1}(1) \equiv \frac{1}{2}\left(1+q_{n}(1)^{2}\right) \leqslant 1$ if $q_{n}(1) \leqslant 1$, and clearly $0 \equiv q_{0}(1) \leqslant 1$.

Let $b$ be the limit of $b_{0} \leqslant b_{1} \leqslant \cdots$. Then $b$ being the limit of positive elements is positive (see 17 VI ), and if $c \in \mathscr{A}$ commutes with $a$, then $c$ commutes with all powers of $a$, and therefore with all $b_{n}$, and thus with $b$. Further, from the recurrence relation $q_{n+1}=\frac{1}{2}\left(a+q_{n}^{2}\right)$ we get $b=\frac{1}{2}\left(a+b^{2}\right)$, and so $-a=-2 b+b^{2}$, giving us $(1-b)^{2}=1-2 b+b^{2}=1-a$.

Let us prove that $b \leqslant 1$. To begin, note that $\left\|b_{n}\right\| \leqslant 1$ for all $n$, by induction, because $0 \equiv\left\|b_{0}\right\| \leqslant 1$, and if $\left\|b_{n}\right\| \leqslant 1$, then $\left\|b_{n+1}\right\| \leqslant \frac{1}{2}\left(\|a\|+\left\|b_{n}\right\|^{2}\right) \leqslant 1$, since $\|a\| \leqslant 1$. Since $b_{n} \geqslant 0$, we get $-1 \leqslant b_{n} \leqslant 1$ for all $n$, and so $b \leqslant 1$.

Let us take a step back for the moment. From what we have proven so far we see that each positive $c \in \mathscr{A}$ is of the form $c \equiv d^{2}$ for some positive $d \in \mathscr{A}$ which commutes with all $e \in \mathscr{A}$ that commute with $c$.

From this we can see that $c_{1} c_{2} \geqslant 0$ for $c_{1}, c_{2} \in \mathscr{A}_{+}$with $c_{1} c_{2}=c_{2} c_{1}$. Indeed, writing $c_{i} \equiv d_{i}^{2}$ with $d_{i}$ as above, we have $d_{1} c_{2}=c_{2} d_{1}$ (because $c_{1} c_{2}=c_{2} c_{1}$ ), and thus $d_{1} d_{2}=d_{2} d_{1}$. It follows that $d_{1} d_{2}$ is self-adjoint, and $c_{1} c_{2}=\left(d_{1} d_{2}\right)^{2}$. Hence $c_{1} c_{2} \geqslant 0$.

We will also need the following corollary. For $c, d \in \mathscr{A}_{+}$with $c \leqslant d$ and $c d=d c$, we have $c^{2} \leqslant d^{2}$. Indeed, $d^{2}-c^{2} \equiv d(d-c)+c(d-c)$ is positive by the previous paragraph.
Let $c \in \mathscr{A} \mathbb{R}$ be such that $c a=a c$ and $a \leqslant 1-c^{2}$. We must show that $b \leqslant 1-c$. Of course, since $b$ is the limit of $b_{1}, b_{2}, \ldots$, it suffices to show that $b_{n} \leqslant 1-c$, and we'll do this by induction. Since $0 \leqslant c^{2} \leqslant 1-a$, we have $\|c\|^{2} \leqslant\|1-a\| \leqslant 1$, and so $-1 \leqslant c \leqslant 1$. Thus $b_{0} \equiv 0 \leqslant 1-c$. Now, suppose that $b_{n} \leqslant 1-c$ for some $n$. Then $b_{n+1}=\frac{1}{2}\left(a+b_{n}^{2}\right) \leqslant \frac{1}{2}\left(\left(1-c^{2}\right)+(1-c)^{2}\right)=1-c$, where we have used that $b_{n}^{2} \leqslant(1-c)^{2}$, because $b_{n} \leqslant 1-c$ by IV.
We'll now show that $b$ is unique in the sense that $b=b^{\prime}$ for any $b^{\prime} \in \mathscr{A}$ with $0 \leqslant b^{\prime} \leqslant 1, b^{\prime} a=a b^{\prime}$ and $\left(1-b^{\prime}\right)^{2}=1-a$. Note that $b^{\prime} \leqslant 1$, because $\left\|1-b^{\prime}\right\|^{2}=$ $\|1-a\| \leqslant 1$, From $a=1-\left(1-b^{\prime}\right)^{2}$, we immediately get $b \leqslant 1-\left(1-b^{\prime}\right)=b^{\prime}$ by V . For the other direction, note that $\left(1-b^{\prime}\right)^{2}=(1-b)^{2} \equiv\left(1-b^{\prime}+\left(b^{\prime}-b\right)\right)^{2}=$ $\left(1-b^{\prime}\right)^{2}+2\left(1-b^{\prime}\right)\left(b^{\prime}-b\right)+\left(b^{\prime}-b\right)^{2}$, which gives $0=2\left(1-b^{\prime}\right)\left(b^{\prime}-b\right)+\left(b^{\prime}-b\right)^{2}$.

Now, since $1-b^{\prime}$ and $b^{\prime}-b$ are positive, and commute, we see that $\left(1-b^{\prime}\right)\left(b^{\prime}-b\right)$ is positive by V , and so $0=2\left(1-b^{\prime}\right)\left(b^{\prime}-b\right)+\left(b^{\prime}-b\right)^{2} \geqslant\left(b^{\prime}-b\right)^{2} \geqslant 0$, which entails $\left(b^{\prime}-b\right)^{2}=0$, and so $\left\|\left(b^{\prime}-b\right)^{2}\right\|=\left\|b^{\prime}-b\right\|^{2}=0$, yielding $b=b^{\prime}$.
VII Exercise Let $a$ be a positive element of a $C^{*}$-algebra $\mathscr{A}$. Show that there is a unique positive element of $\mathscr{A}$ denoted by $\sqrt{a}$ (and by $a^{1 / 2}$ ) with $\sqrt{a}^{2}=a$ and $a \sqrt{a}=\sqrt{a} a$. Show that if $c \in \mathscr{A}$ commutes with $a$, then $c \sqrt{a}=\sqrt{a} c$, and if in addition $c^{*}=c$ and $c^{2} \leqslant a$, then $c \leqslant \sqrt{a}$. Using this, verify:

1. If $a, b \in \mathscr{A}$ are positive, and $a b=b a$, then $a b \geqslant 0$.
2. Let $a \in \mathscr{A}_{+}$. If $b, c \in \mathscr{A}_{\mathbb{R}}$ commute with $a$, then $b \leqslant c$ implies $a b \leqslant a c$.
3. If $a, b \in \mathscr{A}_{\mathbb{R}}$ commute, and $a \leqslant b$, then $a^{2} \leqslant b^{2}$.
4. The requirement in the previous item that $a$ and $b$ commute is essential: there are positive elements $a, b$ of a $C^{*}$-algebra $\mathscr{A}$ with $a \leqslant b$, but $a^{2} \nless b^{2}$. In other words, the square $a \mapsto a^{2}$ on the positive elements of a $C^{*}$-algebra need not be monotone, (but $a \mapsto \sqrt{a}$ is monotone, see 28 IIII ).
(Hint: take $a=\left(\begin{array}{ll}1 & 0 \\ 0 & 0\end{array}\right)$ and $b=a+\frac{1}{2}\left(\begin{array}{ll}1 & 1 \\ 1 & 1\end{array}\right)$ from $M_{2}$.)

24 Definition Given a self-adjoint element $a$ of a $C^{*}$-algebra $\mathscr{A}$, we write

$$
|a|:=\sqrt{a^{2}} \quad a_{+}:=\frac{1}{2}(|a|+a) \quad a_{-}:=\frac{1}{2}(|a|-a) .
$$

We call $a_{+}$the positive part of $a$, and $a_{-}$the negative part.
II Exercise Let $a$ be a self-adjoint element of a $C^{*}$-algebra $\mathscr{A}$.

1. Show that $-|a| \leqslant a \leqslant|a|$, and $\||a|\|=\|a\|$.
2. Prove that $a_{+}$and $a_{-}$are positive, $a=a_{+}-a_{-}$and $a_{+} a_{-}=a_{-} a_{+}=0$.
3. One should not read too much into the notation $|\cdot|$ in the non-commutative case: give an example of self-adjoint elements $a$ and $b$ of a $C^{*}$-algebra with $|a+b| \nless|a|+|b|$.
(Hint: one may take $a=\frac{1}{2}\left(\begin{array}{ll}1 & 1 \\ 1 & 1\end{array}\right)$ and $b=-\left(\begin{array}{ll}1 & 0 \\ 0 & 0\end{array}\right)$.)

III The existence of positive and negative parts in a $C^{*}$-algebra has many pleasant and subtle consequences of which we'll now show one.

Lemma Given an element $a$ of a $C^{*}$-algebra $\mathscr{A}$, we have $a^{*} a \geqslant 0$.
Proof Writing $b:=a\left(\left(a^{*} a\right)_{-}\right)^{1 / 2}$, we have $b^{*} b=\left(\left(a^{*} a\right)_{-}\right)^{1 / 2} a^{*} a\left(\left(a^{*} a\right)_{-}\right)^{1 / 2}=$ $\left(a^{*} a\right)_{-} a^{*} a=-\left(\left(a^{*} a\right)_{-}\right)^{2} \leqslant 0$, and so $b=0$ by 19 III . Hence $\left(a^{*} a\right)_{-}=0$ giving us $a^{*} a=\left(a^{*} a\right)_{+} \geqslant 0$.

Exercise Round up our results regarding positive elements to prove that the following are equivalent for a self-adjoint element $a$ of a $C^{*}$-algebra $\mathscr{A}$.

1. $a$ is positive, that is, $\|a-t\| \leqslant t$ for some $t \geqslant \frac{1}{2}\|a\|$;
2. $\|a-t\| \leqslant t$ for all $t \geqslant \frac{1}{2}\|a\|$;
3. $a \equiv b^{2}$ for some self-adjoint $b \in \mathscr{A}$;
4. $a \equiv c^{*} c$ for some $c \in \mathscr{A}$;
5. $\operatorname{sp}(a) \subseteq[0, \infty)$.

Exercise The fact that $a^{*} a$ is positive for an element $a$ of a $C^{*}$-algebra $\mathscr{A}$ has II some nice consequences of its own needed later on.

1. Show that $b \leqslant c \Longrightarrow a^{*} b a \leqslant a^{*} c a$ for all $b, c \in \mathscr{A}_{\mathbb{R}}$ and $a \in \mathscr{A}$.
2. Show that every mi-map and cp-map is positive.
3. Show that $a \leqslant b^{-1}$ iff $\sqrt{b} a \sqrt{b} \leqslant 1$ iff $\|\sqrt{a} \sqrt{b}\| \leqslant 1$ iff $b \leqslant a^{-1}$ for positive invertible elements $a, b$ of $\mathscr{A}$ (and so $a \leqslant b$ entails $b^{-1} \leqslant a^{-1}$ ).
4. Prove that $(1+a)^{-1} a \leqslant(1+b)^{-1} b$ for $0 \leqslant a \leqslant b$ from $\mathscr{A}$.
(Hint: add $(1+a)^{-1}+(1+b)^{-1}$ to both sides of the inequality.)

Proposition The vector states of $\mathscr{B}(\mathscr{H})$ are order separating (see 21 II ) for III every Hilbert space $\mathscr{H}$.
Proof By 21 VII tt suffices to show that $\|T\|=\sup _{x \in(\mathscr{H})_{1}}|\langle x, T x\rangle|$ for given $T \in$ $\mathscr{B}(\mathscr{H})_{+}$. Since $|\langle x, T x\rangle|=\left\langle T^{1 / 2} x, T^{1 / 2} x\right\rangle=\left\|T^{1 / 2} x\right\|^{2}$ for all $x \in \mathscr{H}$, we have $\|T\|=\left\|T^{1 / 2}\right\|^{2}=\left(\sup _{x \in(\mathscr{H})_{1}}\left\|T^{1 / 2} x\right\|\right)^{2}=\sup _{x \in(\mathscr{H})_{1}}|\langle x, T x\rangle|$.

Corollary For a bounded operator $T$ on a Hilbert space $\mathscr{H}$, we have

1. $T$ is self-adjoint iff $\langle x, T x\rangle$ is real for all $x \in(\mathscr{H})_{1}$;
2. $0 \leqslant T$ iff $0 \leqslant\langle x, T x\rangle$ for all $x \in(\mathscr{H})_{1}$;
3. $\|T\|=\sup _{x \in(\mathscr{H})_{1}}|\langle x, T x\rangle|$ when $T$ is self-adjoint.

VI Proof This follows from 21 V and 21 VII because the vector states on $\mathscr{B}(\mathscr{H})$ are order separating by 111

26 The interaction between the multiplication and order on a $C^{*}$-algebra can be subtle, but when the $C^{*}$-algebra is commutative almost all peculiarities disappear. This is to be expected as any commutative $C^{*}$-algebra is isomorphic to a $C^{*}$-algebra of continuous functions on a compact Hausdorff space (as we'll see in 27 XXVIII .
II Exercise Let $\mathscr{A}$ be a commutative $C^{*}$-algebra. Let $a, b, c \in \mathscr{A}_{\mathbb{R}}$.

1. Show that $|a|$ is the supremum of $a$ and $-a$ in $\mathscr{A}_{\mathbb{R}}$.
2. Show that if $a$ and $b$ have a supremum, $a \vee b$, in $\mathscr{A} \mathbb{R}$, then $c+a \vee b$ is the supremum of $a+c$ and $b+c$.
3. Show that $\mathscr{A}_{\mathbb{R}}$ is a Riesz space, that is, a lattice ordered vector space. (Hint: prove that $\frac{1}{2}(a+b+|a-b|)$ is the supremum of $a$ and $b$ in $\mathscr{A}_{\mathbb{R}}$.)
4. Show that a miu-map $f: \mathscr{A} \rightarrow \mathscr{B}$ between commutative $C^{*}$-algebras preserves finite suprema and infima.

III Exercise Prove the Riesz decomposition lemma: For positive elements $a, b, c$ of a commutative $C^{*}$-algebra $\mathscr{A}$ with $c \leqslant a+b$ we have $c \equiv a^{\prime}+b^{\prime}$ where $0 \leqslant a^{\prime} \leqslant a$ and $0 \leqslant b^{\prime} \leqslant b$.

### 2.4 Representation

### 2.4.1 ... by Continuous Functions

27 Now that we have have a firm grip on the positive elements of a $C^{*}$-algebra we turn to what is perhaps the most important fact about commutative $C^{*}$ algebras: that they are isomorphic to $C^{*}$-algebras of continuous functions on a compact Hausdorff space, via the Gelfand representation.

Setting $\mathscr{A}$ is a commutative $C^{*}$-algebra.
Definition The spectrum of $\mathscr{A}$, denoted by $\operatorname{sp}(\mathscr{A})$, is the set of all miu-maps III $f: \mathscr{A} \rightarrow \mathbb{C}$. We endow $\operatorname{sp}(\mathscr{A})$ with the topology of pointwise convergence.

The Gelfand representation of $\mathscr{A}$ is the miu-map $\gamma: \mathscr{A} \rightarrow C(\operatorname{sp}(\mathscr{A}))$ given by $\gamma(a)(f)=f(a)$.
Exercise Verify that the map $\operatorname{sp}(\mathscr{A}) \rightarrow \mathbb{C}, f \mapsto f(a)$ is indeed continuous for IV every $a \in \mathscr{A}$, and that $\gamma$ is miu.

Remark One might wonder if there is any connection between the spectrum $\operatorname{sp}(\mathscr{A}) \mathrm{V}$ of a commutative $C^{*}$-algebra, and the $\operatorname{spectrum~} \operatorname{sp}(a)$ of one of $\mathscr{A}$ 's elements (from 11 XIX ; and indeed there is as we'll see in XVII (and 28 II ).
Our program for this paragraph is to show that the Gelfand representation $\gamma$ is a miu-isomorphism. In fact, we will show that it gives the unit of an equivalence between the category of commutative $C^{*}$-algebras (with miu-maps) and the opposite of the category of compact Hausdorff spaces (with continuous maps). The first hurdle we take is the injectivity of $\gamma$ - that there are sufficiently many points in the spectrum of a commutative $C^{*}$-algebra, so to speak -, and involves the following special type of order ideal.

Definition A Riesz ideal of $\mathscr{A}$ is an order ideal $I$ such that $a \in I \cap \mathscr{A}_{\mathbb{R}} \Longrightarrow$ $|a| \in I$. A maximal Riesz ideal is a proper Riesz ideal which is maximal among proper Riesz ideals.
Lemma Let $I$ be a Riesz ideal of $\mathscr{A}$. For all $a \in \mathscr{A}$ and $x \in I$ we have $a x \in I$. Proof Since $x=x_{\mathbb{R}}+i x_{\mathbb{I}}$, it suffices to show that $a x_{\mathbb{R}} \in I$ and $a x_{\mathbb{I}} \in I$. Note that $x_{\mathbb{R}}, x_{\mathbb{I}} \in I$, so we might as well assume that $x$ is self-adjoint to begin with. Similarly, using that $x_{+} \in I$ (because $x_{+}=\frac{1}{2}(|x|+x)$ and $|x| \in I$ ) and $x_{-} \in I$, we can reduce the problem to the case that $x$ is positive. We may also assume that $a$ is self-adjoint. Now, since $x \geqslant 0$ and $-\|a\| \leqslant a \leqslant\|a\|$, we have $-\|a\| x \leqslant a x \leqslant\|a\| x$ by 23 VII , and so $a x \in I$, because $\|a\| x \in I$.
Exercise Verify the following facts about Riesz ideals.

1. The least Riesz ideal that contains a self-adjoint element $a$ of $\mathscr{A}$ is

$$
(a)_{m}:=\left\{b \in \mathscr{A}: \exists n \in \mathbb{N}\left[\left|b_{\mathbb{R}}\right|,\left|b_{\mathbb{I}}\right| \leqslant n|a|\right]\right\} .
$$

Moreover, $(a)_{m}=\mathscr{A}$ iff $a$ is invertible, and we have $(a)=(a)_{m}$ when $a \geqslant 0$ (where $(a)$ is the least order ideal that contains $a$, see 22 III). For nonpositive $a$, however, we may have $(a) \neq(a)_{m}$.
2. $I+J$ is a Riesz ideal of $\mathscr{A}$ when $I$ and $J$ are Riesz ideals. (Hint: use 26III) But $I+J$ might not be an order ideal when $I$ and $J$ are order ideals.
3. Each proper Riesz ideal is contained in a maximal Riesz ideal.

XI Lemma A maximal Riesz ideal $I$ of $\mathscr{A}$ is a maximal order ideal.
XII Proof Let $J$ be a proper order ideal with $I \subseteq J$. We must show that $J=I$. Let $a \in J$ be given; we must show that $a \in I$. Since $a_{\mathbb{R}}, a_{\mathbb{I}} \in J$, it suffices to show that $a_{\mathbb{R}}, a_{\mathbb{I}} \in I$, and so we might as well assume that $a$ is self-adjoint to begin with. Similarly, since $|a| \in J$, and it suffices to show that $|a| \in I$, because then $-|a| \leqslant a \leqslant|a|$ entails $a \in I$, we might as well assume that $a$ is positive.

Note that the least ideal ( $a$ ) that contains $a$ is also a Riesz ideal by Hence $I+(a)$ is a Riesz ideal by $\boxtimes$ Since $a \in J$, we have $(a) \subseteq J$, and so $I+(a) \subseteq J$ is proper. It follows that $a \in I+(a)=I$ by maximality of $I$.

XIII Lemma Let $I$ be a maximal Riesz ideal of $\mathscr{A}$. Then there is a miu-map $f: \mathscr{A} \rightarrow \mathbb{C}$ with $\operatorname{ker}(f)=I$.
XIV Proof Since $I$ is a maximal order ideal by XI there is a pu-map $f: \mathscr{A} \rightarrow \mathbb{C}$ with $\operatorname{ker}(f)=I$ by 22 IV . It remains to be shown that $f$ is multiplicative. Let $a, b \in \mathscr{A}$ be given; we must show that $f(a b)=f(a) f(b)$. Surely, since $f$ is unital, we have $f(b-f(b))=f(b)-f(b)=0$, an so $b-f(b) \in \operatorname{ker}(f) \equiv I$. Now, since $I$ is a Riesz ideal, we have $a(b-f(b)) \in I \equiv \operatorname{ker}(f)$ by VIII, and so $0=f(a(b-f(b)))=f(a b)-f(a) f(b)$. Hence $f$ is multiplicative.

XV Proposition Let $a$ be a self-adjoint element of a $C^{*}$-algebra. Then $a$ is not invertible iff there is $f \in \operatorname{sp}(\mathscr{A})$ with $f(a)=0$.
XVI Proof Note that if $a$ is invertible, then $f\left(a^{-1}\right)$ is the inverse of $f(a)$-and so $f(a) \neq 0$-for every $f \in \operatorname{sp}(\mathscr{A})$. For the other, non-trivial, direction, assume that $a$ is not invertible. Then by Х the least Riesz ideal $(a)_{m}$ that contains $a$ is proper, and can be extended to a maximal Riesz ideal $I$. By XIII there is a miu-map $f: \mathscr{A} \rightarrow \mathbb{C}$ with $\operatorname{ker}(f)=I$. Then $f \in \operatorname{sp}(\mathscr{A})$ and $f(a)=0$.
XVII Exercise Show that $\operatorname{sp}(a)=\{f(a): f \in \operatorname{sp}(\mathscr{A})\}$ for each self-adjoint $a \in \mathscr{A}$.
XVIII Exercise Prove that $\|\gamma(a)\|=\|a\|$ for each $a \in \mathscr{A}$ where $\gamma$ is from XXVII.
(Hint: first assume that $a$ is self-adjoint, and use XVII and 1611 . For the general case, use the $C^{*}$-identity.)

Conclude that the Gelfand representation $\gamma: \mathscr{A} \rightarrow C(\operatorname{sp}(\mathscr{A}))$ is injective, and that its range $\{\gamma(a): a \in \mathscr{A}\}$ is a $C^{*}$-subalgebra of $C(\operatorname{sp}(\mathscr{A}))$.

To show that $\gamma$ is surjective, we use the following special case of the Stone- XIX Weierstraß theorem.

Theorem Let $X$ be a compact Hausdorff space, and let $\mathscr{S}$ be a $C^{*}$-subalgebra of $C(X)$ which 'separates the points of $X$ ', that is, for all $x, y \in X$ with $x \neq y$ there is $f \in \mathscr{S}$ with $f(x) \neq f(y)$. Then $\mathscr{S}=C(X)$.
Proof Let $g \in C(X)_{+}$and $\varepsilon>0$. To prove that $\mathscr{S}=C(X)$, it suffices to show XXI that $g \in \mathscr{S}$, and for this, it suffices to find $f \in \mathscr{S}$ with $\|f-g\| \leqslant \varepsilon$, because $\mathscr{S}$ is closed. It is convenient to assume that $g(x)>0$ for all $x \in X$, which we may, without loss of generality, by replacing $g$ by $1+g$.

Let $x, y \in X$ with $x \neq y$ be given. We know there is $f \in \mathscr{S}$ with $f(x) \neq f(y)$. Note that we can assume that $f(x)=0$ (by replacing $f$ by $f-f(x)$ ), and that $f$ is self-adjoint (by replacing $f$ by either $f_{\mathbb{R}}$ or $f_{\mathbb{I}}$ ), and that $f$ is positive (by replacing $f$ by $f_{+}$or $f_{-}$), and that $f(y)=g(y)>0$ (by replacing $f$ by $\frac{g(y)}{f(y)} f$ ), and that $f \leqslant g(y)$ (by replacing $f$ by $f \wedge g(y)$ ).
Let $y \in X$ be given. We will show that there is $f \in \mathscr{S}$ with $0 \leqslant f \leqslant g+\varepsilon$ and $f(y)=g(y)$. Indeed, since $g$ is continuous there is an open neighborhood $V$ of $y$ with $g(y) \leqslant g(x)+\varepsilon$ for all $x \in V$. For each $x \in X \backslash V$ there is $f_{x} \in[0, f(y)]_{\mathscr{S}}$ with $f_{x}(x)=0$ and $f_{x}(y)=g(y)$ by XXII. Since the open subsets $U_{x}:=\{z \in$ $\left.X: f_{x}(z) \leqslant \varepsilon\right\}$ with $x \in X \backslash V$ form an open cover of the closed (and thus compact) subset $X \backslash V$, there are $x_{1}, \ldots, x_{N} \in X \backslash U$ with $U_{x_{1}} \cup \cdots \cup U_{x_{N}} \supseteq X \backslash V$. Define $f:=f_{x_{1}} \wedge \cdots \wedge f_{x_{N}}$. Then $f \in \mathscr{S}, 0 \leqslant f \leqslant g(y), f(y)=g(y)$, and $f(x) \leqslant \varepsilon$ for every $x \in X \backslash V$.

We claim that $f \leqslant g+\varepsilon$. Indeed, if $x \in X \backslash V$, then $f(x) \leqslant \varepsilon \leqslant g(x)+\varepsilon$. If $x \in V$, then $f(x) \leqslant g(y) \leqslant g(x)+\varepsilon$ (by definition of $V$ ). Hence $f \leqslant g+\varepsilon$.
Thus for each $y \in X$ there is $f_{y} \in \mathscr{S}$ with $0 \leqslant f_{y} \leqslant g+\varepsilon$ and $f_{y}(y)=g(y)$. Since $f_{y}$ is continuous at $y$, and $f_{y}(y)=g(y)$, there is an open neighborhood $U_{y}$ of $y$ with $g(y)-\varepsilon \leqslant f_{y}(x)$ for all $x \in U_{y}$. Since these open neighborhoods cover $X$, and $X$ is compact, there are $y_{1}, \ldots, y_{N} \in X$ with $U_{y_{1}} \cup \cdots \cup U_{y_{N}}=X$. Define $f:=f_{y_{1}} \vee \cdots \vee f_{y_{N}}$. Then $f \in \mathscr{S}$, and $g-\varepsilon \leqslant f \leqslant g+\varepsilon$, giving $\|f-g\| \leqslant \varepsilon$.
Lemma The spectrum $\operatorname{sp}(\mathscr{A})$ of $\mathscr{A}$ is a compact Hausdorff space.
Proof Since for each $a \in \mathscr{A}$ and $f \in \operatorname{sp}(\mathscr{A})$ we have $\|f(a)\| \leqslant\|a\|$ by 20 V XXVI we see that $f(a)$ is an element of the compact set $\{z \in \mathbb{C}:|z| \leqslant\|a\|\}$, and so $\operatorname{sp}(\mathscr{A})$ is a subset of

$$
\prod_{a \in \mathscr{A}}\{z \in \mathbb{C}:|z| \leqslant\|a\|\},
$$

which is a compact Hausdorff space (by Tychonoff's theorem, under the product topology it inherits from the space of all functions $\mathscr{A} \rightarrow \mathbb{C}$ ). So to prove that $\operatorname{sp}(\mathscr{A})$ is a compact Hausdorff space, it suffices to show that $\operatorname{sp}(\mathscr{A})$ is closed. In other words, we must show that if $f: \mathscr{A} \rightarrow \mathbb{C}$ is the pointwise limit of a net of miu-maps $\left(f_{i}\right)_{i}$, then $f$ is a miu-map as well. But this is easily achieved using the continuity of addition, involution and multiplication on $\mathbb{C}$, because, for instance, for $a, b \in \mathscr{A}$, we have $f(a b)=\lim _{i} f_{i}(a b)=\lim _{i} f_{i}(a) f_{i}(b)=$ $\left(\lim _{i} f_{i}(a)\right)\left(\lim _{i} f_{i}(b)\right)=f(a) f(b)$.
XXVII Gelfand's Representation Theorem For a commutative $C^{*}$-algebra $\mathscr{A}$, the Gelfand representation, $\gamma: \mathscr{A} \rightarrow C(\operatorname{sp}(\mathscr{A}))$ defined in III is a miu-isomorphism.
XXVIII Proof We already know that $\gamma$ is an injective miu-map (see IV and XVIII). So to prove that $\gamma$ is a miu-isomorphism, it remains to be shown that $\gamma$ is surjective. Since $\operatorname{sp}(\mathscr{A})$ is a compact Hausdorff space (by XXV), and $\gamma(\mathscr{A}) \equiv\{\gamma(a): a \in$ $\mathscr{A}\}$ is a $C^{*}$-subalgebra of $C(\operatorname{sp}(\mathscr{A}))$ (by XVIII), it suffices to show that $\gamma(\mathscr{A})$ separates the points of $\operatorname{sp}(X)$ by XX. This is obvious, because for $f, g \in \operatorname{sp}(\mathscr{A})$ with $f \neq g$ there is $a \in \mathscr{A}$ with $f(a) \equiv \gamma(a)(f) \neq \gamma(a)(g) \equiv g(a)$.

28 While Gelfand's representation theorem is a result about commutative $C^{*}$ algebras, it tells us a lot about non-commutative $C^{*}$-algebras too, via their commutative $C^{*}$-subalgebras.
II Exercise Let $a$ be an element of a (not necessarily commutative) $C^{*}$-algebra $\mathscr{A}$. We are going to use Gelfand's representation theorem to define $f(a)$ for every continuous map $f: \operatorname{sp}(a) \rightarrow \mathbb{C}$ whenever $a$ is contained in some commutative $C^{*}$-algebra. This idea is referred to as the continuous functional calculus.

1. Show that there is a least $C^{*}$-subalgebra $C^{*}(a)$ of $\mathscr{A}$ that contains $a$.

Given $b \in C^{*}(a)$ show that $b c=c b$ for all $c \in \mathscr{A}$ with $a c=c a$.
2. We call $a \in \mathscr{A}$ normal when $C^{*}(a)$ is commutative.

Show that $a$ is normal iff $a a^{*}=a^{*} a$ iff $a_{\mathbb{R}} a_{\mathbb{I}}=a_{\mathbb{I}} a_{\mathbb{R}}$.
3. From now on assume $a$ is normal so that $C^{*}(a)$ is commutative.

Show that $j: \varrho \mapsto \varrho(a), \operatorname{sp}\left(C^{*}(a)\right) \rightarrow \operatorname{sp}(a)$ is a continuous map.
Denoting the composition of the miu-maps

$$
C(\operatorname{sp}(a)) \xrightarrow{f \mapsto f \circ j} C\left(\operatorname{sp}\left(C^{*}(A)\right)\right) \xrightarrow{\cong, \boxed{27 \times X V I I}} C^{*}(a) \xrightarrow{\text { inclusion }} \mathscr{A} .
$$

by $\Phi$, we write $f(a):=\Phi(f)$ for all $f \in C(\operatorname{sp}(a))$.

We have hereby defined, for example, $a^{\alpha}$ when $a \geqslant 0$ and $\alpha \in(0, \infty)$.
From the fact that $\Phi$ is miu some properties of $f(a)$ can be derived. Show, for example, that $a^{\alpha} a^{\beta}=a^{\alpha+\beta}$ for all $\alpha, \beta \in(0, \infty)$ when $a \geqslant 0$.
4. Given $f \in C(\operatorname{sp}(a))$, show that $f(a)$ is the unique element of $C^{*}(a)$ with

$$
\varphi(f(a))=f(\varphi(a))
$$

for all $\varphi \in \operatorname{sp}\left(C^{*}(a)\right)$.
5. (Spectral mapping thm.) Show that $\operatorname{sp}(f(a))=f(\operatorname{sp}(a))$ for $f \in C(\operatorname{sp}(a))$.
6. Show that $\operatorname{sp}(\varrho(a)) \subseteq \operatorname{sp}(a)$ and $\varrho(f(a))=f(\varrho(a))$ for every $f \in C(\operatorname{sp}(a))$ and miu-map $\varrho: \mathscr{A} \rightarrow \mathscr{B}$ into a $C^{*}$-algebra $\mathscr{B}$.
7. Given $f \in C(\operatorname{sp}(a))$ and $g \in C(f(\operatorname{sp}(a)))$ show that $g(f(a))=(g \circ f)(a)$. Show that $\left(a^{\alpha}\right)^{\beta}=a^{\alpha \beta}$ for $\alpha, \beta \in(0, \infty)$ and $a \in \mathscr{A}_{+}$.

Theorem We have $0 \leqslant a \leqslant b \Longrightarrow a^{\alpha} \leqslant b^{\alpha}$ for all positive elements $a$ and $b$ of a $C^{*}$-algebra $\mathscr{A}$, and $\alpha \in(0,1]$.
Proof (Based on [56.) Note that the result is trivial if $a$ and $b$ commute.
It suffices to show that $\left(a+\frac{1}{n}\right)^{\alpha} \leqslant\left(b+\frac{1}{n}\right)^{\alpha}$ for all $n$, because $\left(a+\frac{1}{n}\right)^{\alpha}$ norm converges to $a^{\alpha}$ as $n \rightarrow \infty$. In other words, it suffices to prove $a^{\alpha} \leqslant b^{\alpha}$ under the additional assumption that $a$ and $b$ are invertible. Note that $a^{0}$ and $b^{0}$ are defined for such invertible $a$ and $b$, because the function $(\cdot)^{0}:[0,1] \rightarrow \mathbb{C}$ is only discontinuous at 0 . Writing $E$ for the set of all $\alpha \in[0,1]$ for which $b \mapsto b^{\alpha}$ is monotone on positive, invertible elements of $\mathscr{A}$ we must prove that $E=(0,1]$, and we will in fact show that $E=[0,1]$. Since clearly $0,1 \in E$ it suffices to show that $E$ is convex. We'll do this by showing that $E$ is closed, and $\alpha, \beta \in E \Longrightarrow \frac{1}{2} \alpha+\frac{1}{2} \beta \in E$.
( $E$ is closed) Let $b$ be a positive and invertible element of $\mathscr{A}$. A moment's thought reveals it suffices to prove that $\alpha \mapsto b^{\alpha},[0,1] \rightarrow \mathscr{A}$ is continuous. And indeed it is being the composition of the map $\alpha \mapsto b^{\alpha}:[0,1] \rightarrow C(\operatorname{sp}(b))$, which is norm continuous, and the functional calculus $f \mapsto f(b): C(\operatorname{sp}(b)) \rightarrow \mathscr{A}$, which being a miu-map is norm continous as well.
$\left(\alpha, \beta \in E \Longrightarrow \frac{1}{2} \alpha+\frac{1}{2} \beta \in E\right)$ Let $\alpha, \beta \in E$. Let $a, b \in \mathscr{A}$ be positive VI and invertible with $a \leqslant b$. We must show that $a^{\alpha+\beta / 2} \leqslant b^{\alpha+\beta / 2}$. Since the map
$b^{\alpha+\beta / 4}(\cdot) b^{\alpha+\beta / 4}$ is positive (by 25 II), it suffices to show that $b^{-\frac{\alpha+\beta}{4}} a^{\frac{\alpha+\beta}{2}} b^{-\frac{\alpha+\beta}{4}} \leqslant$ 1 , that is, $\left\|b^{-\frac{\alpha+\beta}{4}} a^{\frac{\alpha+\beta}{2}} b^{-\frac{\alpha+\beta}{4}}\right\| \leqslant 1$.

For this, it seems, we must take a look under the hood of the theory of $C^{*}$-algebras: writing $\varrho(c):=\sup _{\lambda \in \operatorname{sp}(c)}|\lambda|$ for $c \in \mathscr{A}$, we know that $\varrho(c) \leqslant\|c\|$ for any $c$, and $\varrho(c)=\|c\|$ for self-adjoint $c$ by 16 II. Moreover, recall from 191 that $\operatorname{sp}(c d) \backslash\{0\}=\operatorname{sp}(d c) \backslash\{0\}$, and so $\varrho(c d)=\varrho(d c)$ for all $c, d \in \mathscr{A}$. Hence

$$
\begin{aligned}
\left\|b^{-\frac{\alpha+\beta}{4}} a^{\frac{\alpha+\beta}{2}} b^{-\frac{\alpha+\beta}{4}}\right\| & =\varrho\left(b^{-\frac{\alpha+\beta}{4}} a^{\frac{\alpha+\beta}{2}} b^{-\frac{\alpha+\beta}{4}}\right) \\
& =\varrho\left(b^{-\frac{\alpha+\beta}{4}} a^{\frac{\alpha+\beta}{2}} b^{-\frac{\alpha+\beta}{4}} b^{-\frac{\alpha-\beta}{4}} b^{\frac{\alpha-\beta}{4}}\right) \\
& =\varrho\left(b^{\frac{\alpha-\beta}{4}} b^{-\frac{\alpha+\beta}{4}} a^{\frac{\alpha+\beta}{2}} b^{-\frac{\alpha+\beta}{4}} b^{-\frac{\alpha-\beta}{4}}\right) \\
& =\varrho\left(b^{-\beta / 2} a^{\beta / 2} a^{\alpha / 2} b^{-\alpha / 2}\right) \\
& \leqslant\left\|b^{-\beta / 2} a^{\beta / 2}\right\|\left\|a^{\alpha / 2} b^{-\alpha / 2}\right\| \\
& =\left\|b^{-\beta / 2} a^{\beta} b^{-\beta / 2}\right\|^{1 / 2}\left\|b^{-\alpha / 2} a^{\alpha} b^{-\alpha / 2}\right\|^{1 / 2} \\
& \leqslant\left\|b^{-\beta / 2} b^{\beta} b^{-\beta / 2}\right\|^{1 / 2}\left\|b^{-\alpha / 2} b^{\alpha} b^{-\alpha / 2}\right\|^{1 / 2}=1
\end{aligned}
$$

and so we're done.

29 As a cherry on the cake, we use Gelfand's representation theorem 27 XXVII to get an equivalence between the categories $\left(\mathbf{c C}_{\mathrm{MIU}}^{*}\right)^{\text {op }}$ and $\mathbf{C H}$ of continuous maps between compact Hausdorff spaces.

To set the stage, we extend $X \mapsto C(X)$ to a functor $\mathbf{C H} \rightarrow\left(\mathbf{c C}_{\text {MIU }}^{*}\right)^{\text {op }}$ by sending a continuous function $f: X \rightarrow Y$ to the miu-map $C(f): C(Y) \rightarrow C(X)$ given by $C(f)(g)=g \circ f$ for $g \in C(Y)$, and we extend $\mathscr{A} \mapsto \operatorname{sp}(\mathscr{A})$ to a functor $\mathrm{sp}:\left(\mathbf{c C}_{\mathrm{MIU}}^{*}\right)^{\mathrm{op}} \rightarrow \mathbf{C H}$ by sending a miu-map $\varphi: \mathscr{A} \rightarrow \mathscr{B}$ to the continuous $\operatorname{map} \operatorname{sp}(\varphi): \operatorname{sp}(\mathscr{B}) \rightarrow \operatorname{sp}(\mathscr{A})$ given by $\operatorname{sp}(\varphi)(f)=f \circ \varphi$.

The Gelfand representations $\gamma_{\mathscr{A}}: \mathscr{A} \rightarrow C(\operatorname{sp}(\mathscr{A}))$ form a natural isomorphism from $C$ osp to the identity functor on $\left(\mathbf{c C}_{\mathrm{MIU}}^{*}\right)^{\text {op }}$. So to get an equivalence, it suffices to find a natural isomorphism from the identity on $\mathbf{C H}$ to spo $C$, which is provided by the following lemma.

II Lemma Let $X$ be a compact Hausdorff space, and let $\tau: C(X) \rightarrow \mathbb{C}$ be a miu-map. Then there is $x \in X$ with $\tau(f)=f(x)$ for all $f \in C(X)$.
III Proof Define $Z=\left\{x \in X: h(x) \neq 0\right.$ for some $h \in C(X)_{+}$with $\left.\tau(h)=0\right\}$. We'll prove $X \backslash Z$ contains exactly one point, $x_{0}$, and $\tau(f)=f\left(x_{0}\right)$ for all $f$.
IV To see that $X \backslash Z$ contains no more than one point, let $x, y \in X$ with $x \neq y$ be given; we will show that either $x \in Z$ or $y \in Z$. By the usual topological
trickery, we can find $f, g \in C(X)_{+}$with $f g=0, f(x)=1$ and $g(y)=1$. Then $0=\tau(f g)=\tau(f) \tau(g)$, so either $\tau(f)=0$ (and $x \in Z$ ), or $\tau(g)=0$ (and $y \in Z$ ).

That $X \backslash Z$ is non-empty follows from the following result (by taking $f=1$ ). For $f \in C(X)_{+}$with $f(x)>0 \Longrightarrow x \in Z$ for all $x \in X$ we have $\tau(f)=0$. Indeed, for each $x \in X$ with $f(x)>0$ (and so $x \in Z$ ) we can find $h \in C(X)_{+}$ with $\tau(h)=0$ and $h(x) \neq 0$. Then $f(x)<g(x)$ and $\tau(g)=0$ for $g:=\left(\frac{f(x)}{h(x)}+1\right) h$. By compactness, we can find $g_{1}, \ldots, g_{N} \in C(X)_{+}$with $\tau\left(g_{n}\right)=0$, such that for every $x \in X$ there is $n$ with $g(x)<f_{n}(x)$. Writing $g:=g_{1} \vee \cdots \vee g_{N}$, we have $0 \leqslant f \leqslant g$ and $\tau(g)=0$ (because by 26 II $\tau$ preserves finite infima). It follows that $\tau(f)=0$.
We now know that $X \backslash Z$ contains exactly one point, say $x_{0}$. To see that $\tau(f)=$ $f\left(x_{0}\right)$ for $f \in C(X)$, write $g:=\left(f-f\left(x_{0}\right)\right)^{*}\left(f-f\left(x_{0}\right)\right)$ and note that $g(x)>$ $0 \Longrightarrow x \neq x_{0} \Longrightarrow x \in Z$. Thus by V, we get $0=\tau(g)=\left|\tau(f)-f\left(x_{0}\right)\right|^{2}$, and so $\tau(f)=f\left(x_{0}\right)$.
Exercise Let $X$ be a compact Hausdorff space. Show that for every $x \in X$ the map $\delta_{x}: C(X) \rightarrow \mathbb{C}, f \mapsto f(x)$ is miu, and that the map $X \rightarrow \operatorname{sp}(C(X)), x \mapsto$ $\delta_{x}$ is a continuous bijection onto a compact Hausdorff space, and thus a homeomorphism.
Exercise As an application of the equivalence between $\left(\mathbf{c C}_{\text {MIU }}^{*}\right)^{\text {op }}$ and $\mathbf{C H}$, we will show that every injective miu-map between $C^{*}$-algebras is an isometry.

Show that an arrow $f: X \rightarrow Y$ in $\mathbf{C H}$ is mono iff injective, and epi iff surjective (using complete regularity of $Y$ ). Conclude that $f$ is both epi and mono in $\mathbf{C H}$ only if $f$ is an isomorphism (c.q. homeomorphism).

Let $\varrho: \mathscr{A} \rightarrow \mathscr{B}$ be an injective miu-map between $C^{*}$-algebras. Let $a$ be a self-adjoint element of $\mathscr{A}$. Show that $\varrho$ can be restricted to a miu-map $\sigma: C^{*}(a) \rightarrow C^{*}(\varrho(a))$, which is both epi and mono in $\mathbf{c C}_{\mathrm{MIU}}^{*}$. Conclude that $\sigma$ is an isomorphism, and thus $\|\varrho(a)\|=\|a\|$. Use the $C^{*}$-identity to extend the equality $\|\varrho(a)\|=\|a\|$ to (not necessarily self-adjoint) $a \in \mathscr{A}$.
Exercise Let $\varrho: \mathscr{A} \rightarrow \mathscr{B}$ be an injective miu-map. Show that $\varrho(\mathscr{A})$ is closed (using VIII). Conclude that $\varrho(\mathscr{A})$ is a $C^{*}$-subalgebra of $\mathscr{B}$ isomorphic to $\mathscr{A}$.

### 2.4.2 Representation by Bounded Operators

Let us prove that every $C^{*}$-algebra $\mathscr{A}$ is isomorphic to a $C^{*}$-algebra of bounded operators on some Hilbert space. We proceed as follows. To each p-map $\omega: \mathscr{A} \rightarrow$
$\mathbb{C}$ (see 10II) we assign a inner product $[\cdot, \cdot]_{\omega}$ on $\mathscr{A}$, which can be "completed" to a Hilbert space $\mathscr{H}_{\omega}$. Every element $a \in \mathscr{A}$ gives a bounded operator on $\mathscr{H}_{\omega}$ via the action $b \mapsto a b$, which in turn gives a miu-map $\varrho_{\omega}: \mathscr{A} \rightarrow \mathscr{B}\left(\mathscr{H}_{\omega}\right)$. In general $\varrho_{\omega}$ is not injective, but if $\Omega$ is a set of p -maps which separates the points of $\mathscr{A}$, then the composition

$$
\mathscr{A} \xrightarrow{\left\langle\varrho_{\omega}\right\rangle_{\omega \in \Omega}} \bigoplus_{\omega \in \Omega} \mathscr{B}\left(\mathscr{H}_{\omega}\right) \longrightarrow \mathscr{B}\left(\bigoplus_{\omega \in \Omega} \mathscr{H}_{\omega}\right)
$$

does give an injective miu-map $\varrho$, which restricts to an isomorphism 29IX from $\mathscr{A}$ to the $C^{*}$-algebra $\varrho(\mathscr{A})$ of bounded operators on $\bigoplus_{\omega \in \Omega} \mathscr{H}_{\omega}$, see 6 II.

The creation of $\varrho_{\omega}$ from $\omega$ is known as the Gelfand-Naimark-Segal (GNS) construction and will make a reappearance in the theory of von Neumann algebras (in 72 V ).

We take a somewhat utilitarian stance towards the GNS construction here, but there is much more that can be said about it: in the first chapter of my twin brother's thesis, [74], you'll see that the GNS construction has a certain universal property, and that it can be generalized to apply not only to maps of the form $\omega: \mathscr{A} \rightarrow \mathbb{C}$, but also to maps of the form $\varphi: \mathscr{A} \rightarrow \mathscr{B}$.

II Lemma For every p-map $\omega: \mathscr{A} \rightarrow \mathbb{C}$ on a $C^{*}$-algebra $\mathscr{A},[a, b]_{\omega}=\omega\left(a^{*} b\right)$ defines an inner product $[\cdot, \cdot]_{\omega}$ on $\mathscr{A}$ (see 4VIII).
III Proof Note that $[a, a]_{\omega} \equiv \omega\left(a^{*} a\right) \geqslant 0$ for each $a \in \mathscr{A}$, because $a^{*} a \geqslant 0$ (by 24 IV ); and $\overline{[a, b]}_{\omega}=[b, a]_{\omega}$ for $a, b \in \mathscr{A}$, because $\omega$ is involution preserving (by 10 IV ). Finally, it is clear that $[a, \cdot]_{\omega} \equiv \omega\left(a^{*} \cdot\right)$ is linear for each $a \in \mathscr{A}$.
IV Exercise Let $\omega: \mathscr{A} \rightarrow \mathbb{C}$ be a p-map on a $C^{*}$-algebra. Let us for a moment study the semi-norm $\|\cdot\|_{\omega}$ on $\mathscr{A}$ induced by the inner product $[\cdot, \cdot]_{\omega}$ (so $\|a\|_{\omega}=\omega\left(a^{*} a\right)^{1 / 2}$ ), because it plays an important role here, and all throughout the next chapter.

1. Use Cauchy-Schwarz (4XV) to prove Kadison's inequality: for all $a, b \in$ $\mathscr{A}$,

$$
\left|\omega\left(a^{*} b\right)\right|^{2} \leqslant \omega\left(a^{*} a\right) \omega\left(b^{*} b\right) .
$$

2. Show that $\|a b\|_{\omega} \leqslant\|\omega\|\|a\|\|b\|_{\omega}$ for all $a, b \in \mathscr{A}\left(\right.$ using $\left.a^{*} a \leqslant\|a\|^{2}\right)$.

Show that we do not always have $\|a b\|_{\omega} \leqslant\|\omega\|\|a\|_{\omega}\|b\|$.
(Hint: take $a=\left(\begin{array}{ll}0 & 0 \\ 0 & 1\end{array}\right)$ and $b=\frac{1}{2}\left(\begin{array}{ll}1 & 1 \\ 1 & 1\end{array}\right)$ from $\mathscr{A}=M_{2}$, and $\omega\left(\left(\begin{array}{cc}c & d \\ e & f\end{array}\right)\right)=c$.)
Show that neither always $\|a b\|_{\omega} \leqslant\|a\|_{\omega}\|b\|_{\omega}$, or $\left\|a^{*} a\right\|_{\omega}=\|a\|_{\omega}^{2}$.
(Hint: take $a=b=\frac{1}{2}\left(\begin{array}{ll}1 & 1 \\ 1 & 1\end{array}\right)$ from $\mathscr{A}=M_{2}$, and $\omega\left(\left(\begin{array}{cc}c & d \\ e & f\end{array}\right)\right)=c$.)

Give a counterexample to $\left\|a^{*}\right\|_{\omega}=\|a\|_{\omega}$.

Exercise Let us begin by showing how a complex vector space $V$ with inner product $[\cdot, \cdot]$ can be "completed" to a Hilbert space $\mathscr{H}$.

We will take for $\mathscr{H}$ the set of Cauchy sequences on $V$ modulo the following equivalence relation. Two Cauchy sequences $\left(a_{n}\right)_{n}$ and $\left(b_{n}\right)_{n}$ in $V$ are considered equivalent iff $\lim _{n}\left\|a_{n}-b_{n}\right\|=0$. We "embed" $V$ into $\mathscr{H}$ via the map $\eta: V \rightarrow \mathscr{H}$ which sends $a$ to the constant sequence $a, a, a, \ldots$. Note, however, that $\eta$ need not be injective: show that $\eta(a)=\eta(b)$ iff $\|a-b\|=0$ for all $a, b \in V$.

Show that $d\left(\left(a_{n}\right)_{n},\left(b_{n}\right)_{n}\right)=\lim _{n}\left\|a_{n}-b_{n}\right\|$ defines a metric on $\mathscr{H}$, that $\mathscr{H}$ is complete with respect to this metric, and that if $\left(a_{n}\right)_{n}$ is a Cauchy sequence in $V$, then $\left(\eta\left(a_{n}\right)\right)_{n}$ converges to the element $\left(a_{n}\right)_{n}$ of $\mathscr{H}$ (so $V$ is dense in $\mathscr{H}$ ).

Show that every uniformly continuous map $f: V \rightarrow X$ to a complete metric space $X$ can be uniquely extended to a uniformly continuous map $g: \mathscr{H} \rightarrow X$. (We say that $g$ extends $f$ when $f=g \circ \eta$.)

Show that addition, scalar multiplication, and inner product on $V$ (being uniformly continuous) can be uniquely extended to uniformly continuous operations on $\mathscr{H}$, and turn $\mathscr{H}$ into a Hilbert space. (Also verify that the extended inner product agrees with the complete metric we've already put on $\mathscr{H}$.)

Show that every bounded linear map $f: V \rightarrow \mathscr{K}$ to a Hilbert space $\mathscr{K}$ can be uniquely extended to a bounded linear map $g: \mathscr{H} \rightarrow \mathscr{K}$.
(Categorically speaking, Hilbert spaces form a reflexive subcategory of the category of bounded linear maps between complex vector spaces with an inner product.)

## Definition (Gelfand-Naimark-Segal construction)

Let $\omega: \mathscr{A} \rightarrow \mathbb{C}$ be a p-map on a $C^{*}$-algebra $\mathscr{A}$.
Let $\mathscr{H}_{\omega}$ denote the completion of $\mathscr{A}$ endowed with the inner product $[\cdot, \cdot]_{\omega}$ (see III) to a Hilbert space as discussed in V. Recall that we have an "embedding" $\eta_{\omega}: \mathscr{A} \rightarrow \mathscr{H}_{\omega}$ with $\left\langle\eta_{\omega}(a), \eta_{\omega}(b)\right\rangle=[a, b]_{\omega}$ for all $a, b \in \mathscr{A}$.

Since given $a \in \mathscr{A}$ the map $b \mapsto a b, \mathscr{A} \rightarrow \mathscr{A}$ is bounded with respect to $\|\cdot\|_{\omega}$ (because $\|a b\|_{\omega} \leqslant\|\omega\|\|a\|\|b\|_{\omega}$ by $\mathbb{V}$, it can be uniquely extended to a bounded linear map $\mathscr{H}_{\omega} \rightarrow \mathscr{H}_{\omega}$ (by the universal property of $\mathscr{H}_{\omega}$, see $V$ ), which we'll denote by $\varrho_{\omega}(a)$. So $\varrho_{\omega}(a)$ is the unique bounded linear map $\mathscr{H}_{\omega} \rightarrow \mathscr{H}_{\omega}$ with $\varrho_{\omega}(a)\left(\eta_{\omega}(b)\right)=\eta_{\omega}(a b)$ for all $b \in \mathscr{A}$.
Proposition The map $\varrho_{\omega}: \mathscr{A} \rightarrow \mathscr{B}\left(\mathscr{H}_{\omega}\right)$ given by VI is a miu-map.
Proof Let $a_{1}, a_{2} \in \mathscr{A}$ be given. Since $\varrho_{\omega}\left(a_{1}+a_{2}\right) \eta_{\omega}(b)=\eta_{\omega}\left(\left(a_{1}+a_{2}\right) b\right)=$ VIII $\eta_{\omega}\left(a_{1} b\right)+\eta_{\omega}\left(a_{2} b\right)=\left(\varrho_{\omega}\left(a_{1}\right)+\varrho_{\omega}\left(a_{2}\right)\right) \eta_{\omega}(b)$ for all $b \in \mathscr{A}$, and $\left\{\eta_{\omega}(b): b \in \mathscr{A}\right\}$
is dense in $\mathscr{H}_{\omega}$, we see that $\varrho_{\omega}\left(a_{1}+a_{2}\right)=\varrho_{\omega}\left(a_{1}\right)+\varrho_{\omega}\left(a_{2}\right)$. Since similarly $\varrho_{\omega}(\lambda a)=\lambda \varrho_{\omega}(a)$ for $\lambda \in \mathbb{C}$ and $a \in \mathscr{A}$, we see that $\varrho_{\omega}$ is linear.

Since $\varrho_{\omega}(1) \eta_{\omega}(b)=\eta_{\omega}(b)$ for all $b \in \mathscr{A}$, we have $\varrho_{\omega}(1) x=x$ for all $x \in \mathscr{H}_{\omega}$, and so $\varrho_{\omega}$ is unital, $\varrho_{\omega}(1)=1$.

To see that $\varrho_{\omega}$ is multiplicative, note that $\left(\varrho_{\omega}\left(a_{1}\right) \varrho_{\omega}\left(a_{2}\right)\right) \eta_{\omega}(b)=\eta_{\omega}\left(a_{1} a_{2} b\right)=$ $\varrho_{\omega}\left(a_{1} a_{2}\right) \eta_{\omega}(b)$ for all $a_{1}, a_{2}, b \in \mathscr{A}$.

Let $a \in \mathscr{A}$ be given. To show that $\varrho_{\omega}$ is involution preserving it suffices to prove that $\varrho_{\omega}\left(a^{*}\right)$ is the adjoint of $\varrho_{\omega}(a)$. Since $\left\langle\varrho_{\omega}\left(a^{*}\right) \eta_{\omega}(b), \eta_{\omega}(c)\right\rangle \equiv$ $\left[a^{*} b, c\right]_{\omega}=\omega\left(b^{*} a c\right)=[b, a c]_{\omega} \equiv\left\langle\eta_{\omega}(b), \varrho_{\omega}(a) \eta_{\omega}(c)\right\rangle$ for all $b, c \in \mathscr{A}$, and $\left\{\eta_{\omega}(b): b \in\right.$ $\mathscr{A}\}$ is dense in $\mathscr{H}_{\omega}$, we get $\left\langle\varrho_{\omega}\left(a^{*}\right) x, y\right\rangle=\left\langle x, \varrho_{\omega}(a) y\right\rangle$ for all $x, y \in \mathscr{H}_{\omega}$, and so $\varrho_{\omega}\left(a^{*}\right)=\varrho_{\omega}(a)^{*}$.

IX Definition Given a collection $\Omega$ of p-maps $\omega: \mathscr{A} \rightarrow \mathbb{C}$ on a $C^{*}$-algebra $\mathscr{A}$, let $\varrho_{\Omega}: \mathscr{A} \rightarrow \mathscr{B}\left(\mathscr{H}_{\Omega}\right)$ be the miu-map given by $\varrho_{\Omega}(a) x=\sum_{\omega \in \Omega} \varrho_{\omega}(a) x(\omega)$, where $\mathscr{H}_{\Omega}=\bigoplus_{\omega \in \Omega} \mathscr{H}_{\omega}$ (and $\varrho_{\omega}$ is as in VI).
X Proposition For a collection $\Omega$ of positive maps $\mathscr{A} \rightarrow \mathbb{C}$ on a $C^{*}$-algebra $\mathscr{A}$, the following are equivalent.

1. $\varrho_{\Omega}: \mathscr{A} \rightarrow \mathscr{B}\left(\mathscr{H}_{\Omega}\right)$ is injective;
2. $\Omega$ is center separating on $\mathscr{A}$ (see 21 II);
3. $\Omega^{\prime}=\left\{\omega\left(b^{*}(\cdot) b\right): b \in \mathscr{A}, \omega \in \Omega\right\}$ is order separating on $\mathscr{A}$.

In that case, $\varrho_{\Omega}(\mathscr{A})$ is a $C^{*}$-subalgebra of $\mathscr{B}\left(\mathscr{H}_{\Omega}\right)$, and $\varrho_{\Omega}$ restricts to a miuisomorphism from $\mathscr{A}$ to $\varrho_{\Omega}(\mathscr{A})$.
XI Proof It is clear that 3 entails 2 ,
XII 2 Let $a \in \mathscr{A}$ with $\varrho_{\Omega}(a)=0$ be given. We must show that $a=0$ (in order to show that $\varrho_{\Omega}$ is injective), and for this it is enough to prove that $a^{*} a=0$. Let $b \in \mathscr{A}$ and $\omega \in \Omega$ be given. Since $\Omega$ is center separating, it suffices to show that $0=\omega\left(b^{*} a^{*} a b\right) \equiv\|a b\|_{\omega}^{2}$. Since $\varrho_{\Omega}(a)=0$, we have $\varrho_{\omega}(a)=0$, thus $0=\varrho_{\omega}(a) \eta_{\omega}(b)=\eta_{\omega}(a b)$, and so $\|a b\|_{\omega}=0$. Hence $\varrho_{\Omega}$ is injective.
XIII 11 Let $a \in \mathscr{A}$ with $\omega\left(b^{*} a b\right) \geqslant 0$ for all $\omega \in \Omega$ and $b \in \mathscr{A}$ be given. We must show that $a \geqslant 0$. Since $\varrho_{\Omega}$ is injective, we know by 29 IX that $\varrho_{\Omega}(\mathscr{A})$ is a $C^{*}$-subalgebra of $\mathscr{B}\left(\mathscr{H}_{\Omega}\right)$, and $\varrho_{\Omega}$ restricts to a miu-isomorphism from $\mathscr{A}$ to $\varrho_{\Omega}(\mathscr{A})$. So in order to prove that $a \geqslant 0$, it suffices to show that $\varrho_{\Omega}(a) \geqslant 0$, and for this we must prove that $\varrho_{\omega}(a) \geqslant 0$ for given $\omega \in \Omega$. Since the vector states on $\mathscr{H}_{\omega}$ are order separating by 25 III it suffices to show that $\left\langle x, \varrho_{\omega}(a) x\right\rangle \geqslant 0$ for given $x \in \mathscr{H}_{\omega}$. Since $\left\{\eta_{\omega}(b): b \in \mathscr{A}\right\}$ is dense in $\mathscr{H}_{\omega}$, we only need to prove
that $0 \leqslant\left\langle\eta_{\omega}(b), \varrho_{\omega}(a) \eta_{\omega}(b)\right\rangle \equiv \omega\left(b^{*} a b\right)$ for given $b \in \mathscr{A}$, but this is true by assumption.
Theorem (Gelfand-Naimark) Every $C^{*}$-algebra $\mathscr{A}$ is miu-isomorphic to a $C^{*}$ - XIV algebra of operators on a Hilbert space.
Proof Since the states on $\mathscr{A}$ are separating $(22 \mathrm{VIII})$, and therefore center separating, the miu-map $\varrho_{\Omega}: \mathscr{A} \rightarrow \mathscr{B}\left(\mathscr{H}_{\Omega}\right)$ (defined in $\triangle$ restricts to a miuisomorphism from $\mathscr{A}$ onto the $C^{*}$-subalgebra $\varrho(\mathscr{A})$ of $\mathscr{B}\left(\mathscr{H}_{\Omega}\right)$ by X

### 2.5 Matrices over $C^{*}$-algebras

We have seen (in 4) that the $N \times N$-matrices ( $N$ being a natural number) over the complex numbers $\mathbb{C}$ form a $C^{*}$-algebra (denoted by $M_{N}$ ) by interpreting them as bounded operators on the Hilbert space $\mathbb{C}^{N}$, and proving that the bounded operators $\mathscr{B}(\mathscr{H})$ on any Hilbert space $\mathscr{H}$ form a $C^{*}$-algebra.

In this paragraph, we'll prove the analogous and more general result that the $N \times N$-matrices over a $C^{*}$-algebra $\mathscr{A}$ form a $C^{*}$-algebra by interpreting them as adjoinable module maps on the Hilbert $\mathscr{A}$-module $\mathscr{A}^{N}$, see 321 and 32 XIII.

Definition An ( $\mathscr{A}$-valued) inner product on a right $\mathscr{A}$-module $X\left(\mathscr{A}\right.$ being a $C^{*}$ algebra) is a map $\langle\cdot, \cdot\rangle: X \times X \rightarrow \mathscr{A}$ such that, for all $x, y \in X,\langle x, \cdot\rangle: X \rightarrow \mathscr{A}$ is a module map, $\langle x, x\rangle \geqslant 0$, and $\langle x, y\rangle=\langle y, x\rangle^{*}$. We say that such an inner product is definite if $\langle x, x\rangle=0 \Longrightarrow x=0$ for all $x \in X$.

A pre-Hilbert $\mathscr{A}$-module $X$ (where $\mathscr{A}$ is always assumed to be a $C^{*}$-algebra) is a right $\mathscr{A}$-module endowed with a definite inner product. Such $X$ is called a Hilbert $\mathscr{A}$-module when it is complete with respect to the norm we'll define in X.

Let $X$ and $Y$ be pre-Hilbert $\mathscr{A}$-module. We say that a map $T: X \rightarrow Y$ is adjoint to a map $S: Y \rightarrow X$ when

$$
\langle T x, y\rangle=\langle x, S y\rangle \quad \text { for all } x \in X \text { and } y \in Y .
$$

In that case, we call $T$ adjointable. It is not difficult to see that $T$ must be linear, and a module map, and adjoint to exactly one $S$, which we denote by $T^{*}$.
(Note that we did not require that $T$ is bounded, and in fact, it need not be, see 35IX However, if $T$ is bounded, then so is $T^{*}$, see Х, and if either $X$ or $Y$ is complete, then $T$ is automatically bounded, see 35 VI )

The vector space of adjoinable bounded module maps $T: X \rightarrow Y$ is denoted by $\mathscr{B}^{a}(X, Y)$, and we write $\mathscr{B}^{a}(X)=\mathscr{B}^{a}(X, X)$.

II Example We endow $\mathscr{A}^{N}$ (where $\mathscr{A}$ is a $C^{*}$-algebra and $N$ is a natural number) with the inner product $\langle x, y\rangle=\sum_{n} x_{n}^{*} y_{n}$, making it a Hilbert $\mathscr{A}$-module.

III Exercise Let $S$ and $T$ be adjoinable operators on a pre-Hilbert $\mathscr{A}$-module.

1. Show that $T^{*}$ is adjoint to $T$ (and so $T^{* *}=T$ ).
2. Show that $(T+S)^{*}=T^{*}+S^{*}$ and $(\lambda S)^{*}=\bar{\lambda} S^{*}$ for $\lambda \in \mathbb{C}$.
3. Show that $S T$ is adjoint to $T^{*} S^{*}$ (and so $\left.(S T)^{*}=T^{*} S^{*}\right)$.

IV Exercise Although a bounded linear map between Hilbert spaces is always adjoinable (see 55), a bounded module map between Hilbert $\mathscr{A}$-modules might have no adjoint as is revealed by the following example (based on [54, p. 447).

Prove that $J:=\{f \in C[0,1]: f(0)=0\}$ is a closed right ideal of $C[0,1]$, and thus a Hilbert $C[0,1]$-module.

Show that the inclusion $T: J \rightarrow C[0,1]$ is a bounded module map, which has no adjoint by proving that there is no $b \in J$ with $\langle b, a\rangle=T a \equiv a$ for all $a \in J$ (for if $T$ had an adjoint $T^{*}$, then $\left\langle T^{*} 1, a\right\rangle=\langle 1, T a\rangle=a$ for all $a \in J$ ).
$\checkmark$ Remark Note that part of the problem here is the lack of the obvious analogue to Riesz' representation theorem 5 IX for Hilbert $\mathscr{A}$-modules. One solution (taken in the literature) is to simply add Riesz' representation theorem as axiom giving us the self-dual Hilbert $\mathscr{A}$-modules. For those who like to keep Riesz' representation theorem a theorem, I'd like to mention that it is also possible to assume instead that the Hilbert $\mathscr{A}$-module is complete with respect to a suitable uniformity, as in done in my twin brother's thesis, 74 , see 149 V .

VI Proposition (Cauchy-Schwarz) We have $\langle x, y\rangle\langle y, x\rangle \leqslant\|\langle y, y\rangle\|\langle x, x\rangle$ for every inner product $\langle\cdot, \cdot\rangle$ on a right $\mathscr{A}$-module $X$, and $x, y \in X$.

VII Remark The symmetry-breaking norm symbols "||" cannot simply be removed from this version of Cauchy-Schwarz, because $0 \leqslant\langle x, y\rangle\langle y, x\rangle \leqslant\langle y, y\rangle\langle x, x\rangle$ would imply that $\langle y, y\rangle\langle x, x\rangle$ is positive, and self-adjoint, and thus that $\langle y, y\rangle$ and $\langle x, x\rangle$ commute, which is not always the case.
VIII Proof Let $\omega: \mathscr{A} \rightarrow \mathbb{C}$ be a state of $\mathscr{A}$. Since the states on $\mathscr{A}$ are order separating 22 VIII), it suffices to show that $\omega(\langle x, y\rangle\langle y, x\rangle) \leqslant\|\langle y, y\rangle\| \omega(\langle x, x\rangle)$. Noting
that $(u, v) \mapsto \omega(\langle u, v\rangle)$ is a complex-valued inner product on $X$, we compute

$$
\begin{array}{lll}
\omega(\langle x, y\rangle\langle y, x\rangle)^{2} & \\
\quad=\omega(\langle x, y\langle y, x\rangle\rangle)^{2} & \\
\quad \leqslant \omega(\langle x, x\rangle) \omega(\langle y\langle y, x\rangle, y\langle y, x\rangle\rangle) & \text { by Cauchy-Schwarz, 4XV } \\
\quad=\omega(\langle x, x\rangle) \omega(\langle x, y\rangle\langle y, y\rangle\langle y, x\rangle) & \\
& \leqslant \omega(\langle x, x\rangle) \omega(\langle x, y\rangle\langle y, x\rangle)\|\langle y, y\rangle\| & \text { since }\langle y, y\rangle \leqslant\|\langle y, y\rangle\| .
\end{array}
$$

It follows (also when $\omega(\langle x, y\rangle\langle y, x\rangle)=0)$, that

$$
\omega(\langle x, y\rangle\langle y, x\rangle) \leqslant\|\langle y, y\rangle\| \omega(\langle x, x\rangle),
$$

and so we're done.
Exercise Let $X$ be a pre-Hilbert $\mathscr{A}$-module. Verify that

1. $\|x\|=\|\langle x, x\rangle\|^{1 / 2}$ defines a norm $\|\cdot\|$ on $X$, and
2. $\|x b\| \leqslant\|x\|\|b\|$ and $\|\langle x, y\rangle\| \leqslant\|x\|\|y\|$ for all $x, y \in X$ and $b \in \mathscr{A}$.

Lemma For a linear map $T: X \rightarrow Y$ between pre-Hilbert $\mathscr{A}$-modules, and $B>0$, the following are equivalent.

1. $\|T x\| \leqslant B\|x\|$ for all $x \in X$ (that is, $T$ is bounded by $B$ );
2. $\|\langle y, T x\rangle\| \leqslant B\|y\|\|x\|$ for all $x \in X, y \in Y$.

Moreover, if $T$ is adjoinable, and bounded, then $\left\|T^{*}\right\|=\|T\|$.
Proof If $\|T x\| \leqslant B\|x\|$ for all $x \in X$, then $T$ is bounded, $\|T\| \leqslant B$, and therefore $\|\langle y, T x\rangle\| \leqslant\|y\|\|T x\| \leqslant B\|y\|\|x\|$ for all $x \in X$ and $y \in Y$ using V ,

On the other hand, if 2 holds, and $x \in X$ is given, then we have $\|T x\|^{2}=$ $\|\langle T x, T x\rangle\| \leqslant B\|T x\|\|x\|$, entailing $\|T x\| \leqslant B\|x\|$ (also when $\|T x\|=0$ ).

If $T$ is adjoinable, and bounded, then $\left\|\left\langle x, T^{*} y\right\rangle\right\|=\|\langle y, T x\rangle\| \leqslant\|T\|\|y\|\|x\|$ for all $x \in X, y \in Y$, so $\left\|T^{*}\right\| \leqslant\|T\|$, giving us that $T^{*}$ is bounded. Since by a similar reasoning $\|T\| \leqslant\left\|T^{*}\right\|$, we get $\|T\|=\left\|T^{*}\right\|$.
Exercise Show that $\left\|T^{*} T\right\|=\|T\|^{2}$ for every adjoinable bounded map $T$ on a XII pre-Hilbert $\mathscr{A}$-module. (Hint: adapt the proof of 4 XVI .)

XIII Proposition The adjoinable bounded module maps on a Hilbert $\mathscr{A}$-module form a $C^{*}$-algebra $\mathscr{B}^{a}(X)$ with composition as multiplication, adjoint as involution, and the operator norm as norm.
XIV Proof Considering 4 VII and XII the only thing that remains to be shown is that $\mathscr{B}^{a}(X)$ is closed (with respect to the operator norm) in the set of all bounded linear maps $\mathscr{B}(X)$. So let $T: X \rightarrow X$ be a bounded linear map which is the limit of a sequence $T_{1}, T_{2}, \ldots$ of adjoinable bounded module maps.

To see that $T$ has an adjoint, note that $\left\|T_{n}^{*}-T_{m}^{*}\right\|=\left\|\left(T_{n}-T_{m}\right)^{*}\right\|=$ $\left\|T_{n}-T_{m}\right\|$ for all $n, m$, and so $T_{1}^{*}, T_{2}^{*}, \ldots$ is a Cauchy sequence, and converges to some bounded operator $S$ on $X$. Since for $x, y \in X$ and $n$,

$$
\begin{aligned}
\|\langle S x, y\rangle-\langle x, T y\rangle\| & \leqslant\left\|\left\langle\left(S-T_{n}^{*}\right) x, y\right\rangle\right\|+\left\|\left\langle x,\left(T_{n}-T\right) y\right\rangle\right\| \\
& \leqslant\left\|S-T_{n}^{*}\right\|\|x\|\|y\|+\left\|T_{n}-T\right\|\|x\|\|y\|,
\end{aligned}
$$

we see that $\langle S x, y\rangle=\langle x, T y\rangle$, so $S$ is the adjoint of $T$, and $T$ is adjoinable.
XV Exercise Let $X$ be a Hilbert $\mathscr{A}$-module. Show that the vector states of $\mathscr{B}^{a}(X)$ are order separating (see 21 II ). Conclude that for an adjoinable operator $T$ on $X$

1. $T$ is self-adjoint iff $\langle x, T x\rangle$ is self-adjoint for all $x \in(X)_{1}$;
2. $0 \leqslant T$ iff $0 \leqslant\langle x, T x\rangle$ for all $x \in(X)_{1}$;
3. $\|T\|=\sup _{x \in(X)_{1}}\|\langle x, T x\rangle\|$ when $T$ is self-adjoint.
(Hint: adapt the proofs of 25 III and 25 V .)
XVI Corollary The operator $T^{*} T$ is positive in $\mathscr{B}^{a}(X)$ for every adjoinable operator $T: X \rightarrow Y$ between Hilbert $\mathscr{A}$-modules.
XVII Proof $\left\langle x, T^{*} T x\right\rangle=\langle T x, T x\rangle \geqslant 0$ for all $x \in X$, and so $T^{*} T \geqslant 0$ by 25 V ,
33 Exercise Let us consider matrices over a $C^{*}$-algebra $\mathscr{A}$.
4. Show that every $N \times M$-matrix $A$ (over $\mathscr{A}$ ) gives a bounded module $\operatorname{map} \underline{A}: \mathscr{A}^{N} \rightarrow \mathscr{A}^{M}$ via $\underline{A}\left(a_{1}, \ldots, a_{N}\right)=A\left(a_{1}, \ldots, a_{N}\right)$, which is adjoint to $\underline{A^{*}}$ (where $A^{*}=\left(A_{j i}^{*}\right)_{i j}$ is conjugate transpose).
5. Show that $A \mapsto \underline{A}$ gives a linear bijection between the vector space of $N \times$ $M$-matrices over $\mathscr{A}$ and the vector space of adjoinable bounded module maps $\mathscr{B}^{a}\left(\mathscr{A}^{N}, \mathscr{A}^{M}\right)$.
6. Show that $\underline{A} \circ \underline{B}=\underline{A B}$ when $A$ is an $N \times M$ and $B$ an $M \times K$ matrix.
7. Conclude that the vector space $M_{N} \mathscr{A}$ of $N \times N$-matrices over $\mathscr{A}$ is a $C^{*}-$ algebra with matrix multiplication (as multiplication), conjugate transpose as involution, and the operator norm (as norm, so $\|A\|=\|\underline{A}\|$ ).

Exercise Let us describe the positive $N \times N$ matrices over a $C^{*}$-algebras $\mathscr{A}$.

1. Show that an $N \times N$ matrix $A$ over $\mathscr{A}$ is positive iff $0 \leqslant \sum_{i, j} a_{i}^{*} A_{i j} a_{j}$ for all $a_{1}, \ldots, a_{N} \in \mathscr{A}$. (Hint: use 25 III.)
2. Show that the matrix $\left(\left\langle x_{i}, x_{j}\right\rangle\right)_{i j}$ is positive for all vectors $x_{1}, \ldots, x_{N}$ from a pre-Hilbert $\mathscr{A}$-module $X$.
3. Show that the matrix $\left(a_{i}^{*} a_{j}\right)_{i j}$ is positive for all $a_{1}, \ldots, a_{N} \in \mathscr{A}$.

Exercise Let $f: \mathscr{A} \rightarrow \mathscr{B}$ be a linear map between $C^{*}$-algebras.

1. Show that applying $f$ entry-wise to a $N \times N$ matrix $A$ over $\mathscr{A}$ (yielding the matrix $\left(f\left(A_{i j}\right)\right)_{i j}$ over $\left.\mathscr{B}\right)$ gives a linear map, which we'll denote by $M_{N} f: M_{N} \mathscr{A} \rightarrow M_{N} \mathscr{B}$.
2. The map $M_{N} f$ inherits some traits of $f$ : show that if $f$ is unital, then $M_{N} f$ unital; if $f$ is multiplicative, then $M_{N} f$ is multiplicative; and if $f$ is involution preserving, then so is $M_{N} f$.
3. However, show that $M_{n} f$ need not be positive when $f$ is positive, and that $M_{n} f$ need not be bounded, when $f$ is.

Let us briefly return to the completely positive maps (defined in 10 II ), to show that a map $f$ between $C^{*}$-algebras is completely positive precisely when $M_{N} f$ is positive for all $N$, and to give some examples of completely positive maps.

We also prove two lemmas stating special properties of completely positive maps (setting them apart from plain positive maps), that'll come in very handy later on. The first one is a variation on Cauchy-Schwarz (XIV), and the second one concerns the points at which a cpu-map is multiplicative (XVIII).

Completely positive maps are often touted as a good models for quantum processes (over plain positive maps) with an argument involving the tensor product, and while we agree, we submit that the absence of analogues of XIV and XVIII for positive maps is already enough to make complete positivity indispensable.

II Lemma For a linear map $f: \mathscr{A} \rightarrow \mathscr{B}$ between $C^{*}$-algebras, and natural number $N$, the following are equivalent.

1. $M_{N} f: M_{N} \mathscr{A} \rightarrow M_{N} \mathscr{B}$ is positive;
2. $\sum_{i j} b_{i}^{*} f\left(a_{i}^{*} a_{j}\right) b_{j} \geqslant 0$ for all $a \equiv\left(a_{1}, \ldots, a_{N}\right) \in \mathscr{A}^{N}$ and $b \in \mathscr{B}^{N}$;
3. the matrix $\left(f\left(a_{i}^{*} a_{j}\right)\right)_{i j}$ is positive in $M_{N} \mathscr{B}$ for all $a \in \mathscr{A}^{N}$.

III Proof Recall that $M_{N} f$ is positive iff $\left(M_{N} f\right)(C)$ is positive for all $C \in\left(M_{N} \mathscr{A}\right)_{+}$. The trick is to note that such $C$ can be written as $C \equiv A^{*} A$ for some $A \in M_{N} \mathscr{A}$, and thus as $C \equiv\left(a_{1}^{T}\right)^{*} a_{1}^{T}+\cdots+\left(a_{N}^{T}\right)^{*} a_{N}^{T}$, where $a_{n} \equiv\left(A_{n 1}, \ldots, A_{n N}\right)$ is the $n$-th row of $A$. Hence $M_{N} f$ is positive iff $\left(M_{N} f\right)\left(\left(a^{T}\right)^{*} a^{T}\right) \equiv\left(f\left(a_{i}^{*} a_{j}\right)\right)_{i, j}$ is positive for all tuples $a \in \mathscr{A}^{N}$. Since $B \in M_{N} \mathscr{B}$ is positive iff $\langle b, B b\rangle \geqslant 0$ for all $b \in \mathscr{B}^{N}$, we conclude: $M_{N} f$ is positive iff $0 \leqslant\left\langle b,\left(M_{N} f\right)\left(\left(a^{T}\right)^{*} a^{T}\right) b\right\rangle=\sum_{i j} b_{i}^{*} f\left(a_{i}^{*} a_{j}\right) b_{j}$ for all $a \in \mathscr{A}^{N}$ and $b \in \mathscr{B}^{N}$.
IV Exercise Conclude from 团 that a linear map $f$ between $C^{*}$-algebras is completely positive iff $M_{N} f$ is positive for all $N$ iff for all $N$ and $a \in \mathscr{A}^{N}$ the matrix $\left(f\left(a_{i}^{*} a_{j}\right)\right)_{i, j}$ is positive in $M_{N} \mathscr{B}$.

Deduce that the composition of cp-maps is completely positive.
Show that a mi-map $f$ is completely positive. (Hint: $M_{N} f$ is a mi-map too.)
V Exercise Show that given a $C^{*}$-algebra $\mathscr{A}$, the following maps are completely positive:

1. $b \mapsto a^{*} b a: \mathscr{A} \rightarrow \mathscr{A}$ for every $a \in \mathscr{A}$;
2. $T \mapsto S^{*} T S: \mathscr{B}^{a}(X) \rightarrow \mathscr{B}^{a}(Y)$ for every adjoinable operator $S: Y \rightarrow X$ between Hilbert $\mathscr{A}$-modules;
3. $T \mapsto\langle x, T x\rangle, \mathscr{B}^{a}(X) \rightarrow \mathscr{A}$ for every element $x$ of a Hilbert $\mathscr{A}$-module $X$.

VI Exercise Show that the product of a family of $C^{*}$-algebras $\left(\mathscr{A}_{i}\right)_{i}$ in the category $\mathbf{C}_{\text {CPSU }}^{*}$ (see 10 II ) is given by $\bigoplus_{i} \mathscr{A}_{i}$ with the same projections as in 181

Show that the equaliser of miu-maps $f, g: \mathscr{A} \rightarrow \mathscr{B}$ in $\mathbf{C}_{\text {CPSU }}^{*}$ is the inclusion of the $C^{*}$-subalgebra $\{a \in \mathscr{A}: f(a)=g(a)\}$ of $\mathscr{A}$ into $\mathscr{A}$.
VII Lemma Let $\mathscr{A}$ be a commutative $C^{*}$-algebra, and let $N$ be a natural number. The set of matrices of the form $\sum_{k} a_{k} B_{k}$, where $a_{1}, \ldots, a_{K} \in \mathscr{A}_{+}$and $B_{1}, \ldots, B_{K} \in M_{N}(\mathbb{C})_{+}$, is norm dense in $\left(M_{N} \mathscr{A}\right)_{+}$.

Proof Since $\mathscr{A}$ is isomorphic to $C(X)$ for some compact Hausdorff space $X$
VIII (by 27 XXVIII$)$ ), we may as well assume that $\mathscr{A} \equiv C(X)$.

Let $A \in M_{N}(C(X))_{+}$and $\varepsilon>0$ be given. We're looking for $g_{1}, \ldots, g_{K} \in$ $C(X)_{+}$and $B_{1}, \ldots, B_{K} \in\left(M_{N}\right)_{+}$with $\left\|A-\sum_{k} g_{k} B_{k}\right\| \leqslant \varepsilon$. Since $A(x):=$ $\left(A_{i j}(x)\right)_{i j}$ gives a continuous map $X \rightarrow M_{N}$, the sets $U_{x}=\{y \in X: \| A(x)-$ $A(y) \|<\varepsilon\}$ form an open cover of $X$. By compactness of $X$ this cover has a finite subcover; there are $x_{1}, \ldots, x_{K} \in X$ with $U_{x_{1}} \cup \cdots \cup U_{x_{K}}=X$.

Let $y \in X$ be given. Since $y \in U_{x_{k}}$ for some $k$, there is, by complete regularity of $X$, a function $f_{y} \in(C(X))_{+}$with $f_{y}(y)>0$ and $\operatorname{supp}\left(f_{y}\right) \subseteq U_{x_{k}}$. Since the open subsets $\operatorname{supp}\left(f_{y}\right)$ cover $X$ there are (by compactness of $X$ ) finitely many $y_{1}, \ldots y_{L}$ with $X=\operatorname{supp}\left(f_{y_{1}}\right) \cup \cdots \cup \operatorname{supp}\left(f_{y_{L}}\right)$, and so $\sum_{\ell} f_{y_{\ell}}>0$. Let us group together the $f_{y_{\ell}}$ s: pick for each $\ell$ an $k_{\ell}$ with $\operatorname{supp}\left(f_{y_{\ell}}\right) \subseteq U_{x_{k_{\ell}}}$, and let $g_{k}:=\sum\left\{f_{\ell}: k_{\ell}=k\right\}$. Then $g_{k} \in(C(X))_{+}, \operatorname{supp}\left(g_{k}\right) \subseteq U_{k}$, and $\sum_{k} g_{k}>0$. Upon replacing $g_{k}$ with $\left(\sum_{\ell} g_{\ell}\right)^{-1} g_{k}$ if necessary, we see that $\sum_{k} g_{k}=1$.

Since $\operatorname{supp}\left(g_{k}\right) \subseteq U_{x_{k}}$, we have $-\varepsilon \leqslant A(x)-A\left(x_{k}\right) \leqslant \varepsilon$ for all $x \in \operatorname{supp}\left(g_{k}\right)$, and so $-g_{k}(x) \varepsilon \leqslant g_{k}(x) A(x)-g_{k}(x) A\left(x_{k}\right) \leqslant g_{k}(x) \varepsilon$ for all $x \in X$, that is, $-g_{k} \varepsilon \leqslant g_{k} A-g_{k} A\left(x_{k}\right) \leqslant g_{k} \varepsilon$. Summing yields $-\varepsilon \leqslant A-\sum_{k} g_{k} A\left(x_{k}\right) \leqslant \varepsilon$, and so $\left\|A-\sum_{k} g_{k} A\left(x_{k}\right)\right\| \leqslant \varepsilon$.

Proposition Let $f: \mathscr{A} \rightarrow \mathscr{B}$ be a positive map between $C^{*}$-algebras. If either $\mathscr{A}$ or $\mathscr{B}$ is commutative, then $f$ is completely positive.
Proof Suppose that $\mathscr{B}$ is commutative, and let $a_{1}, \ldots, a_{N} \in \mathscr{A}, b_{1}, \ldots, b_{N} \in \mathscr{B}$ be given. We must show that $\sum_{i, j} b_{i}^{*} f\left(a_{i}^{*} a_{j}\right) b_{j}$ is positive. This follows from the observation that $\omega\left(\sum_{i, j} b_{i}^{*} f\left(a_{i}^{*} a_{j}\right) b_{j}\right)=\omega\left(f\left(\sum_{i, j}\left(a_{i} \omega\left(b_{i}\right)\right)^{*} a_{j} \omega\left(b_{j}\right)\right)\right) \geqslant 0$ for every $\omega \in \operatorname{sp}(\mathscr{A})$.
Suppose instead that $\mathscr{A}$ is commutative, and let $A \in\left(M_{N} \mathscr{A}\right)_{+}$be given for some natural number $N$. We must show that $\left(M_{N} f\right)(A)$ is positive in $M_{N} \mathscr{B}$. By VII, the problem reduces to the case that $A \equiv a B$ where $a \in \mathscr{A}_{+}$and $B \in\left(M_{N}\right)_{+}$. Since $\left(M_{N} f\right)(a B) \equiv f(a) B$ is clearly positive in $M_{N} \mathscr{B}$, we are done.
Lemma For a positive matrix $A \equiv\left(\begin{array}{cc}p & a \\ a^{*} & q\end{array}\right)$ over a $C^{*}$-algebra $\mathscr{A}$ we have

$$
a^{*} a \leqslant\|p\| q \quad \text { and } \quad a a^{*} \leqslant\|q\| p .
$$

In particular, if $p=0$ or $q=0$, then $a=a^{*}=0$.
Proof Since $(x, y) \mapsto\langle x, A y\rangle$ gives an $\mathscr{A}$-valued inner product on $\mathscr{A}^{2}$,

$$
\begin{aligned}
a a^{*} & =\left\langle\binom{ 1}{0}, A\binom{0}{1}\right\rangle\left\langle\binom{ 0}{1}, A\binom{1}{0}\right\rangle \\
& \leqslant\left\|\left\langle\binom{ 0}{1}, A\binom{0}{1}\right\rangle\right\|\left\langle\binom{ 1}{0}, A\binom{1}{0}\right\rangle=\|q\| p
\end{aligned}
$$

by Cauchy-Schwarz (see 32 VI).
By a similar reasoning, we get $a^{*} a \leqslant\|p\| q$.
XIV Lemma We have $f\left(a^{*} b\right) f\left(b^{*} a\right) \leqslant\left\|f\left(b^{*} b\right)\right\| f\left(a^{*} a\right)$ for every p-map $f: \mathscr{A} \rightarrow \mathscr{B}$ between $C^{*}$-algebras and $a, b \in \mathscr{A}$, provided that $M_{2} f$ is positive.
XV Proof Since writing $x \equiv(a, b) \in \mathscr{A}^{2}$, the $2 \times 2$ matrix $\left(x^{T}\right)^{*} x^{T} \equiv\left(\begin{array}{cc}a^{*} a & a^{*} b \\ b^{*} a & b^{*} b\end{array}\right)$ in $M_{2} \mathscr{A}$ is positive, the $2 \times 2$ matrix $T:=\left(\begin{array}{cc}f\left(a^{*} a\right) & f\left(a^{*} b\right) \\ f\left(b^{*} a\right) & f\left(b^{*} b\right)\end{array}\right)$ in $M_{2} \mathscr{B}$ is positive. Thus we get $f\left(a^{*} b\right) f\left(b^{*} a\right) \leqslant\left\|f\left(b^{*} b\right)\right\| f\left(a^{*} a\right)$ by XII.
XVI Corollary $\|f\|=\|f(1)\|$ for every cp-map $f: \mathscr{A} \rightarrow \mathscr{B}$ between $C^{*}$-algebras.
XVII Proof Let $a \in \mathscr{A}$ be given. It suffices to show that $\|f(a)\| \leqslant\|f(1)\|\|a\|$ so that $\|f\| \leqslant\|f(1)\|$, because we already know that $\|f(1)\| \leqslant\|f\|\|1\|=\|f\|$. Since $\left\|f\left(a^{*} a\right)\right\| \leqslant\|f(1)\|\left\|a^{*} a\right\|$ by 20 II, we have $\|f(a)\|^{2}=\left\|f(a)^{*} f(a)\right\|=$ $\left\|f\left(a^{*} 1\right) f\left(1^{*} a\right)\right\| \leqslant\left\|f\left(1^{*} 1\right)\right\|\left\|f\left(a^{*} a\right)\right\| \leqslant\|f(1)\|\|f(1)\|\left\|a^{*} a\right\|=\|f(1)\|^{2}\|a\|^{2}$ by XIV, and so $\|f(a)\| \leqslant\|f(1)\|\|a\|$.
XVIII Lemma (Choi [10]) We have $f(a)^{*} f(a) \leqslant f\left(a^{*} a\right)$ for every cpu-map $f: \mathscr{A} \rightarrow \mathscr{B}$ between $C^{*}$-algebras, and $a \in \mathscr{A}$. Moreover, if $f\left(a^{*} a\right)=f(a)^{*} f(a)$ for some $a \in$ $\mathscr{A}$, then $f(b a)=f(b) f(a)$ for all $b \in \mathscr{A}$.
XIX Proof By XIV we have $f(a)^{*} f(a)=f\left(a^{*} 1\right) f\left(1^{*} a\right) \leqslant\left\|f\left(1^{*} 1\right)\right\| f\left(a^{*} a\right)=f\left(a^{*} a\right)$, where we used that $f$ is unital, viz. $f(1)=1$.

Let $a, b \in \mathscr{A}$ be given, and assume that $f\left(a^{*} a\right)=f(a)^{*} f(a)$. Instead of $f(b a)=f(b) f(a)$ we'll prove that $f\left(a^{*} b\right)=f(a)^{*} f(b)$ (but this is nothing more than a reformulation). Since $M_{2} f$ is cp, we have, writing $A \equiv\left(\begin{array}{ll}a & b \\ 0 & 0\end{array}\right)$,

$$
\begin{aligned}
\left(\begin{array}{ll}
f(a)^{*} f(a) & f(a)^{*} f(b) \\
f(b)^{*} f(a) & f(b)^{*} f(b)
\end{array}\right) & =\left(M_{2} f\right)(A)^{*}\left(M_{2} f\right)(A) \\
& \leqslant\left(M_{2} f\right)\left(A^{*} A\right)=\left(\begin{array}{ll}
f\left(a^{*} a\right) & f\left(a^{*} b\right) \\
f\left(b^{*} a\right) & f\left(b^{*} b\right)
\end{array}\right) .
\end{aligned}
$$

Hence (using that $\left.f\left(a^{*} a\right)=f(a)^{*} f(a)\right)$ the following matrix is positive.

$$
\left(\begin{array}{cc}
0 & f\left(a^{*} b\right)-f(a)^{*} f(b) \\
f\left(b^{*} a\right)-f(b)^{*} f(a) & f\left(b^{*} b\right)-f(b)^{*} f(b)
\end{array}\right)
$$

But then by XII we have $f\left(a^{*} b\right)-f(a)^{*} f(b)=0$.

### 2.6 Towards von Neumann Algebras

Let us work towards the subject of the next chapter, von Neumann algebras, by
pointing out two special properties of $\mathscr{B}(\mathscr{H})$ on which the definition of a von Neumann algebra is based, namely that

1. any norm-bounded directed subset of self-adjoint operators on $\mathscr{H}$ has a supremum $\left(\right.$ in $\left.\mathscr{B}(\mathscr{H})_{\mathbb{R}}\right)$, and
2. all vector functionals $\langle x,(\cdot) x\rangle: \mathscr{B}(\mathscr{H}) \rightarrow \mathbb{C}$ preserve these suprema.

We'll end the chapter by showing in 39 IX that every functional on $\mathscr{B}(\mathscr{H})$ that preserves the aforementioned suprema is a (possibly infinite) sum of vector functionals.

### 2.6.1 Directed Suprema

Theorem (Uniform Boundedness) A set $\mathscr{F}$ of bounded linear maps from a complete normed vector space $\mathscr{X}$ to a normed vector space $\mathscr{Y}$ is bounded in the sense that $\sup _{T \in \mathscr{F}}\|T\|<\infty$ provided that $\sup _{T \in \mathscr{F}}\|T x\|<\infty$ for all $x \in \mathscr{X}$.
Proof Based on 66.
Let $r>0$ and $T \in \mathscr{F}$ be given. Writing $B_{r}(x)=\{y \in \mathscr{X}:\|x-y\| \leqslant r\}$ for the ball around $x \in \mathscr{X}$ with radius $r$, note that $r\|T\|=\sup _{\xi \in B_{r}(0)}\|T \xi\|$ almost by definition of the operator norm. We will need the less obvious fact that $r\|T\| \leqslant \sup _{\xi \in B_{r}(x)}\|T \xi\|$ for every $x \in \mathscr{X}$.

To see why this is true, note that for $\xi \in B_{r}(0)$ either $\|T \xi\| \leqslant\|T(x+\xi)\|$ or $\|T \xi\| \leqslant\|T(x-\xi)\|$, because we would otherwise have $2\|T \xi\|=\| T(x+\xi)-$ $T(x-\xi)\|\leqslant\| T(x+\xi)\|+\| T(x-\xi)\|<2\| T \xi \|$. Hence $r\|T\|=\sup _{\xi \in B_{r}(0)}\|T \xi\| \leqslant$ $\sup _{\xi \in B_{r}(x)}\|T \xi\|$.
Suppose towards a contradiction that $\sup _{T \in \mathscr{F}}\|T\|=\infty$, and pick $T_{1}, T_{2}, \ldots$ with $\left\|T_{n}\right\| \geqslant n 3^{n}$. Using IV. choose $x_{1}, x_{2}, \ldots$ in $\mathscr{X}$ with $\left\|x_{n}-x_{n-1}\right\| \leqslant 3^{-n}$ and $\left\|T_{n} x_{n}\right\| \geqslant \frac{2}{3} 3^{-n}\left\|T_{n}\right\|$, so that $\left(x_{n}\right)_{n}$ is a Cauchy sequence, and therefore converges to some $x \in \mathscr{X}$. Note that $\left\|x-x_{n}\right\| \leqslant \frac{1}{2} 3^{-n}$ (because $\sum_{k=0}^{\infty} 3^{-k}=\frac{3}{2}$ ), and so $\left\|T_{n} x\right\| \geqslant\left\|T_{n} x_{n}\right\|-\left\|T_{n}\left(x_{n}-x\right)\right\| \geqslant \frac{2}{3} 3^{-n}\left\|T_{n}\right\|-\frac{1}{2} 3^{-n}\left\|T_{n}\right\| \geqslant \frac{1}{6} n$, which contradicts the assumption that $\sup _{T \in \mathscr{F}}\|T x\|<\infty$.
Theorem Let $T: X \rightarrow Y$ be an adjoinable map between pre-Hilbert $\mathscr{A}$ modules. If either $X$ or $Y$ is complete, then $T$ and $T^{*}$ are bounded.
Proof We may assume without loss of generality that $X$ is complete (by swapping $T$ for $T^{*}$ and $X$ with $Y$ if necessary).

Note that for every $y \in Y$, the linear map $\langle y, T \cdot\rangle \equiv\left\langle T^{*} y, \cdot\right\rangle: Y \rightarrow \mathscr{A}$ is bounded, because $\left\|\left\langle T^{*} y, x\right\rangle\right\| \leqslant\left\|T^{*} y\right\|\|x\|$ for all $x \in X$ (see 32 Vl ).

Since on the other hand, $\|\langle y, T x\rangle\| \leqslant\|y\|\|T x\| \leqslant\|T x\|$ for all $x \in X$ and $y \in Y$ with $\|y\| \leqslant 1$, we have $\sup _{\|y\| \leqslant 1}\|\langle y, T x\rangle\| \leqslant\|T x\|<\infty$ for all $x \in X$, and thus $B:=\sup _{\|y\| \leqslant 1}\|\langle y, T \cdot\rangle\|<\infty$ by $\Pi$.

It follows that $\|\langle y, T x\rangle\| \leqslant B\|y\|\|x\|$ for all $y \in Y$ and $x \in X$, and thus $T$ and $T^{*}$ are bounded, by 32 X
VIII Remark As a special case of the preceding theorem we get the fact, known as the Hellinger-Toeplitz theorem, that every symmetric operator on a Hilbert space is bounded.
IX Example The condition that either $X$ or $Y$ be complete may not be dropped: the linear map $T: c_{00} \rightarrow c_{00}$ given by $T \alpha=\left(n \alpha_{n}\right)_{n}$ for $\alpha \in c_{00}$ is self-adjoint, but not bounded, because $T$ maps $\left(1, \frac{1}{2}, \ldots, \frac{1}{n}, 0,0, \ldots\right)$ having 2 -norm below $\frac{\pi}{\sqrt{6}}$ to $(1,1, \ldots, 1,0,0, \ldots)$, which has 2 -norm equal to $\sqrt{n}$.

36 Definition A Hilbert $\mathscr{A}$-module $X$ is self-dual when every bounded module map $r: X \rightarrow \mathscr{A}$ is of the form $r \equiv\langle y,(\cdot)\rangle$ for some $y \in X$.

II Example By Riesz' representation theorem (5IX) every Hilbert space is selfdual.

III Exercise Show that given a $C^{*}$-algebra $\mathscr{A}$ the Hilbert $\mathscr{A}$-module $\mathscr{A}^{N}$ of $N$ tuples is self dual.
IV Definition Let us say that a (bounded) form on Hilbert $\mathscr{A}$-modules $X$ and $Y$ is a map $[\cdot, \cdot]: X \times Y \rightarrow \mathscr{A}$ such that $[x, \cdot]: Y \rightarrow \mathscr{A}$ and $[\cdot, y]^{*}: X \rightarrow \mathscr{A}$ are (bounded) module maps for all $x \in X$ and $y \in Y$.
$\vee \quad$ Proposition For every bounded form $[\cdot, \cdot]: X \times Y \rightarrow \mathscr{A}$ on self-dual Hilbert $\mathscr{A}$ modules $X$ and $Y$ there is a unique adjoinable bounded module map $T: X \rightarrow Y$. with $[x, y] \equiv\langle T x, y\rangle$ for all $x \in X$ and $y \in Y$.
VI Proof Let $x \in X$ be given. Since $[x, \cdot]: Y \rightarrow \mathscr{A}$ is a a bounded module map, and $Y$ is self-dual, there is a unique $T x \in Y$ with $[x, y]=\langle T x, y\rangle$ for all $y \in Y$, giving a map $T: X \rightarrow Y$. For a similar reason we get a map $S: Y \rightarrow X$ with $\langle S y, x\rangle=[x, y]^{*}$ for all $x \in X$ and $y \in Y$. Since $S$ and $T$ are clearly adjoint, they are bounded module maps by 35 VI .

37 Another consequence of 35 I is this:
II Proposition Given a net $\left(y_{\alpha}\right)_{\alpha}$ in a Hilbert space $\mathscr{H}$ for which $\left\langle y_{\alpha}, x\right\rangle$ is Cauchy
and bounded for every $x \in \mathscr{H}$, there is a unique $y \in \mathscr{H}$ with $\langle y, x\rangle=\lim _{\alpha}\left\langle y_{\alpha}, x\right\rangle$ for all $y \in \mathscr{H}$.
Proof To obtain $x$, we want to apply Riesz' representation theorem $\sqrt{5 \mathrm{IX})}$ to the linear map $f: \mathscr{H} \rightarrow \mathbb{C}$ defined by $f(x)=\lim _{\alpha}\left\langle y_{\alpha}, x\right\rangle$, but must first show that $f$ is bounded. For this it suffices to show that $\sup _{\alpha}\left\|\left\langle y_{\alpha},(\cdot)\right\rangle\right\|<\infty$, and this follows by 35 II from the assumption that $\sup _{\alpha}\left|\left\langle y_{\alpha}, x\right\rangle\right|<\infty$ for every $x \in \mathscr{H}$.

By Riesz' representation theorem 5 (IX), there is a unique $x \in \mathscr{H}$ with $\langle y, x\rangle=f(x) \equiv \lim _{\alpha}\left\langle y_{\alpha}, x\right\rangle$ for all $x \in \mathscr{H}$, and so we're done.
Remark The condition in $\Pi$ that the net $\left(\left\langle y_{\alpha}, x\right\rangle\right)_{\alpha}$ be bounded for every $x$ may not be omitted (even though $\left(\left\langle y_{\alpha}, x\right\rangle\right)_{\alpha}$ being Cauchy is eventually bounded).

To see this, consider a linear map $f: \mathscr{H} \rightarrow \mathbb{C}$ on a Hilbert space $\mathscr{H}$ which is not bounded. We claim that there is a net $\left(y_{\alpha}\right)_{\alpha}$ in $\mathscr{H}$ with $f(x)=\lim _{\alpha}\left\langle y_{\alpha}, x\right\rangle$ for all $x \in \mathscr{H}$, and so there can be no $y \in \mathscr{H}$ with $\langle y, x\rangle=\lim _{\alpha}\left\langle y_{\alpha}, x\right\rangle$ for all $x \in \mathscr{H}$, because that would imply that $f$ is bounded.

To create this net, note that $f$ is bounded on the span $\langle F\rangle$ of every finite subset $F \equiv\left\{x_{1}, \ldots, x_{n}\right\}$ of vectors from $\mathscr{H}$, and so by Riesz' representation theorem 5IX applied to $f$ restricted to closed subspace $\langle F\rangle$ of $\mathscr{H}$ there is a unique $y_{F} \in\langle F\rangle$ such that $f(x)=\left\langle y_{F}, x\right\rangle$ for all $x \in\langle F\rangle$.

These $y_{F}$ 's form a net in $\mathscr{H}$ (when we order the finite subsets $F$ of $\mathscr{H}$ by inclusion), which approximates $f$ in the sense that $f(x)=\lim _{F}\left\langle y_{F}, x\right\rangle$ for every $x \in \mathscr{H}$, (because $f(x)=\left\langle y_{F}, x\right\rangle$ for every $F$ with $\{x\} \subseteq F$ ).
Definition Let $\mathscr{H}$ be a Hilbert space.

1. The weak operator topology (WOT) on $\mathscr{B}(\mathscr{H})$ is the least topology with respect to which $T \mapsto\langle x, T x\rangle, \mathscr{B}(\mathscr{H}) \rightarrow \mathbb{C}$ is continuous for every $x \in \mathscr{H}$. So a net $\left(T_{\alpha}\right)_{\alpha}$ converges to $T$ in $\mathscr{B}(\mathscr{H})$ with respect to the weak operator topology iff $\left\langle x, T_{\alpha} x\right\rangle \rightarrow\langle x, T x\rangle$ as $\alpha \rightarrow \infty$ for all $x \in \mathscr{H}$.
2. The strong operator topology (SOT) on $\mathscr{B}(\mathscr{H})$ is the least topology with respect to which $T \mapsto\|T x\| \equiv\left\langle x, T^{*} T x\right\rangle^{1 / 2}$ is continuous for every $x \in \mathscr{H}$.
So a net $\left(T_{\alpha}\right)_{\alpha}$ converges to $T$ in $\mathscr{B}(\mathscr{H})$ with respect to the strong operator topology iff $\left\|T_{\alpha} x-T x\right\| \rightarrow 0$ as $\alpha \rightarrow \infty$ for all $x \in \mathscr{H}$.

Remark Although we'll only make use of the weak operator topology we have nonetheless included the definition of the strong operator topology here for comparison with the ultrastrong topology that appears in the next chapter.

VII Lemma Let $\left(T_{\alpha}\right)_{\alpha}$ be a net of bounded operators on a Hilbert space $\mathscr{H}$ such that $\left(\left\langle x, T_{\alpha} x\right\rangle\right)$ is Cauchy and bounded for every $x \in \mathscr{H}$.

Then $\left(T_{\alpha}\right)_{\alpha}$ WOT-converges to some bounded operator $T$ in $\mathscr{B}(\mathscr{H})$.
VIII Proof Let $x, y \in \mathscr{H}$ be given. Since by a simple computation

$$
\left\langle y, T_{\alpha} x\right\rangle=\frac{1}{4} \sum_{k=0}^{3} i^{k}\left\langle i^{k} y+x, T_{\alpha}\left(i^{k} y+x\right)\right\rangle,
$$

$\left(\left\langle y, T_{\alpha} x\right\rangle\right)_{\alpha}$ is bounded for every $y \in \mathscr{H}$, and so by П there is $T x \in \mathscr{H}$ with $\langle y, T x\rangle=\lim _{\alpha}\left\langle y, T_{\alpha} x\right\rangle$ for all $y \in \mathscr{H}$, giving us a linear map $T: \mathscr{H} \rightarrow \mathscr{H}$. It is clear that $\left(T_{\alpha}\right)_{\alpha}$ WOT-converges to $T$, provided that $T$ is bounded.

So to complete the proof, we must show that $T$ is bounded, and we'll do this by showing that $T$ has an adjoint (see 35 VI ). Note that $\left\langle x, T_{\alpha}^{*} x\right\rangle=\overline{\left\langle x, T_{\alpha} x\right\rangle}$ is Cauchy and bounded (with $\alpha$ running), so by a similar reasoning as before (but with $T_{\alpha}^{*}$ instead of $T_{\alpha}$ ) we get a map $S: \mathscr{H} \rightarrow \mathscr{H}$ with $\langle x, S y\rangle=\lim _{\alpha}\left\langle x, T_{\alpha}^{*} y\right\rangle$ for all $x, y \in \mathscr{H}$, which will be adjoint to $T$, which is therefore bounded.

IX Proposition Let $\mathscr{H}$ be a Hilbert space, and $\mathscr{D}$ an upwards directed subset of $\mathscr{B}(\mathscr{H})_{\mathbb{R}}$ with $\sup _{T \in \mathscr{D}}\langle x, T x\rangle<\infty$ for all $x \in \mathscr{H}$. Then

1. $(T)_{T \in \mathscr{D}}$ converges in the weak operator topology to some $T^{\prime}$ in $(\mathscr{B}(\mathscr{H}))_{\mathbb{R}}$,
2. $T^{\prime}$ is the supremum of $\mathscr{D}$ in $(\mathscr{B}(\mathscr{H}))_{\mathbb{R}}$, and
3. $\left\langle x, T^{\prime} x\right\rangle=\sup _{T \in \mathscr{D}}\langle x, T x\rangle$ for all $x \in \mathscr{H}$.
$\times$ Proof Let $x \in \mathscr{H}$. Since $\langle x,(\cdot) x\rangle: \mathscr{B}(\mathscr{H}) \rightarrow \mathbb{C}$ is positive we see that $(\langle x, T x\rangle)_{T \in \mathscr{D}}$ is an increasing net in $\mathbb{R}$, bounded from above (by assumption), and therefore converges to $\sup _{T \in \mathscr{D}}\langle x, T x\rangle$. In particular, $(T)_{T \in \mathscr{D}}$ is WOTCauchy, and "WOT-bounded", and thus (by VII) WOT-converges to some selfadjoint $T^{\prime}$ from $\mathscr{B}(\mathscr{H})$.

Since $(\langle x, T x\rangle)_{T \in \mathscr{D}}$ converges both to $\left\langle x, T^{\prime} x\right\rangle$, and to $\sup _{T \in \mathscr{D}}\langle x, T x\rangle$, we conclude that $\left\langle x, T^{\prime} x\right\rangle=\sup _{T \in \mathscr{D}}\langle x, T x\rangle$ for every $x \in \mathscr{H}$. In particular, $\langle x, T x\rangle \leqslant\left\langle x, T^{\prime} x\right\rangle$ for all $x \in \mathscr{H}$ and $T \in \mathscr{D}$, and thus $T \leqslant T^{\prime}$ for all $T \in \mathscr{D}$.

Let $S$ be a self-adjoint bounded operator on $\mathscr{H}$ with $T \leqslant S$ for all $T \in \mathscr{D}$. To prove that $T^{\prime}$ is the supremum of $\mathscr{D}$, we must show that $T^{\prime} \leqslant S$. Let $x \in \mathscr{H}$ be given. Since $\langle x, T x\rangle \leqslant\langle x, S x\rangle$ for each $T \in \mathscr{D}$ (because $T \leqslant S$ ), we have $\left\langle x, T^{\prime} x\right\rangle \equiv \sup _{T \in \mathscr{D}}\langle x, T x\rangle \leqslant\langle x, S x\rangle$, and therefore $T^{\prime} \leqslant S$ by 25 V
XI Definition Let $\mathscr{H}$ be a Hilbert space. The supremum of a (norm) bounded directed subset $\mathscr{D}$ in $(\mathscr{B}(\mathscr{H}))_{\mathbb{R}}($ which exists by $\mathbb{X}$ is denoted by $\bigvee \mathscr{D}$.

### 2.6.2 Normal Functionals

Definition Given a Hilbert space $\mathscr{H}$ a p-map $\omega: \mathscr{B}(\mathscr{H}) \rightarrow \mathbb{C}$ is called normal when $\omega(\bigvee \mathscr{D})=\bigvee_{T \in \mathscr{D}} \omega(T)$ for every bounded directed subset $\mathscr{D}$ of $\mathscr{B}(\mathscr{H})_{\mathbb{R}}$.
Example All vector functionals $\langle x,(\cdot) x\rangle$ are normal by 37 IX .
Exercise To show that a positive linear functional is normal, it suffices to show II that it preserves directed suprema of effects: show that given a Hilbert space $\mathscr{H}$ a positive map $\omega: \mathscr{B}(\mathscr{H}) \rightarrow \mathbb{C}$ is normal provided that $\omega(\bigvee \mathscr{D})=\bigvee_{T \in \mathscr{D}} \omega(T)$ for every directed subset $\mathscr{D}$ of $[0,1]_{\mathscr{B}(\mathscr{H})}$.
Lemma Every sequence $x_{1}, x_{2}, \ldots$ in a Hilbert space $\mathscr{H}$ with $\sum_{n}\left\|x_{n}\right\|^{2}<\infty$ gives a np-map $\omega: \mathscr{B}(\mathscr{H}) \rightarrow \mathbb{C}$ defined by $\omega(T)=\sum_{n}\left\langle x_{n}, T x_{n}\right\rangle$.
Proof Given $T \in \mathscr{B}(\mathscr{H})$ we have $\left|\left\langle x_{n}, T x_{n}\right\rangle\right| \leqslant\left\|x_{n}\right\|^{2}\|T\|$ by Cauchy-Schwarz 4 XV , so $\sum_{n}\left|\left\langle x_{n}, T x_{n}\right\rangle\right| \leqslant\|T\| \sum_{n}\left\|x_{n}\right\|^{2}$, which means that $\sum_{n}\left\langle x_{n}, T x_{n}\right\rangle$ converges, and so we may define $\omega$ as above.

It is easy to see that $\omega$ is linear and positive, so we'll only show that $\omega$ is normal. We must prove that $\omega(\bigvee \mathscr{D})=\bigvee_{T \in \mathscr{D}} \omega(T)$ for every bounded directed subset of $(\mathscr{B}(\mathscr{H}))_{\mathbb{R}}$. By 111 we may assume without loss of generality that $\mathscr{D} \subseteq[0,1]_{\mathscr{B}(\mathscr{H})}$. This has the benefit that $\left\langle x_{n}, T x_{n}\right\rangle$ is positive for all $n$ and $T \in \mathscr{D}$, so that their sum (over $n$ ) is given by a supremum over partial sums, viz. $\sum_{n}\left\langle x_{n}, T x_{n}\right\rangle=\bigvee_{N} \sum_{n=1}^{N}\left\langle x_{n}, T x_{n}\right\rangle$. Completing the proof is now simply a matter of interchanging suprema,

$$
\begin{aligned}
\bigvee_{T \in \mathscr{D}} \omega(T) & =\bigvee_{T \in \mathscr{D}} \bigvee_{N} \sum_{n=1}^{N}\left\langle x_{n}, T x_{n}\right\rangle \\
& =\bigvee_{N} \bigvee_{T \in \mathscr{D}} \sum_{n=1}^{N}\left\langle x_{n}, T x_{n}\right\rangle \\
& =\bigvee_{N} \sum_{n=1}^{N}\left\langle x_{n},(\bigvee \mathscr{D}) x_{n}\right\rangle=\omega(\bigvee \mathscr{D}),
\end{aligned}
$$

where we used that $\sum_{n=1}^{N}\left\langle x_{n},(\cdot) x_{n}\right\rangle$ is normal.
Exercise The following observations regarding a net $\left(x_{\alpha}\right)_{\alpha}$ in a Hilbert space $\mathscr{H}$ will be useful later on.

1. Show that $\sum_{\alpha}\left\|x_{\alpha}\right\|^{2}<\infty$ if and only if $\sum_{\alpha}\left\langle x_{\alpha},(\cdot) x_{\alpha}\right\rangle$ converges with respect to the operator norm to some bounded functional on $\mathscr{B}(\mathscr{H})$.
2. Given some $x \in \mathscr{H}$, show that $x_{\alpha}$ converges to $x$ if and only if $\left\langle x_{\alpha},(\cdot) x_{\alpha}\right\rangle$ operator-norm converges to $\langle x,(\cdot) x\rangle$.
(For the "if" part it may be convenient to first prove that $\left\langle x_{\alpha}, x\right\rangle \rightarrow\langle x, x\rangle$ by considering the bounded operator $|x\rangle\langle x|$ on $\mathscr{B}(\mathscr{H})$.)

39 The final project of this chapter is to show that each normal positive functional $\omega$ on a $\mathscr{B}(\mathscr{H})$ is of the form $\omega \equiv \sum_{n=0}^{\infty}\left\langle x_{n},(\cdot) x_{n}\right\rangle$ for some $x_{1}, x_{2}, \ldots$ with $\sum_{n}\left\|x_{n}\right\|^{2}<\infty$. For this we'll need some more nuggets from the theory of Hilbert spaces.
II Definition A subset $\mathscr{E}$ of a Hilbert space is called orthonormal if $\left\langle e, e^{\prime}\right\rangle=0$ for all $e, e^{\prime} \in \mathscr{E}$ with $e \neq e^{\prime}$, and $\langle e, e\rangle=1$ for all $e \in \mathscr{E}$. We say that $\mathscr{E}$ is maximal when $\mathscr{E}$ is maximal among all orthonormal subsets of $\mathscr{H}$ ordered by inclusion, and in that case we call $\mathscr{E}$ an orthonormal basis for $\mathscr{H}$ for reasons that will be become clear in IV below.
III Remark Clearly, by Zorn's lemma, each Hilbert space has an orthonormal basis.
IV Proposition Given an orthonormal subset $\mathscr{E}$ of a Hilbert space $\mathscr{H}$, and $x \in \mathscr{H}$,

1. (Bessel's inequality) $\sum_{e \in \mathscr{E}}|\langle e, x\rangle|^{2} \leqslant\|x\|^{2}$;
2. $\sum_{e \in \mathscr{E}}\langle e, x\rangle e$ converges in $\mathscr{H}$,
3. $\sum_{e \in \mathscr{E}}\langle e, x\rangle e=x$ if $\mathscr{E}$ is maximal, and
4. (Parseval's identity) $\sum_{e \in \mathscr{E}}|\langle e, x\rangle|^{2}=\|x\|^{2}$ if $\mathscr{E}$ is maximal.
$\checkmark$ Proof 1 Since for finite subset $\mathscr{F}$ of $\mathscr{E}$ we have $0 \leqslant\left\|x-\sum_{e \in \mathscr{F}}\langle e, x\rangle e\right\|^{2}=$ $\|x\|^{2}-2 \sum_{e \in \mathscr{F}}\langle e, x\rangle\langle x, e\rangle+\sum_{e, e^{\prime} \in \mathscr{F}}\left\langle x, e^{\prime}\right\rangle\left\langle e^{\prime}, e\right\rangle\langle e, x\rangle=\|x\|^{2}-\sum_{e \in \mathscr{F}}|\langle e, x\rangle|^{2}$, and so $\sum_{e \in \mathscr{F}}|\langle e, x\rangle|^{2} \leqslant\|x\|^{2}$, we get $\sum_{e \in \mathscr{E}}|\langle e, x\rangle|^{2} \leqslant\|x\|^{2}$.

2 From the observation that $\left\|\sum_{e \in \mathscr{F}}\langle e, x\rangle e\right\|^{2}=\sum_{e \in \mathscr{F}}|\langle e, x\rangle|^{2}$ for any finite $\mathscr{F} \subseteq \mathscr{E}$, and the fact that $\sum_{e \in \mathscr{E}}|\langle e, x\rangle|^{2}$ converges (by the previous point), one deduces that $\left(\sum_{e \in \mathscr{F}}\langle e, x\rangle e\right)_{\mathscr{F}}$ is Cauchy, and so $\sum_{e \in \mathscr{E}}\langle e, x\rangle e$ converges.
3) Writing $y:=\sum_{e \in \mathscr{E}}\langle e, x\rangle e$ we must show that $x=y$. If it were not so, if $x \neq y$, then $e^{\prime}:=\|x-y\|^{-1}(x-y)$ satisfies $\left\langle e^{\prime}, e^{\prime}\right\rangle=1$ and $\left\langle e^{\prime}, e\right\rangle=0$ for all $e \in \mathscr{E}$, and so may be added to $\mathscr{E}$ to yield an orthonormal basis $\mathscr{E} \cup\left\{e^{\prime}\right\}$ extending $\mathscr{E}$ contradicting $\mathscr{E}_{\mathrm{S}}$ maximality.

4 Finally, $\|x\|^{2}=\langle x, x\rangle=\sum_{e, e^{\prime} \in \mathscr{E}}\left\langle x, e^{\prime}\right\rangle\left\langle e^{\prime}, e\right\rangle\langle e, x\rangle=\sum_{e \in \mathscr{E}}|\langle e, x\rangle|^{2}$.
VI Exercise Let $\mathscr{E}$ be an orthonormal basis of a Hilbert space $\mathscr{H}$.

1. Show that $\sum_{e \in \mathscr{E}}|e\rangle\langle e|$ converges to 1 in the weak operator topology.
2. Show that $\sum_{e \in \mathscr{E}}|e\rangle\langle e|=1$ also in the sense that the directed set of partial sums $\sum_{e \in \mathscr{F}}|e\rangle\langle e|$, where $\mathscr{F}$ is a finite subset of $\mathscr{E}$, has 1 as its supremum.
3. Conclude that $\omega(1)=\sum_{e \in \mathscr{E}} \omega(|e\rangle\langle e|)$ for every np-map $\omega: \mathscr{B}(\mathscr{H}) \rightarrow \mathbb{C}$.

Lemma Given a Hilbert space $\mathscr{H}$ with orthonormal basis $\mathscr{E}$, we have

$$
\omega(A)=\sum_{e, e^{\prime} \in \mathscr{E}}\left\langle e, A e^{\prime}\right\rangle \omega\left(|e\rangle\left\langle e^{\prime}\right|\right)
$$

for every normal p-map $\omega: \mathscr{B}(\mathscr{H}) \rightarrow \mathbb{C}$ and $A \in \mathscr{B}(\mathscr{H})$.
Proof Let $\mathscr{F}$ be a finite subset of $\mathscr{E}$, and write $P=\sum_{e \in \mathscr{F}}|e\rangle\langle e|$. Since $P A P=\sum_{e, e^{\prime} \in \mathscr{F}}\left\langle e, A e^{\prime}\right\rangle|e\rangle\left\langle e^{\prime}\right|$ it suffices to show that $\omega(A-P A P)$ vanishes as $\mathscr{F}$ increases. Note that $P^{*} P=P$ and $\left(P^{\perp}\right)^{*} P^{\perp}=P^{\perp}$. Further, since $\|P\| \leqslant 1$, and $A-P A P=P^{\perp} A+P A P^{\perp}$, we have, by Kadison's inequality,

$$
\begin{aligned}
|\omega(A-P A P)| & \leqslant\left|\omega\left(P^{\perp} A\right)\right|+\left|\omega\left(P A P^{\perp}\right)\right| \\
& \leqslant \omega\left(P^{\perp}\right)^{1 / 2} \omega\left(A^{*} A\right)^{1 / 2}+\omega\left(P A A^{*} P\right)^{1 / 2} \omega\left(P^{\perp}\right)^{1 / 2} \\
& \leqslant 2\|A\| \omega(1)^{1 / 2} \omega\left(P^{\perp}\right)^{1 / 2}
\end{aligned}
$$

But since $\sum_{e \in \mathscr{E}} \omega(|e\rangle\langle e|)=\omega(1)$ by VI we see that $\omega\left(P^{\perp}\right) \rightarrow 0$ as $\mathscr{F} \rightarrow \infty$.
Theorem Let $\mathscr{H}$ be a Hilbert space. Every normal p-map $\omega: \mathscr{B}(\mathscr{H}) \rightarrow \mathbb{C}$ is of the form $\omega=\sum_{n}\left\langle x_{n},(\cdot) x_{n}\right\rangle$ where $x_{1}, x_{2}, \ldots \in \mathscr{H}$ with $\sum_{n}\left\|x_{n}\right\|^{2}=\|\omega\|$.
Proof By 36 V there is a unique $\varrho \in \mathscr{B}(\mathscr{H})$ with $\omega(|y\rangle\langle x|)=\langle x, \varrho y\rangle$ for all $x, y \in$ $\mathscr{H}$, because $(x, y) \mapsto \omega(|y\rangle\langle x|), \mathscr{H} \times \mathscr{H} \rightarrow \mathbb{C}$ is a bounded form in the sense of 36 IV . Note that $\varrho$ is positive by 25 V because $\langle x, \varrho x\rangle=\omega(|x\rangle\langle x|) \geqslant 0$ for all $x \in \mathscr{H}$. Now, let $\mathscr{E}$ be an orthonormal basis for $\mathscr{H}$. Since $\omega$ is normal, VI gives us $\omega(1)=\sum_{e \in \mathscr{E}} \omega(|e\rangle\langle e|)=\sum_{e \in \mathscr{E}}\langle e, \varrho e\rangle=\sum_{e \in \mathscr{E}}\|\sqrt{\varrho} e\|^{2}$, so that $\omega^{\prime}:=$ $\sum_{e \in \mathscr{E}}\langle\sqrt{\varrho} e,(\cdot) \sqrt{\varrho} e\rangle$ defines a normal positive functional on $\mathscr{B}(\mathscr{H})$ by 38 VI . Thus, we are done if can show that $\omega^{\prime}=\omega$, (because $\sqrt{\varrho} e$ is non-zero for at most countably many $e \in \mathscr{E})$. To this end, note that $\omega(|x\rangle\langle x|)=\langle\sqrt{\varrho} x, \sqrt{\varrho} x\rangle=$ $\sum_{e \in \mathscr{E}}\langle\sqrt{\varrho} x, e\rangle\langle e, \sqrt{\varrho} x\rangle=\sum_{e \in \mathscr{E}}\langle\sqrt{\varrho} e, \mid x\rangle\langle x \mid \sqrt{\varrho} e\rangle=\omega^{\prime}(|x\rangle\langle x|)$ for each $x \in$ $\mathscr{H}$, and so $\omega(|x\rangle\langle y|)=\omega^{\prime}(|x\rangle\langle y|)$ for all $x, y \in \mathscr{H}$ by polarization, and thus $\omega=$ $\omega^{\prime}$ by VII .

In this chapter we've studied the algebraic structure of the space $\mathscr{B}(\mathscr{H})$ of
bounded operators on a Hilbert space $\mathscr{H}$ abstractly via the notion of the $C^{*}$ algebra. We've seen not only that every $C^{*}$-algebra is miu-isomorphic to a
$C^{*}$-subalgebra of such a $\mathscr{B}(\mathscr{H})$ (in 30 XIV ), but also that any commutative $C^{*}$ algebra is miu-isomorphic to the space $C(X)$ of continuous functions on some compact Hausdorff space (in 27 XXVII). But there's more to $\mathscr{B}(\mathscr{H})$ than just being a $C^{*}$-algebra: it has the two additional properties of having suprema of bounded directed subsets (see 37 IX ), and having a faithful collection of normal functionals (viz. the vector functionals, 25 III ). This leads us to the study of von Neumann algebras - the topic of the next chapter.

## Chapter 3

## Von Neumann Algebras

We have arrived at the main subject of this thesis, the special class of $C^{*}$ algebras called von Neumann algebras (see definition 42 below) that are characterised by the existence of certain directed suprema and an abundance of functionals that preserve these suprema. While all $C^{*}$-algebras and the cpsumaps between them may perhaps serve as models for quantum data types and processes, respectively, we focus for the purposes of this thesis our attention on the subcategory $\mathbf{W}_{\text {CPSU }}^{*}$ of von Neumann algebras and the cpsu-maps between them that preserve these suprema (called normal maps, see 44XV), because

1. $\mathbf{W}_{\text {CPSU }}^{*}$ is a model of the quantum lambda calculus (in a way that $\mathbf{C}_{\mathrm{CPSU}}^{*}$ is not, see 125 X , and
2. we were able to axiomatise the sequential product $(b \mapsto \sqrt{a} b \sqrt{a})$ in $\mathbf{W}_{\text {CPSU }}^{*}$ (but not in $\mathbf{C}_{\text {CPSU }}^{*}$ ) see 1061

Both these are reserved for the next chapter; in this chapter we'll (re)develop the theory we needed to prove them.

The archetypal von Neumann algebra is the $C^{*}$-algebra $\mathscr{B}(\mathscr{H})$ of bounded operators on a Hilbert space $\mathscr{H}$. In fact, the original 50 . 70 and common 1243 definition of a von Neumann algebra is a $C^{*}$-subalgebra $\mathscr{A}$ of a $\mathscr{B}(\mathscr{H})$ that is closed in a "suitable topology" such as the strong or weak operator topology (see 37 V ). Most authors make the distinction between such rings of operators (called von Neumann algebras) and the $C^{*}$-algebras miu-isomorphic to them (called $W^{*}$-algebras), but we won't bother and call them all von Neumann
algebras. Partly because it seems difficult to explain to someone picturing a quantum data type the meaning of the weak operator topology and the Hilbert space $\mathscr{H}$, we'll use Kadison's characterisation 42] of von Neumann algebras as $C^{*}$-algebras with a certain dcpo-structure (c.f. 37 IX ) and sufficiently many Scott-continuous functionals (c.f. 38 II ) as our definition instead, see 42 .

But we also use Kadison's definition just to see to what extent the representation of von Neumann algebras as rings of operators (see 48 VIII ) can be avoided when erecting the basic theory. Instead we'll put the directed suprema and normal positive functionals on centre stage. All the while our treatment doesn't stray too far from the beaten path, and borrows many arguments from the standard texts 43 , 62 ; but most of them had to be tweaked in places, and some demanded a complete overhaul.

The material on von Neumann algebras is less tightly knit as the theory of $C^{*}$-algebras, and so after the basics we deal with four topics more or less in linear order (instead of intertwined.)

The great abundance of projections (elements $p$ with $p^{*} p=p$ ) in von Neumann algebras - a definite advantage over $C^{*}$-algebras-is the first topic. We'll see for example that the existence of norm bounded directed suprema in a von Neumann algebra $\mathscr{A}$ allows us to show that there is a least projection $\lceil a\rceil$ above any effect $a$ from $\mathscr{A}$ given by $\lceil a\rceil=\bigvee_{n} a^{1 / 2^{n}}$ (see 561 ; and also that any element of a von Neumann algebra can be written as a norm limit of linear combinations of projections (in 65 IV ). Many a result about von Neumann algebras can be proven by an appeal to projections.

The second topic concerns two topologies that are instrumental for the more delicate results and constructions: the ultraweak topology induced by the normal positive functionals $\omega: \mathscr{A} \rightarrow \mathbb{C}$, and the ultrastrong topology induced by the associated seminorms $\|\cdot\|_{\omega}$ (see 42). We'll show among other things that a von Neumann algebra is complete with respect to the ultrastrong topology and bounded complete with respect to the ultraweak topology (see 771).

This completeness allows us to define, for example, for any pair $a, b$ of elements from a von Neumann algebra $\mathscr{A}$ with $a^{*} a \leqslant b^{*} b$ an element $a / b$ with $a=(a / b) b$ (see 811 -this is the third topic. Taking $b=\sqrt{a^{*} a}$ we obtain the famous polar decomposition $a=\left(a / \sqrt{a^{*} a}\right) \sqrt{a^{*} a}$ (see 821 , which is usually proven for a bounded operator on a Hilbert space first).

The fourth, and final topic, is ultraweakly continuous functionals on a von Neumann algebra: we'll show in 90 II that any centre separating collection (21II) of normal positive functionals $\Omega$ on a von Neumann algebra completely determines the normal positive functionals, which will be important for the definition of the tensor product of von Neumann algebras in the next chapter, see 108 II

### 3.1 The Basics

### 3.1.1 Definition and Counterexamples

Definition A $C^{*}$-algebra $\mathscr{A}$ is a von Neumann algebra when

1. every bounded directed subset $D$ of self-adjoint elements of $\mathscr{A}$ (so $D \subseteq$ $\mathscr{A}_{\mathbb{R}}$ ) has a supremum $\bigvee D$ in $\mathscr{A}_{\mathbb{R}}$, and
2. if $a$ is a positive element of $\mathscr{A}$ with $\omega(a)=0$ for every normal (see below) positive linear map $\omega: \mathscr{A} \rightarrow \mathbb{C}$, then $a=0$ 团

A positive linear map $\omega: \mathscr{A} \rightarrow \mathbb{C}$ is called normal if $\omega(\bigvee D)=\bigvee_{d \in D} \omega(d)$ for every bounded directed subset of self-adjoint elements of $D$ which has a supremum $\bigvee D$ in $\mathscr{A}_{\mathbb{R}}$.
The ultraweak topology on $\mathscr{A}$ is the least topology on $\mathscr{A}$ that makes all normal positive linear maps $\omega: \mathscr{A} \rightarrow \mathbb{C}$ continuous. The ultrastrong topology on $\mathscr{A}$ is the least topology on $\mathscr{A}$ that makes $a \mapsto \omega\left(a^{*} a\right)$ continuous for every np$\operatorname{map} \omega: \mathscr{A} \rightarrow \mathbb{C}$.

Remark We work with the ultraweak and ultrastrong topology in tandem,
III because neither is ideal, and they tend to be complementary: for example, $a \mapsto a^{*}$ is ultraweakly continuous but not ultrastrongly (see 43II point 4), while $a \mapsto|a|$ is ultrastrongly continuous (see 74 III ) but not ultraweakly 43 II , point (6). This doesn't prevent the ultraweak topology from being weaker than the ultrastrong topology: a net that converges ultrastrongly converges ultraweakly as well, see 431. To see this, and when dealing with the ultrastrong topology in general, it is useful to note that every np-functional $\omega$ on a von Neumann algebra $\mathscr{A}$ gives rise to an inner product $[a, b]_{\omega}=\omega\left(a^{*} b\right)$ and seminorm $\|a\|_{\omega}=[a, a]_{\omega}^{1 / 2}=\omega\left(a^{*} a\right)^{1 / 2}$ (as in 30 II and 30 IV .

## Examples

1. $\mathbb{C}$ and $\{0\}$ are clearly von Neumann algebras.

[^4]2. The $C^{*}$-algebra $\mathscr{B}(\mathscr{H})$ of bounded operators on a Hilbert space $\mathscr{H}$ is a von Neumann algebra: $\mathscr{B}(\mathscr{H})$ has bounded directed suprema of selfadjoint elements by 37 IX and the vector states (and thus all normal functionals) are order separating (and thus faithful) by 25 III
3. The direct sum $\bigoplus_{i} \mathscr{A}_{i}$ (see 3 V ) of a family $\left(\mathscr{A}_{i}\right)_{i}$ of von Neumann algebras is itself a von Neumann algebra.
(While we're not quite ready to define morphisms between von Neumann algebras, we can already spoil that the direct sum gives the categorical product of von Neumann algebras once we do, see 47 IV .)
4. A $C^{*}$-subalgebra $\mathscr{B}$ of a von Neumann algebra $\mathscr{A}$ is called a von Neumann subalgebra (and is itself a von Neumann algebra) if for every bounded directed subset $D$ of self-adjoint elements from $\mathscr{B}$ we have $\bigvee D \in \mathscr{B}$ (where the supremum is taken in $\mathscr{A}_{\mathbb{R}}$ ).
5. We'll see in 65 III that given a subset $S$ of a von Neumann algebra $\mathscr{A}$ the set $S^{\square}=\{a \in \mathscr{A}: \forall s \in S[$ as $=s a]\}$ called the commutant of $S$ is a von Neumann subalgebra of $\mathscr{A}$ when $S$ is closed under involution.
6. We'll see in 49 IV that the $N \times N$-matrices over a von Neumann algebra $\mathscr{A}$ form a von Neumann algebra.
7. We'll see in $51 \times$ that the bounded measurable functions on a finite complete measure space $X$ (modulo the negligible ones) form a commutative von Neumann algebra $L^{\infty}(X)$.

43 Exercise Let $\mathscr{A}$ be a von Neumann algebra.

1. Show that $|\omega(a)| \leqslant\|a\|_{\omega}\|\omega\|^{1 / 2}$ for every np-map $\omega: \mathscr{A} \rightarrow \mathbb{C}$ and $a \in \mathscr{A}$.
2. Show that when a net $\left(a_{\alpha}\right)_{\alpha}$ in $\mathscr{A}$ converges ultrastrongly to $a \in \mathscr{A}$ it does so ultraweakly, too.
3. Show that an ultraweakly closed subset $C$ of $\mathscr{A}$ is also ultrastrongly closed.

II Exercise We give some counterexamples in $\mathscr{B}\left(\ell^{2}\right)$ to plausible propositions to sharpen your understanding of the ultrastrong and ultraweak topologies, and so that you may better appreciate the strange manoeuvres we'll need to pull off later on.

1. First some notation: given $n, m \in \mathbb{N}$, we denote by $|n\rangle\langle m|$ the bounded operator on $\ell^{2}$ given by $(|n\rangle\langle m|)(f)(n)=f(m)$ and $(|n\rangle\langle m|)(f)(k)=0$ for $k \neq n$ and $f \in \ell^{2}$.
Verify the following computation rules, where $k, \ell, m, n \in \mathbb{N}$.

$$
(|n\rangle\langle m|)^{*}=|m\rangle\langle n|, \quad|n\rangle\langle m||\ell\rangle\langle k|= \begin{cases}|n\rangle\langle k| & \text { if } m=\ell \\ 0 & \text { otherwise }\end{cases}
$$

2. Show that $\bigvee_{N} \sum_{n=0}^{N}|n\rangle\langle n|=1$.

Conclude that $(|n\rangle\langle n|)_{n}$ converges ultrastrongly (and ultraweakly) to 0 .
Thus ultrastrong (and ultraweak) convergence does not imply norm convergence, which isn't unexpected. But we also see that if a sequence $\left(b_{n}\right)_{n}$ converges ultrastrongly (or ultraweakly) to some $b$, then $\left(\left\|b_{n}\right\|\right)_{n}$ doesn't even have to converge to $\|b\|$.
(Note that $(|n\rangle\langle n|)_{n}$ resembles a 'moving bump'.)
3. Note that when a net $\left(a_{\alpha}\right)_{\alpha}$ converges ultrastrongly to $a$, then $\left(a_{\alpha}^{*} a_{\alpha}\right)_{\alpha}$ is norm-bounded and converges ultraweakly to $a^{*} a$.
The converse does not hold: show that (already in $\mathbb{C}$ ) $e^{i n}$ does not converge ultraweakly (nor ultrastrongly) as $n \rightarrow \infty$, while $1 \equiv e^{-i n} e^{i n}$ is normbounded and converges ultraweakly to 1 as $n \rightarrow \infty$.
4. Show that $(|0\rangle\langle n|)_{n}$ converges ultrastrongly (and ultraweakly) to 0 .

Deduce that $(|n\rangle\langle 0|)_{n}$ converges ultraweakly to 0 , but doesn't converge ultrastrongly at all.
Conclude that $a \mapsto a^{*}$ is not ultrastrongly continuous on $\mathscr{B}\left(\ell^{2}\right)$.
(This has the annoying side-effect that it is not immediately clear that the ultrastrong closure of a $C^{*}$-subalgebra of a von Neumann algebra is a von Neumann subalgebra; we'll deal with this by showing that the ultrastrong closure coincides with the ultraweak closure in 73 VIII )
5. Show that the unit ball $\left(\mathscr{B}\left(\ell^{2}\right)\right)_{1}$ of $\mathscr{B}\left(\ell^{2}\right)$ is not ultrastrongly compact by proving that $(|0\rangle\langle n|)_{n}$ has no ultrastrongly convergent subnet.
(But we'll see in 77 III that the unit ball of a von Neumann algebra is ultraweakly compact.)
6. Show that $|n\rangle\langle 0|+|0\rangle\langle n|$ converges ultraweakly to 0 as $n \rightarrow \infty$, while $(|n\rangle\langle 0|+|0\rangle\langle n|)^{2} \equiv|0\rangle\langle 0|+|n\rangle\langle n|$ converges ultraweakly to $|0\rangle\langle 0|$.
Conclude that $a \mapsto a^{2}$ is not ultraweakly continuous on $\mathscr{B}\left(\ell^{2}\right)$.
Conclude that $a, b \mapsto a b$ is not jointly ultraweakly continuous on $\mathscr{B}\left(\ell^{2}\right)$.
Prove that $||n\rangle\langle 0|+| 0\rangle\langle n||=| 0\rangle\langle 0|+|n\rangle\langle n|$.
Conclude that $a \mapsto|a|$ is not ultraweakly continuous on $\left(\mathscr{B}\left(\ell^{2}\right)\right)_{\mathbb{R}}$.
(We'll see in $74 \mid$ that $a \mapsto|a|$ is ultrastrongly continuous on self-adjoint elements.)
7. Let us consider the two extensions of $|\cdot|$ to arbitrary elements, namely $a \mapsto$ $\sqrt{a^{*} a}=:|a|_{s}$ and $a \mapsto \sqrt{a a^{*}}=:|a|_{r}$ (for support and range, c.f. 59 VII ).
Prove that $|0\rangle\langle 0|+|0\rangle\langle n|$ converges ultrastrongly to $|0\rangle\langle 0|$ as $n \rightarrow \infty$.
Show that $||0\rangle\langle 0|+| 0\rangle\left.\langle n|\right|_{s}=|0\rangle\langle 0|+|0\rangle\langle n|+|n\rangle\langle 0|+|n\rangle\langle n|$ converges ultraweakly to $||0\rangle\langle 0||_{s} \equiv|0\rangle\langle 0|$ as $n \rightarrow \infty$, but not ultrastrongly.
Show that $||0\rangle\langle 0|+| 0\rangle\left.\langle n|\right|_{r}=\sqrt{2}|0\rangle\langle 0|$.
Conclude that $|\cdot|_{s}$ and $|\cdot|_{r}$ are not ultrastrongly continuous on $\mathscr{B}\left(\ell^{2}\right)$.
8. Show that $1+|n\rangle\langle 0|+|0\rangle\langle n|$ is positive, and converges ultraweakly to 1 as $n \rightarrow \infty$, while the squares $1+|n\rangle\langle n|+|0\rangle\langle 0|+2|n\rangle\langle 0|+2|0\rangle\langle n|$ converge ultraweakly to $1+|0\rangle\langle 0|$ (as $n \rightarrow \infty)$.
Hence $a \mapsto a^{2}$ and $a \mapsto \sqrt{a}$ are not ultraweakly continuous on $\mathscr{B}\left(\ell^{2}\right)_{+}$.
9. For the next counterexample, we need a growing moving bump, which still converges ultraweakly. Sequences won't work here:
Show that $n|n\rangle\langle n|$ does not converge ultraweakly as $n \rightarrow \infty$.
Show that $n|f(n)\rangle\langle f(n)|$ does not converge ultraweakly as $n \rightarrow \infty$ for every strictly monotone (increasing) map $f: \mathbb{N} \rightarrow \mathbb{N}$.
So we'll resort to a net. Let $D$ be the directed set which consists of pairs $(n, f)$, where $n \in \mathbb{N} \backslash\{0\}$ and $f: \mathbb{N} \rightarrow \mathbb{N}$ is monotone, ordered by $(n, f) \leqslant(m, g)$ iff $n \leqslant m$ and $f \leqslant g$.
Show that the net $(n|f(n)\rangle\langle f(n)|)_{n, f \in D}$ converges ultrastrongly to 0 .
So a net which converges ultrastrongly need not be bounded! (The cure for this pathology is Kaplansky's density theorem, see 74 IV .)
Show that $\frac{1}{n}|f(n)\rangle\langle 0|$ converges ultrastrongly to 0 as $D \ni(n, f) \rightarrow \infty$.

Show that the product $|f(n)\rangle\langle 0|=(n|f(n)\rangle\langle f(n)|)\left(\frac{1}{n}|f(n)\rangle\langle 0|\right)$ does not converge ultrastrongly as $D \ni(n, f) \rightarrow \infty$.
Conclude that multiplication $a, b \mapsto a b$ is not jointly ultrastrongly continuous on $\mathscr{B}\left(\ell^{2}\right)$, even when $b$ is restricted to a bounded set.
(Nevertheless we'll see that multiplication is ultrastrongly continuous when $a$ is restricted to a bounded set in 45 VI .)
10. Show that $a_{n, f}=\frac{1}{n}(|f(n)\rangle\langle 0|+|0\rangle\langle f(n)|)+n|f(n)\rangle\langle f(n)|$ converges ultrastrongly to 0 as $D \ni(n, f) \rightarrow \infty$, while $a_{n, f}^{2}$ does not.
Hence $a \mapsto a^{2}$ is not ultrastrongly continuous on $\mathscr{B}\left(\ell^{2}\right)_{\mathbb{R}}$.
11. Let us show that $\mathscr{B}\left(\ell^{2}\right)$ is not ultraweakly complete.

Show that there is an unbounded linear map $f: \ell^{2} \rightarrow \mathbb{C}$ (perhaps using the fact that every vector space has a basis by the axiom of choice), and that for each finite dimensional linear subspace $S$ of $\ell^{2}$ there is a unique vector $x_{S} \in S$ with $f(x)=\left\langle x_{S}, y\right\rangle$ for all $y \in S$ (using 5IX).
Consider the net $\left(|e\rangle\left\langle x_{S}\right|\right)_{S}$ where $S$ ranges over the finite dimensional subspaces of $\ell^{2}$ ordered by inclusion, and $e$ is some fixed vector in $\ell^{2}$ with $\|e\|=1$.
Let $\omega: \mathscr{B}\left(\ell^{2}\right) \rightarrow \mathbb{C}$ be an np-map, so $\omega \equiv \sum_{n}\left\langle y_{n},(\cdot) y_{n}\right\rangle$ for $y_{1}, y_{2}, \ldots \in \ell^{2}$ with $\sum_{n}\left\|y_{n}\right\|^{2}<\infty$, see 39 IX .
Show that $\omega\left(|e\rangle\left\langle x_{S}\right|-|e\rangle\left\langle x_{T}\right|\right)=\left\langle x_{S}-x_{T}, \sum_{n} y_{n}\left\langle y_{n}, e\right\rangle\right\rangle=0$ when $S$ and $T$ are finite dimensional linear subspaces of $\ell_{2}$ which contain the vector $\sum_{n} y_{n}\left\langle y_{n}, e\right\rangle$.
Conclude that $\left(|e\rangle\left\langle x_{S}\right|\right)_{S}$ is ultraweakly Cauchy.
Show that if $\left(|e\rangle\left\langle x_{S}\right|\right)_{S}$ converges ultraweakly to some $A$ in $\mathscr{B}\left(\ell^{2}\right)$, then we have $\langle e, A y\rangle=f(y)$ for all $y \in \ell^{2}$.

Conclude that $\left(|e\rangle\left\langle x_{S}\right|\right)_{S}$ does not converge ultraweakly, and that $\mathscr{B}\left(\ell^{2}\right)$ is not ultraweakly complete.
(Nevertheless, we'll see that every von Neumann algebra is ultrastrongly complete, and that every norm-bounded ultraweakly Cauchy net in a von Neumann converges, in 771)

### 3.1.2 Elementary Theory

44 The basic facts concerning von Neumann algebras cccchdgnvuvucvlvnujgbferfthndivnkgbkjveven
relationship between multiplication and the order structure. For example, while it is clear that translation and scaling on a von Neumann algebra are ultraweakly (and ultrastrongly) continuous, the fact that multiplication is ultraweakly (and ultrastrongly) continuous in each coordinate is less obvious (see 45 IV ). Quite surprisingly, this problem reduces to the ultraweak continuity of $b \mapsto a^{*} b a$ by the following identity.

II Exercise Show that for elements $a, b, c$ of a $C^{*}$-algebra,

$$
a^{*} c b=\frac{1}{4} \sum_{k=0}^{3} i^{k}\left(i^{k} a+b\right)^{*} c\left(i^{k} a+b\right) .
$$

(Note that this identity is a variation on the polarization identity for inner products, see 4XV.)

III Lemma Let $\left(x_{\alpha}\right)_{\alpha \in D}$ be a net of effects of a von Neumann algebra $\mathscr{A}$, which converges ultraweakly to 0 . Let $\left(b_{\alpha}\right)_{\alpha \in D}$ be a net of elements with $\left\|b_{\alpha}\right\| \leqslant 1$ for all $\alpha$. Then $\left(x_{\alpha} b_{\alpha}\right)_{\alpha}$ converges ultraweakly to 0 .
IV Proof Let $\omega: \mathscr{A} \rightarrow \mathbb{C}$ be an np-map. We have, for each $\alpha$,

$$
\begin{aligned}
\left|\omega\left(x_{\alpha} b_{\alpha}\right)\right|^{2} & =\left|\omega\left(\sqrt{x_{\alpha}} \sqrt{x_{\alpha}} b_{\alpha}\right)\right|^{2} & & \text { since } x_{\alpha} \geqslant 0 \\
& \leqslant \omega\left(x_{\alpha}\right) \omega\left(b_{\alpha}^{*} x_{\alpha} b_{\alpha}\right) & & \text { by Kadison's inequality, 30IV } \\
& \leqslant \omega\left(x_{\alpha}\right) \omega\left(b_{\alpha}^{*} b_{\alpha}\right) & & \text { since } x_{\alpha} \leqslant 1 \\
& \leqslant \omega\left(x_{\alpha}\right) \omega(1) & & \text { since } b_{\alpha}^{*} b_{\alpha} \leqslant 1 .
\end{aligned}
$$

Thus, since $\left(\omega\left(x_{\alpha}\right)\right)_{\alpha}$ converges to 0 , we see that $\left(\omega\left(x_{\alpha} b_{\alpha}\right)\right)_{\alpha}$ converges to 0 , and so $\left(x_{\alpha} b_{\alpha}\right)_{\alpha}$ converges ultraweakly to 0 .
$\checkmark$ Exercise Let $D$ be a bounded directed set of self-adjoint elements of a von Neumann algebra $\mathscr{A}$, and let $a \in \mathscr{A}$.
VI Show that the net $(d)_{d \in D}$ converges ultraweakly to $\bigvee D$.
VII Use IIII to show that $(d a)_{d}$ converges ultraweakly to $(\bigvee D) a$, and that $\left(a^{*} d\right)_{d}$ converges ultraweakly to $a^{*}(\bigvee D)$.

VIII Proposition Let $a$ be an element of a von Neumann algebra $\mathscr{A}$. Then

$$
\bigvee_{d \in D} a^{*} d a=a^{*}(\bigvee D) a
$$

for every bounded directed subset $D$ of self-adjoint elements of $\mathscr{A}$.

Proof If $a$ is invertible, then the (by 25II) order preserving map $b \mapsto a^{*} b a$ has an order preserving inverse (namely $b \mapsto\left(a^{-1}\right)^{*} b a^{-1}$ ), and therefore preserves all suprema.
The general case reduces to the case that $a$ is invertible in the following way. There is (by 11 VI$) \lambda>0$ such that $\lambda+a$ is invertible. Then as $d$ increases

$$
a^{*} d a \equiv(\lambda+a)^{*} d(\lambda+a)-\lambda^{2} d-\lambda a^{*} d-\lambda d a
$$

converges ultraweakly to $a^{*}(\bigvee D) a$, because $\left((\lambda+a)^{*} d(\lambda+a)\right)_{d}$ converges ultraweakly to $(\lambda+a)^{*}(\bigvee D)(\lambda+a)$ by $\triangle$ X and V , and $\left(a^{*} d+d a\right)_{d}$ converges ultraweakly to $a^{*}(\bigvee D)+(\bigvee D) a$ by VII . Since $\left(a^{*} d a\right)_{d}$ converges to $\bigvee_{d \in D} a^{*} d a$ too, we could conclude that $\bigvee_{d \in D} a^{*} d a=a^{*}(\bigvee D) a$ if we would already know that the ultraweak topology is Hausdorff. At the moment, however, we must content ourselves with the conclusion that $\omega\left(a^{*}(\bigvee D) a-\bigvee_{d \in D} a^{*} d a\right)=0$ for every np-functional $\omega$ on $\mathscr{A}$. But since $a^{*}(\bigvee D) a-\bigvee_{d \in D} a^{*} d a$ happens to be positive, we conclude that $a^{*}(\bigvee D) a-\bigvee_{d \in D} a^{*} d a=0$ nonetheless.
Exercise Show that the set of np-functionals on a von Neumann algebra $\mathscr{A}$ is not only faithful but also order separating using 30 X . Deduce

1. that the ultraweak and ultrastrong topologies are Hausdorff,
2. that $\mathscr{A}_{+}, \mathscr{A}_{\mathbb{R}}$ and $[0,1]_{\mathscr{A}}$ are ultraweakly (and ultrastrongly) closed,
3. and that the unit ball $(\mathscr{A})_{1}$ is ultrastrongly closed.
(We'll see only later on, in 73 VIII , that $(\mathscr{A})_{1}$ is ultraweakly closed as well.)

Exercise Let $D$ be a directed subset of self-adjoint elements of a von Neumann XII algebra $\mathscr{A}$, and let $a \in \mathscr{A}$.
Show that if $a d=d a$ for all $d \in D$, then $a(\bigvee D)=(\bigvee D) a$.
Use 则 to show that $(\bigvee D-d)^{2}$ converges ultraweakly to 0 as $D \ni d \rightarrow \infty$.
Conclude that $(d)_{d \in D}$ converges ultrastrongly to $\bigvee D$.
Exercise Show that for a positive linear map $f: \mathscr{A} \rightarrow \mathscr{B}$ between von Neumann XV algebras, the following are equivalent.

1. $f$ is ultraweakly continuous;
2. $f$ is ultraweakly continuous on $[0,1]_{\mathscr{A}}$;
3. $f(\bigvee D)=\bigvee_{d \in D} f(d)$ for each bounded directed $D \subseteq \mathscr{A}_{\mathbb{R}}$;
4. $\omega \circ f: \mathscr{A} \rightarrow \mathbb{C}$ is normal for each np-map $\omega: \mathscr{B} \rightarrow \mathbb{C}$.

In that case we say that $f$ is normal.
Conclude that $b \mapsto a^{*} b a, \mathscr{A} \rightarrow \mathscr{A}$ is ultraweakly continuous for every element $a$ of a von Neumann algebra $\mathscr{A}$.

45 Exercise Show that if a positive linear map $f: \mathscr{A} \rightarrow \mathscr{B}$ between von Neumann algebras is ultrastrongly continuous (on $[0,1]_{\mathscr{A}}$ ), then $f$ is normal. (Hint: use that a bounded directed set $D \subseteq \mathscr{A}_{\mathbb{R}}$ converges ultrastrongly to $\bigvee D$.)

The converse does not hold: give an example of a map $f$ which is normal, but not ultrastrongly continuous. (Hint: transpose.)
II Proposition An ncp-map $f: \mathscr{A} \rightarrow \mathscr{B}$ between von Neumann algebras is ultrastrongly continuous.
III Proof To show that $f$ is ultrastrongly continuous it suffices to show that $f$ is ultrastrongly continuous at 0 . So let $\left(b_{\alpha}\right)_{\alpha}$ be a net in $\mathscr{A}$ which converges ultrastrongly to 0 ; we must show that $\left(f\left(b_{\alpha}\right)\right)_{\alpha}$ converges ultrastrongly to 0 , viz. that $\left(f\left(b_{\alpha}\right)^{*} f\left(b_{\alpha}\right)\right)_{\alpha}$ converges ultraweakly to 0 . Since $f\left(b_{\alpha}\right)^{*} f\left(b_{\alpha}\right) \leqslant f\left(b_{\alpha}^{*} b_{\alpha}\right)\|f(1)\|$ by 34 XIV. it suffices to show that $\left(f\left(b_{\alpha}^{*} b_{\alpha}\right)\right)_{\alpha}$ converges ultraweakly to 0 , but this follows from the facts that $f$ is ultraweakly continuous and $\left(b_{\alpha}^{*} b_{\alpha}\right)_{\alpha}$ converges ultraweakly to 0 (since $\left(b_{\alpha}\right)_{\alpha}$ converges ultrastrongly to 0 ).
IV Exercise Let $\mathscr{A}$ be a von Neumann algebra. Conclude (using 11 and 34 V ) that the map $a \mapsto b^{*} a b, \mathscr{A} \rightarrow \mathscr{A}$ is ultrastrongly continuous for every element $b \in \mathscr{A}$.

Use this, and 44II to show that $b \mapsto a b, b a: \mathscr{A} \rightarrow \mathscr{A}$ are ultraweakly and ultrastrongly continuous for every element $a$ of a von Neumann algebra $\mathscr{A}$.
$\checkmark$ We saw in 43 II that the multiplication on a von Neumann algebra is not jointly ultraweakly continuous, even on a bounded set. Neither is $a, b \mapsto a b$ jointly ultrastrongly continuous, even when $b$ is restricted to a bounded set; but it is jointly ultrastrongly continuous when $a$ is restricted to a bounded set:
VI Proposition Let $\left(a_{\alpha}\right)_{\alpha}$ and $\left(b_{\alpha}\right)_{\alpha}$ be nets in a von Neumann algebra $\mathscr{A}$ with the same index set that converge ultrastrongly to $a, b \in \mathscr{A}$, respectively. Then the net $\left(a_{\alpha} b_{\alpha}\right)_{\alpha}$ converges ultrastrongly to $a b$ provided that $\left(a_{\alpha}\right)_{\alpha}$ is bounded.
VII Proof Let $\omega: \mathscr{A} \rightarrow \mathbb{C}$ be an np-functional. Since

$$
\begin{aligned}
\left\|a b-a_{\alpha} b_{\alpha}\right\|_{\omega} & \leqslant\left\|\left(a-a_{\alpha}\right) b\right\|_{\omega}+\left\|a_{\alpha}\left(b-b_{\alpha}\right)\right\|_{\omega} \\
& \leqslant\left\|a-a_{\alpha}\right\|_{\omega\left(b^{*}(\cdot) b\right)}+\left\|a_{\alpha}\right\|\left\|b-b_{\alpha}\right\|_{\omega}
\end{aligned}
$$

vanishes as $\alpha \rightarrow \infty$, we see that $\left(a_{\alpha} b_{\alpha}\right)_{\alpha}$ converges ultrastrongly to $a b$.

We can now prove a bit more about the ultrastrong and ultraweak topologies.
Exercise Show that a net $\left(b_{\alpha}\right)_{\alpha}$ in a von Neumann algebra $\mathscr{A}$ converges ultrastrongly to an element $b$ of $\mathscr{A}$ if and only if both $b_{\alpha}^{*} b_{\alpha} \longrightarrow b^{*} b$ and $b_{\alpha} \longrightarrow b$ ultraweakly as $\alpha \rightarrow \infty$.

Exercise Show that for a positive linear map $\omega: \mathscr{A} \rightarrow \mathbb{C}$ on a von Neumann algebra $\mathscr{A}$ the following are equivalent

1. $\omega$ is normal;
2. $\omega$ is ultraweakly continuous;
3. $\omega$ is ultrastrongly continuous.
(Hint: combine 44 XV and 45II)
Enter the eponymous hero(s) of this thesis.
Definition We denote the category of normal cpsu-maps by $\mathbf{W}_{\text {CPSU }}^{*}$, and its subcategory of nmiu-maps by $\mathbf{W}_{\text {MIU }}^{*}$. (We omit the " N " for the sake of brevity.) Though arguably $\mathbf{W}_{\text {MIU }}^{*}$ is a good candidate for being called the category of von Neumann algebra, the title of this thesis refers to $\mathbf{W}_{\text {CPSU }}^{*}$. Indeed, it's the ncpsu-maps between von Neumann algebras that stand to model the arbitrary quantum processes, and it's the category of these quantum processes we want to mine for abstract structure. This is mostly a task for the next chapter, though. For now we'll just establish that $\mathbf{W}_{\text {Cpsu }}^{*}$ has all products, $\mathbb{I V}$. certain equalisers, V. and that $\left(\mathbf{W}_{\text {CPSU }}^{*}\right)^{\text {op }}$ is an effectus, see VI.

Exercise Show that given a family $\left(\mathscr{A}_{i}\right)_{i}$ of von Neumann algebras the direct sum $\bigoplus_{i} \mathscr{A}_{i}$ from 3 V is a von Neumann algebra and the projections $\pi_{j}: \bigoplus_{i} \mathscr{A}_{i} \rightarrow \mathscr{A}_{j}$ are normal. Moreover, show that this makes $\bigoplus_{i} \mathscr{A}_{i}$ into the product of the $\mathscr{A}_{i}$ in the categories $\mathbf{W}_{\text {MIU }}^{*}$ and $\mathbf{W}_{\text {CPSU }}^{*}$ (see 10 VII and 34 VII .
Exercise Show that given nmiu-maps $f, g: \mathscr{A} \rightarrow \mathscr{B}$ between von Neumann aland the inclusion $e: \mathscr{E} \rightarrow \mathscr{A}$ is the equaliser of $f$ and $g$ in the categories $\mathbf{W}_{\text {MIU }}^{*}$ and $\mathbf{W}_{\text {CPSU }}^{*}$ (see 10 VIII and 34 VI ).
Let us briefly indicate what makes $\left(\mathbf{W}_{\text {CPSU }}^{*}\right)^{\text {op }}$ an effectus; for a precise formulation and proof of this fact we refer to 2,7 ( or $180 \mathrm{~V}, 180 \mathrm{VII}$, and 180 X ahead). Note that the sum $f+g$ of two ncpsu-maps $f, g: \mathscr{A} \rightarrow \mathscr{B}$ between von Neumann algebras is again an ncpsu-map iff $f(1)+g(1) \leqslant 1$. The partial addition on
ncpsu-maps thereby defined has, aside from some fairly obvious properties (summarized by the fact that the category $\mathbf{W}_{\text {CPSU }}^{*}$ is $\mathbf{P C M}$-enriched), the following special trait: given ncpsu-maps $f: \mathscr{A} \rightarrow \mathscr{D}$ and $g: \mathscr{B} \rightarrow \mathscr{D}$ with $f(1)+g(1) \leqslant 1$ we may form an ncpsu-map $[f, g]: \mathscr{A} \times \mathscr{B} \rightarrow \mathscr{D}$ by $[f, g](a, b)=f(a)+g(b)$, and, moreover, every ncpsu-map $\mathscr{A} \times \mathscr{B} \rightarrow \mathscr{D}$ is of this form. This observation, which gives the product of $\mathbf{W}_{\text {CPSU }}^{*}$ a coproduct-like quality without forcing it to be a biproduct (which it's not), makes ( $\left.\mathbf{W}_{\text {CPSU }}^{*}\right)^{\text {op }}$ a FinPAC (see 180 VIII ).

For $\left(\mathbf{W}_{\text {CPSU }}^{*}\right)^{\text {op }}$ to be an effectus, we need a second ingredient: the complex number $\mathbb{C}$. Since the ncpsu-maps $p: \mathbb{C} \rightarrow \mathscr{A}$ are all of the form $\lambda \mapsto \lambda a$ for some effect $a \in[0,1]_{\mathscr{A}}$, the ncpsu-maps $p: \mathbb{C} \rightarrow \mathscr{A}$ (called predicates in this context) are not only endowed with a partial addition, but even form an effect algebra. This, combined with the observation that an ncpsu-map $f: \mathscr{A} \rightarrow \mathscr{B}$ is constant zero iff $f(1)=0$, makes $\left(\mathbf{W}_{\text {CPSU }}^{*}\right)^{\text {op }}$ an effectus in partial form (see 180 VIII ).

As you can see, there's nothing deep underlying $\left(\mathbf{W}_{\text {CPSU }}^{*}\right)^{\text {op }}$ being an effectus. In that respect effectus theory resembles topology: just as a topology provides a basis for notions such as compactness, connectedness, meagreness, and homotopy, so does an effectus provide a framework to study aspects of computation such as side effects (223 II) and purity (173 VII).

48 Let us quickly prove that every von Neumann algebra is isomorphic to a von Neumann algebra of operators on a Hilbert space (see VIII).

II Exercise Let $\Omega$ be a collection of np-functionals on a von Neumann algebra $\mathscr{B}$ that is faithful (see 21 II ). Show that a positive linear map $f: \mathscr{A} \rightarrow \mathscr{B}$ is normal iff $\omega \circ f$ is normal for all $\omega \in \Omega$.

III Proposition Given an np-map $\omega: \mathscr{A} \rightarrow \mathbb{C}$ on a von Neumann algebra $\mathscr{A}$, the $\operatorname{map} \varrho_{\omega}: \mathscr{A} \rightarrow \mathscr{B}\left(\mathscr{H}_{\omega}\right)$ from 30 VI is normal.
IV Proof Since by definition of $\mathscr{H}_{\omega}$ the vectors of the form $\eta_{\omega}(a)$ where $a \in \mathscr{A}$ are dense in $\mathscr{H}_{\omega}$, the vector functionals $\left\langle\eta_{\omega}(a),(\cdot) \eta_{\omega}(a)\right\rangle$ form a faithful collection of np-functionals on $\mathscr{B}\left(\mathscr{H}_{\omega}\right)$. Thus by $\prod$ it suffices to show given $a \in \mathscr{A}$ that $\left\langle\eta_{\omega}(a), \varrho_{\omega}(\cdot) \eta_{\omega}(a)\right\rangle \equiv \omega\left(a^{*}(\cdot) a\right)$ is normal, which it is, by 44 VIII
$\vee$ Exercise Show that the map $\varrho_{\Omega}$ from 30 IX is normal for every collection $\Omega$ of np-maps $\mathscr{A} \rightarrow \mathbb{C}$ on a von Neumann algebra $\mathscr{A}$.

VI Lemma Let $f: \mathscr{A} \rightarrow \mathscr{B}$ be an injective nmiu-map between von Neumann algebras. Then the image $f(\mathscr{A})$ is a von Neumann subalgebra of $\mathscr{B}$, and $f$ restricts to a nmiu-isomorphism from $\mathscr{A}$ to $f(\mathscr{A})$.

VII Proof We already know by 29 IX that $f(\mathscr{A})$ is a $C^{*}$-subalgebra of $\mathscr{A}$, and that $f$
restricts to a miu-isomorphism $f^{\prime}: \mathscr{A} \rightarrow f(\mathscr{A})$. The only thing left to show is that $f(\mathscr{A})$ is a von Neumann subalgebra of $\mathscr{B}$, because a miu-isomorphism between von Neumann algebras (being an order isomorphism) will automatically be a nmiu-isomorphism. Let $D$ be a bounded directed subset of $f(\mathscr{A})$. Note that $S:=\left(f^{\prime}\right)^{-1}(D)$ is a bounded directed subset of $\mathscr{A}$, and so $\bigvee D \equiv \bigvee f(S)=$ $f(\bigvee S)$, because $f$ is normal. Thus $\bigvee f(D) \in f(\mathscr{A})$, and so $f(\mathscr{A})$ is a von Neumann subalgebra of $\mathscr{B}$.

Theorem (normal Gelfand-Naimark) Every von Neumann algebra $\mathscr{A}$ is nmiuisomorphic to von Neumann algebra of operators on a Hilbert space.

Proof Recall that an element $a \in \mathscr{A}$ is zero iff $\omega(a)=0$ for all np-maps $\omega: \mathscr{A} \rightarrow \mathbb{C}$. It follows that the collection $\Omega$ of all np-maps $\mathscr{A} \rightarrow \mathbb{C}$ obeys the condition of 30 X , and so the miu-map $\varrho_{\Omega}: \mathscr{A} \rightarrow \mathscr{B}\left(\mathscr{H}_{\Omega}\right)$ (from 30 IX ) is injective. Since $\varrho_{\Omega}$ is also normal by $\bar{V}$, we see by $\bar{V}$ that $\varrho_{\Omega}$ restricts to a nmiu-isomorphism from $\mathscr{A}$ to the von Neumann subalgebra $\varrho_{\Omega}(\mathscr{A})$ of $\mathscr{B}\left(\mathscr{H}_{\Omega}\right)$.

### 3.1.3 Examples

## Matrices over von Neumann algebras

We'll show that the $C^{*}$-algebra of $N \times N$-matrices $M_{N}(\mathscr{A})$ over a von Neumann algebra $\mathscr{A}$ is itself a von Neumann algebra, and to this end, we prove something a bit more more general.

Theorem Given a von Neumann algebra $\mathscr{A}$, the $C^{*}$-algebra $\mathscr{B}^{a}(X) 32$ XIII of bounded adjointable module maps on a self-dual (361) Hilbert $\mathscr{A}$-module $X$ is a von Neumann algebra, and $\langle x,(\cdot) x\rangle: \mathscr{B}^{a}(X) \rightarrow \mathscr{A}$ is normal for every $x \in X$.
Proof We'll first show that a bounded directed subset $\mathscr{D}$ of $\mathscr{B}^{a}(X)_{\mathbb{R}}$ has a supremum (in $\mathscr{B}^{a}(X)_{\mathbb{R}}$ ). To obtain a candidate for this supremum, we first define a bounded form $[\cdot, \cdot]: X \times X \rightarrow \mathscr{A}$ in the sense of 36 IV and apply 36 V , To this end note that given $x \in X$ the subset $\{\langle x, T x\rangle: T \in \mathscr{D}\}$ of $\mathscr{A}_{\mathbb{R}}$ is bounded and directed, and so (since $\mathscr{A}$ is a von Neumann algebra) has a supremum. Since the the net $(\langle x, T x\rangle)_{T \in \mathscr{D}}$ converges ultraweakly to this supremum by 44 VI , we see that $\langle y, T z\rangle=\frac{1}{4} \sum_{k=0}^{3} i^{k}\left\langle y+i^{k} z, T\left(y+i^{k} z\right)\right\rangle$ converges ultraweakly to some element $[y, z]$ of $\mathscr{A}$ as $T \rightarrow \infty$ for all $y, z \in X$, giving us a form $[\cdot, \cdot]$ on $X$. Since $\|\langle y, T z\rangle\| \leqslant \sup _{T^{\prime} \in \mathscr{D}}\left\|T^{\prime}\right\|\|y\|\|z\|$ for all $T \in \mathscr{D}$ by 32 X , and thus $\|[y, z]\| \leqslant \sup _{T^{\prime} \in \mathscr{D}}\left\|T^{\prime}\right\|\|y\|\|z\|$ for all $y, z \in X$, we see that
the form $[\cdot, \cdot]$ is bounded. Since $X$ is self dual, there is by $36 \mathrm{~V} S \in \mathscr{B}^{a}(X)$ with $[y, z]=\langle y, S z\rangle$ for all $y, z \in X$; we'll show that $S$ is the supremum of $\mathscr{D}$.

To begin, given $T \in \mathscr{D}$ we have $\langle x, T x\rangle \leqslant \bigvee_{T^{\prime} \in \mathscr{D}}\left\langle x, T^{\prime} x\right\rangle=[x, x]=\langle x, S x\rangle$ for all $x \in X$, and so $T \leqslant S$ by 32 XV , that is, $S$ is an upper bound for $\mathscr{D}$. Given another upper bound $S^{\prime} \in \mathscr{B}^{a}(X)_{\mathbb{R}}$ of $\mathscr{D}$ (so $T \leqslant S^{\prime}$ for all $T \in \mathscr{D}$ ) we have $\langle x, T x\rangle \leqslant\left\langle x, S^{\prime} x\right\rangle$ and so $\langle x, S x\rangle=[x, x]=\bigvee_{T \in \mathscr{D}}\langle x, T x\rangle \leqslant\left\langle x, S^{\prime} x\right\rangle$ for all $x \in X$ implying that $S \leqslant S^{\prime}$. Hence $S$ is the supremum of $\mathscr{D}$ in $\mathscr{B}^{a}(X)_{\mathbb{R}}$. Note that since $\langle x, S x\rangle=\bigvee_{T \in \mathscr{D}}\langle x, T x\rangle$ we immediately see that $\langle x,(\cdot) x\rangle: \mathscr{B}^{a}(X) \rightarrow \mathscr{A}$ preserves bounded directed suprema for every $x \in X$.

It remains to be shown that there are sufficiently many np-functionals on $\mathscr{B}^{a}(X)$ in the sense that $T \in\left(\mathscr{B}^{a}(X)\right)_{+}$is zero when $\omega(T)=0$ for every npfunctional $\omega: \mathscr{B}^{a}(X) \rightarrow \mathbb{C}$. This is indeed the case for such an operator $T$, because $\xi(\langle x,(\cdot) x\rangle)$ is an np-functional on $\mathscr{B}^{a}(X)$ for every $x \in X$ and an np-functional $\xi: \mathscr{A} \rightarrow \mathbb{C}$, implying that $\xi(\langle x, T x\rangle)=0$, and $\langle x, T x\rangle=0$, and so $T=0$.

IV Exercise Let $\mathscr{A}$ be a von Neumann algebra, and let $N$ be a natural number.

1. Show that the $C^{*}$-algebra $M_{N}(\mathscr{A})$ of $N \times N$-matrices over $\mathscr{A}$ (see 331) is a von Neumann algebra.
2. Show that the map $A \mapsto \sum_{i j} a_{i}^{*} A_{i j} a_{j}: M_{N} \mathscr{A} \rightarrow \mathscr{A}$ is normal and completely positive, and that the map $A \mapsto \sum_{i j} a_{i}^{*} A_{i j} b_{j}: M_{N} \mathscr{A} \rightarrow \mathscr{A}$ is ultrastrongly and ultraweakly continuous for all $a_{1}, \ldots, a_{N}, b_{1}, \ldots, b_{N} \in \mathscr{A}$.
In particular, $A \mapsto A_{i j}: M_{N} \mathscr{A} \rightarrow \mathscr{A}$ is ultraweakly and ultrastrongly continuous for all $i, j$.
Show that a net $\left(A_{\alpha}\right)_{\alpha}$ in $M_{N} \mathscr{A}$ converges ultraweakly (ultrastrongly) to $B \in M_{N} \mathscr{A}$ iff $\left(A_{\alpha}\right)_{i j}$ converges ultraweakly (ultrastrongly) to $B_{i j}$ as $\alpha \rightarrow$ $\infty$ for all $i, j$.
3. Given an ncp-map $f: \mathscr{A} \rightarrow \mathscr{B}$ between von Neumann algebras, show that the cp-map $M_{N} f: M_{N} \mathscr{A} \rightarrow M_{N} \mathscr{B}$ from 33 III is normal.

## Commutative von Neumann algebras

50 Another important source of examples of von Neumann algebras is measure theory: we'll show that the bounded measurable functions on a finite complete
measure space $X$ form a commutative von Neumann algebra $L^{\infty}(X)$ when functions that are equal almost everywhere are identified (see 51 IX . In fact, we'll see in 70 III that every commutative von Neumann algebra is nmiu-isomorphic to a direct sum of $L^{\infty}(X)$ s. This is not only interesting in its own right, but will also be used in the next chapter to show that the only von Neumann algebras that can be endowed with a 'duplicator' are of the form $\ell^{\infty}(X)$ for some set $X$ (see 127 III).

We should probably mention that $L^{\infty}(X)$ can be defined for any measure space $X$, and is precisely a von Neumann algebra when $X$ is localisable see 63. This has the advantage that any commutative von Neumann algebra is nmiuisomorphic to a single $L^{\infty}(X)$ for some localisable measure space $X$, but since it has no other advantages relevant to this text we restrict ourselves to complete finite measure spaces.

We'll assume the reader is reasonably familiar with the basics of measure theory, and we'll only show a selection of results that we deemed important. For the other details, we refer to volumes 1 and 2 of [16]. Nevertheless, we'll recall some basic definitions to fix terminology, which is sometimes simpler than in [16] (because we're dealing with finite complete measure spaces), and sometimes modified to the complex-valued case (c.f. 133C of 16 ). A motivated reader will have no problem adapting the results from [16] to our setting.

Let $X$ be a finite and complete measure space. We'll denote the $\sigma$-algebra of measurable subsets of $X$ by $\Sigma_{X}$, and the measure by $\mu_{X}: \Sigma_{X} \rightarrow[0, \infty$ ) (or $\mu$ when no confusion is expected). That $X$ is finite means that $\mu(X)<\infty$ (which doesn't mean that the set $X$ is finite), and that $X$ is complete means that every subset $A$ of a negligible subset $B$ of $X$ is itself negligible. (Recall that $N \subseteq X$ is negligible when $N \in \Sigma_{X}$ and $\mu(N)=0$.) A function $f: X \rightarrow \mathbb{C}$ is measurable when the inverse image $f^{-1}(U)$ of any open subset $U$ of $\mathbb{C}$ is measurable (which happens precisely when both $x \mapsto f(x)_{\mathbb{R}}, x \mapsto f(x)_{\mathbb{I}}: X \rightarrow \mathbb{R}$ are measurable in the sense of 121 C of $[16]$ ). An important example of a measurable function on $X$ is the indicator function $\mathbf{1}_{A}$ of a measurable subset $A$ of $X$ (which is equal to 1 on $A$ and 0 elsewhere.)

The bounded measurable functions $f: X \rightarrow \mathbb{C}$ form a $C^{*}$-subalgebra of $\mathbb{C}^{X}$ that we'll denote by $\mathcal{L}^{\infty}(X)$. The space $\mathcal{L}^{\infty}(X)$ is not only closed with respect to the (supremum) norm on $\mathbb{C}^{X}$, but also with respect to coordinatewise limits of sequences (c.f. 121F of [16]). As a result, the coordinatewise (countable) supremum $\bigvee_{n} f_{n}$ of a bounded ascending sequence $f_{1} \leqslant f_{2} \leqslant \cdots$ in $\mathcal{L}^{\infty}(X)_{\mathbb{R}}$ is again in $\mathcal{L}^{\infty}(X)$, and is fact the supremum of $\left(f_{n}\right)_{n}$ in $\mathcal{L}^{\infty}(X)$. However $\mathcal{L}^{\infty}(X)$
might still not be a von Neumann algebra because not every bounded directed subset of $\mathcal{L}^{\infty}(X)_{\mathbb{R}}$ might have a supremum as we'll show presently; this is why we'll move from $\mathcal{L}^{\infty}(X)$ to $L^{\infty}(X)$ in a moment.

III For a counterexample to $\mathcal{L}^{\infty}(X)$ being always a von Neumann algebra we take $X$ to be the unit interval $[0,1]$ with the Lebesgue measure. Let $A$ be a non-measurable subset of $[0,1]$ (see 134B of [16]). The indicator functions $\mathbf{1}_{F}$ where $F$ is a finite subset of $A$ form a bounded directed subset $D$ of $\mathcal{L}^{\infty}([0,1])_{\mathbb{R}}$ that - so we claim - has no supremum. Indeed, note that since $f \in \mathcal{L}^{\infty}([0,1])_{\mathbb{R}}$ is an upper bound for $D$ iff $\mathbf{1}_{A} \leqslant f$, the least upper bound $h$ for $D$ would be the least bounded measurable function above $\mathbf{1}_{A}$. Surely, $h \neq \mathbf{1}_{A}$ for such $h$ (because otherwise $A$ would be measurable), so $h(x)>\mathbf{1}_{A}(x)$ for some $x \in[0,1]$. But then $h-\left(h(x)-\mathbf{1}_{A}(x)\right) \mathbf{1}_{\{x\}}<h$ is an upper bound for $D$ too contradicting the minimality of $h$. Whence $\mathcal{L}^{\infty}([0,1])$ is not a von Neumann algebra.

IV To deal with $L^{\infty}(X)$ we need to know a bit more about $\mathcal{L}^{\infty}(X)$, namely that the measure on $X$ can be extended to a an integral $\int: \mathcal{L}^{\infty}(X) \rightarrow \mathbb{C}$ (see 122 M of $[16]^{\dagger}$ with the following properties.

1. $\int\left(\mathbf{1}_{A}\right)=\mu(A)$ for every measurable subset $A$ of $X$.
2. $\int: \mathcal{L}^{\infty}(X) \rightarrow \mathbb{C}$ is a positive linear map (see 122 O of 16 ).
3. $\int \bigvee_{n} f_{n}=\bigvee_{n} \int f_{n}$ for every bounded sequence $f_{1} \leqslant f_{2} \leqslant \cdots$ in $\mathcal{L}^{\infty}(X)_{\mathbb{R}}$. (This is a special case of Levi's theorem, see 123A of [16.)

Unsurprisingly, the integral interacts poorly with the uncountable directed suprema that do exist in $\mathcal{L}^{\infty}(X)$ : for example, the set $D:=\left\{f \in[0,1]_{\mathcal{L}^{\infty}(X)}: \int f=0\right\}$ is directed, bounded, and has supremum 1 , but $\bigvee_{f \in D} \int f=0<1=\int \bigvee D$. What is surprising is that the lifting of $\int$ to $L^{\infty}(X)$ will be normal.
$\vee$ But let us first define $L^{\infty}(X)$. We say that $f, g \in \mathcal{L}^{\infty}(X)$ are equal almost everywhere and write $f \approx g$ when $f(x)=g(x)$ for almost all $x \in X$ (that is, $\{x \in X: f(x) \neq g(x)\}$ is negligible). It is easily seen that $\approx$ is an equivalence relation; we denote the equivalence class of an function $f \in \mathcal{L}^{\infty}(X)$ by $f^{\circ}$, and the set of equivalence classes by $L^{\infty}(X):=\left\{f^{\circ}: f \in \mathcal{L}^{\infty}(X)\right\}$, which becomes a commutative $C^{*}$-algebra when endowed with the same operations as $\mathcal{L}^{\infty}(X)$,

[^5]but with a slightly modified norm given by, for $\mathfrak{f} \equiv f^{\circ} \in L^{\infty}(X)$,
\[

$$
\begin{aligned}
\|\mathfrak{f}\| & =\min \left\{\|g\|: g \in \mathcal{L}^{\infty}(X) \text { and } g^{\circ}=\mathfrak{f}\right\} \\
& =\min \{\lambda \geqslant 0:|f(x)| \leqslant \lambda \text { for almost all } x \in X\} .
\end{aligned}
$$
\]

This is called the essential supremum norm. To see that $L^{\infty}(X)$ is complete one can use the fact that $\mathcal{L}^{\infty}(X)$ is complete in a slightly more general sense than discussed before: when a bounded sequence $f_{1}, f_{2}, \ldots$ in $\mathcal{L}^{\infty}(X)$ converges coordinatewise for almost all $x \in X$ to some bounded function $f: X \rightarrow \mathbb{C}$, this function $f$ is itself measurable (and so $f \in \mathcal{L}^{\infty}(X)$, c.f. 121F of 16$]$ ).

Another consequence of this is that a bounded ascending sequence $f_{1}^{\circ} \leqslant$ $f_{2}^{\circ} \leqslant \cdots$ in $L^{\infty}(X)$ (so $f_{1}, f_{2}, \ldots \in \mathcal{L}^{\infty}(X)$, and $f_{1}(x) \leqslant f_{2}(x) \leqslant \cdots$ for almost all $x \in X$ ) has a supremum $\bigvee_{n} f_{n}^{\circ}$ in $\mathcal{L}^{\infty}(X)$. Indeed, we'll have $\bigvee_{n} f_{n}^{\circ}=g^{\circ}$ for any bounded map $g: X \rightarrow \mathbb{C}$ with $g(x)=\bigvee_{n} f_{n}(x)$ for almost all $x \in X$.
Now, let us return to the integral. Since $\int f=\int g$ for all $f, g \in \mathcal{L}^{\infty}(X)$ with $f \approx g$ we get a map $\int: L^{\infty}(X) \rightarrow \mathbb{C}$ given by $\int f^{\circ}=\int f$. Clearly, $\int$ is positive and linear, and by (a slightly less special case of) Levi's theorem (123A of 16 ) we see that $\int \bigvee_{n} \mathfrak{f}_{n}=\bigvee_{n} \int \mathfrak{f}_{n}$ for any bounded ascending sequence $\mathfrak{f}_{1} \leqslant \mathfrak{f}_{2} \leqslant \cdots$ in $L^{\infty}(X)_{\mathbb{R}}$. Note that $\int: L^{\infty}(X) \rightarrow \mathbb{C}$ is also faithful, because if $\int f^{\circ}=\int f=0$ for some $f \in \mathcal{L}^{\infty}(X)$, then $f(x)=0$ for almost all $x \in X$, and so $f^{\circ}=0$. Now, the fact that $L^{\infty}(X)$ is a von Neumann algebra follows from the following general and rather surprising observation.

Proposition Let $\mathscr{A}$ be a $C^{*}$-algebra, and let $\tau: \mathscr{A} \rightarrow \mathbb{C}$ be a faithful positive map. If every bounded ascending sequence $a_{1} \leqslant a_{2} \leqslant \cdots$ of self-adjoint elements from $\mathscr{A}$ has a supremum $\bigvee_{n} a_{n}($ in $\mathscr{A} \mathbb{R})$ and $\tau\left(\bigvee_{n} a_{n}\right)=\bigvee_{n} \tau\left(a_{n}\right)$, then $\mathscr{A}$ is a von Neumann algebra, and $\tau$ is normal.

Proof Our first task is to show that a bounded directed subset $D$ of self-adjoint elements of $\mathscr{A}$ has a supremum $\bigvee D$ in $\mathscr{A} \mathbb{R}$. Since $\bigvee_{d \in D} \tau(d)$ is a supremum in $\mathbb{R}$ we can find $a_{1} \leqslant a_{2} \leqslant \cdots$ in $D$ with $\bigvee_{n} \tau\left(a_{n}\right)=\bigvee_{d \in D} \tau(d)$. We'll show that $\bigvee_{n} a_{n}$ is the supremum of $D$. Surely, any upper bound of $D$ being also an upper bound for $a_{1} \leqslant a_{2} \leqslant \cdots$ is above $\bigvee_{n} a_{n}$, so the only thing that we need to show is that $\bigvee_{n} a_{n}$ is an upper bound of $D$. So let $b \in D$ be given. The trick is to pick a sequence $b_{1} \leqslant b_{2} \leqslant \cdots$ in $D$ with $b \leqslant b_{1}$ and $a_{n} \leqslant b_{n}$ for all $n$ (which exists on account of $D$ 's directedness). Then $\bigvee_{n} a_{n} \leqslant \bigvee_{n} b_{n}$, and $\bigvee_{d \in D} \tau(d)=\bigvee_{n} \tau\left(a_{n}\right)=\tau\left(\bigvee_{n} a_{n}\right) \leqslant \tau\left(\bigvee_{n} b_{n}\right)=\bigvee_{n} \tau\left(b_{n}\right) \leqslant \bigvee_{d \in D} \tau(d)$, so $\tau\left(\bigvee_{n} a_{n}\right)=\tau\left(\bigvee_{n} b_{n}\right)$, which implies that $\bigvee_{n} a_{n}=\bigvee_{n} b_{n}$ as $\tau$ is faithful. Since then $b \leqslant b_{1} \leqslant \bigvee_{n} b_{n}=\bigvee_{n} a_{n}$ we see that $\bigvee_{n} a_{n}$ is an upper bound (and thus the supremum) of $D$. Moreover, since $\bigvee_{d \in D} \tau(d) \leqslant \tau(\bigvee D)=\tau\left(\bigvee_{n} a_{n}\right)=$
$\bigvee_{n} \tau\left(a_{n}\right) \leqslant \bigvee_{d \in D} \tau(d)$, we see that $\bigvee_{d \in D} \tau(d)=\tau(\bigvee D)$, and so $\tau$ is normal. Since $\tau$ is faithful and normal, $\mathscr{A}$ is a von Neumann algebra.

IX Corollary Given a finite complete measure space $X$ the $C^{*}$-algebra $L^{\infty}(X)$ is a commutative von Neumann algebra, and the assignment $f \mapsto \int f$ gives a faithful normal positive map $\int: L^{\infty}(X) \rightarrow \mathbb{C}$.

52 We'll show that any commutative von Neumann algebra $\mathscr{A}$ that admits a faithful np-functional $\omega: \mathscr{A} \rightarrow \mathbb{C}$ is nmiu-isomorphic to $L^{\infty}(X)$ for some finite complete measure space $X$. It makes sense to regard this result as a von Neumann algebra analogue of Gelfand's theorem for commutative $C^{*}$-algebras, (see 27 XXVII that any commutative $C^{*}$-algebra is miu-isomorphic to $C(Y)$ for some compact Hausdorff space $Y$.) But one should not take the comparison too far too lightly: while Gelfand's theorem readily yields a clean equivalence between commutative $C^{*}$-algebras and compact Hausdorff spaces (see 29), the fact that $L^{\infty}\left(X_{1}\right) \cong$ $L^{\infty}\left(X_{2}\right)$ for finite complete measure spaces $X_{1}$ and $X_{2}$ does not even imply that $X_{1}$ and $X_{2}$ have the same cardinality ${ }^{\ddagger}$ Obtaining an equivalence between commutative von Neumann algebras and measure spaces is nonetheless possible after a suitable non-trivial modification to the category of measure spaces (as is shown by Robert Furber in as of yet unpublished work.)

We obtain our finite complete measure space $X$ from the commutative von Neumann algebra $\mathscr{A}$ by taking for $X$ the compact Hausdorff space $\operatorname{sp}(\mathscr{A})$ of all miu-functionals on $\mathscr{A}$, and declaring that a subset $A$ of $X \equiv \operatorname{sp}(\mathscr{A})$ is measurable when $A$ is clopen up to a meagre subset (defined below, Пl). It takes some effort to show that this yields a $\sigma$-algebra in $\operatorname{sp}(\mathscr{A})$, and that the faithful np-functional $\omega: \mathscr{A} \rightarrow \mathbb{C}$ gives a finite complete measure on $\operatorname{sp}(\mathscr{A})$, but once this is achieved it's easily seen that $\mathscr{A} \cong C(\operatorname{sp}(\mathscr{A})) \cong L^{\infty}(\operatorname{sp}(\mathscr{A}))$.
II Definition Let $X$ be a topological space.

1. A subset $A$ of $X$ is called meagre when $A \subseteq \bigcup_{n} B_{n}$ for some closed subsets $B_{1} \subseteq B_{2} \subseteq \cdots$ of $X$ with empty interior (so $B_{n}^{\circ}=\varnothing$ for all $n$.)
2. Given $A, B \subseteq X$ we write $A \approx B$ when $A \cup B \backslash A \cap B$ is meagre.

[^6]3. We say that $A \subseteq B$ is almost clopen when $A \approx C$ for some clopen $C \subseteq X$.

Exercise Given a topological space $X$, verify the following facts.

1. A countable union $\bigcup_{n} A_{n}$ of meagre subsets $A_{1}, A_{2}, \ldots \subseteq X$ is meagre.
2. A subset of a meagre set is meagre.
3. $\bar{U} \approx U$ for every open subset $U$ of $X$.
(Hint: show that $\bar{U} \backslash U$ is closed with empty interior.)
4. $\bigcup_{n} A_{n} \approx \bigcup_{n} B_{n}$ for all $A_{1}, A_{2}, \ldots, B_{1}, B_{2}, \ldots \subseteq X$ with $A_{n} \approx B_{n}$.
5. $A \backslash B \approx A^{\prime} \backslash B^{\prime}$ for all $A, A^{\prime}, B, B^{\prime} \subseteq X$ with $A \approx A^{\prime}$ and $B \approx B^{\prime}$.
6. If $A, B \subseteq X$ are almost clopen, then $A \cup B$ and $A \backslash B$ are almost clopen.

The fact that the almost clopen subsets of the $\operatorname{spectrum} \operatorname{sp}(\mathscr{A})$ of a commutative von Neumann algebra $\mathscr{A}$ are closed under countable unions (and thus form a $\sigma$-algebra) relies on a special topological property of $\operatorname{sp}(\mathscr{A})$ that is described in IIII below.
Exercise Let $\mathscr{A}$ be a commutative von Neumann algebra. Using the fact that the Gelfand representation $\gamma_{\mathscr{A}}: \mathscr{A} \rightarrow C(\operatorname{sp}(\mathscr{A}))$ from 27 III is a miuisomorphism by 27 XXVII and thus an order isomorphism, show that $C(\operatorname{sp}(\mathscr{A}))$ is a commutative von Neumann algebra that is nmiu-isomorphic to $\mathscr{A}$ via $\gamma_{\mathscr{A}}$.
Proposition The spectrum $\operatorname{sp}(\mathscr{A})$ of a commutative von Neumann algebra $\mathscr{A}$ is extremally disconnected: the closure $\bar{U}$ of an open subset $U$ of $\operatorname{sp}(\mathscr{A})$ is open. Proof (Based on §6.1 of 69.)

Let $U$ be an open subset of $\operatorname{sp}(\mathscr{A})$, and let $\mathbf{1}_{U}$ be the indicator function of $U$. The set $D=\left\{f \in C(\operatorname{sp}(\mathscr{A})): f \leqslant \mathbf{1}_{U}\right\}$ is directed and bounded and so has a supremum $\bigvee D$ in $C(\operatorname{sp}(\mathscr{A}))$ since $C(\operatorname{sp}(\mathscr{A}))$ is a von Neumann algebra by 11 . Note that $0 \leqslant \bigvee D \leqslant 1$. We'll prove that $\bigvee D=\mathbf{1}_{\bar{U}}$, because this entails that $\mathbf{1}_{\bar{U}}$ is continuous, so that $\bar{U}$ is both open and closed.

Let $x \in U$ be given. By Urysohn's lemma (see 15.6 of 76], using here that $\operatorname{sp}(\mathscr{A})$ being a compact Hausdorff space, $27 \times X V$, is normal by 17.10 of 76]) there is $f \in[0,1]_{C(\operatorname{sp}(\mathscr{A}))}$ with $f(x)=1$ and $f(y)=0$ for all $y \in \operatorname{sp}(X) \backslash U$. It
follows that $f \in D$, and $f \leqslant \bigvee D \leqslant 1$, so that $1=f(x) \leqslant(\bigvee D)(x) \leqslant 1$, and $(\bigvee D)(x)=1$. By continuity of $\bigvee D$, we get $(\bigvee D)(x)=1$ for all $x \in \bar{U}$.

Let $y \in \operatorname{sp}(\mathscr{A}) \backslash U$ be given. Again by Urysohn's lemma there is $f \in$ $[0,1]_{C(\operatorname{sp}(\mathscr{A}))}$ with $f(y)=0$ and $f(x)=1$ for all $x \in \bar{U}$. Since $g \leqslant \mathbf{1}_{U} \leqslant f$ for every $g \in D$, we get $\bigvee D \leqslant f$, and so $0 \leqslant(\bigvee D)(y) \leqslant f(y)=0$, which implies that $(\bigvee D)(y)=0$. Hence $(\bigvee D)(y)=0$ for all $y \in \operatorname{sp}(\mathscr{A}) \backslash U$.

All in all we have $\bigvee D=\mathbf{1}_{\bar{U}}$, and so $\bar{U}$ is open.
V Corollary The almost clopen subsets of an extremally disconnected topological space $X$ form a $\sigma$-algebra.
VI Proof In light of 52 III it remains only to be shown that the union $\bigcup_{n} A_{n}$ of almost clopen subsets $A_{1}, A_{2}, \ldots$ is almost clopen. Let $C_{1}, C_{2}, \ldots \subseteq X$ be clopen with $A_{n} \approx C_{n}$ for each $n$. Then $\bigcup_{n} A_{n} \approx \bigcup_{n} C_{n}$, and $C:=\bigcup_{n} C_{n}$ is open (but not necessarily closed). Since $C \approx \bar{C}$ (by 52 III ), and $\bar{C}$ is clopen (as $X$ is extremally disconnected) we get $\bigcup_{n} A_{n} \approx \bar{C}$, so $\bigcup_{n} A_{n}$ is almost clopen.

54 The final ingredient we need to prove the main result, XI, of this section is the observation that an almost clopen subset of a compact Hausdorff space is equivalent to precisely one clopen, which follows from the following famous theorem.

II Baire category theorem A meagre subset of a compact Hausdorff space has empty interior.
III Proof Let $A$ be a meagre subset of a compact Hausdorff space $X$. So there are closed $B_{1} \subseteq B_{2} \subseteq \ldots$ with $A \subseteq \bigcup_{n} B_{n}$ and $B_{n}^{\circ}=\varnothing$ for all $n$. Then $U_{n}:=X \backslash B_{n}$ is an open dense subset of $X$ for each $n$. Since $A^{\circ} \subseteq\left(\bigcup_{n} B_{n}\right)^{\circ}=X \backslash\left(\overline{\bigcap_{n} U_{n}}\right)$ it suffices to show that $\bigcap_{n} U_{n}$ is dense in $X$. That is, given a non-empty open subset $V$ of $X$ we must show that $V \cap \bigcap_{n} U_{n} \neq \varnothing$.

Write $V_{1}:=V$. Since $U_{1}$ is open and dense, and $V_{1}$ is open and not empty, we have $U_{1} \cap V_{1} \neq \varnothing$. Since $X$ is regular (see e.g. 76]) we can find an open and non-empty subset $V_{2}$ of $X$ with $\bar{V}_{2} \subseteq U_{1} \cap V_{1}$. Continuing this process we obtain non-empty open subsets $V \equiv V_{1} \supseteq V_{2} \supseteq \cdots$ of $X$ with $\bar{V}_{n+1} \subseteq U_{n} \cap V_{n}$ for all $n$, and so $\bar{V}_{1} \supseteq V_{1} \supseteq \bar{V}_{2} \supseteq V_{2} \supseteq \cdots$. Since $X$ is compact, $\bigcap_{n} \bar{V}_{n}$ can not be empty, and neither will be $V \cap \bigcap_{n} U_{n} \supseteq \bigcap_{n} \bar{V}_{n}$.
Lemma For open subsets $U$ and $V$ of a compact Hausdorff space $X$,

$$
U \approx V \quad \Longleftrightarrow \quad \bar{U} \approx \bar{V} \quad \Longleftrightarrow \quad \bar{U}=\bar{V} .
$$

Proof As $U \approx \bar{U}$ by 52 III the only thing that is not obvious is that $\bar{U} \approx \bar{V} \Longrightarrow \vee$ $\bar{U}=\bar{V}$. So suppose that $\bar{U} \approx \bar{V}$. Then $U \backslash \bar{V}$ is empty, because it is an open subset of the meagre set $\bar{U} \cup \bar{V} \backslash \bar{U} \cap \bar{V}$ (which has empty interior by \||) In other words, we have $U \subseteq \bar{V}$, and thus $\bar{U} \subseteq \bar{V}$. Similarly, $\bar{V} \subseteq \bar{U}$, and so $\bar{V}=\bar{U}$.
Corollary Given an almost clopen subset $A$ of a compact Hausdorff space $X \quad$ VI there is precisely one clopen $C$ with $A \approx C$.
Proof When $C \approx A \approx C^{\prime}$ for clopen subsets $C, C^{\prime} \subseteq X$, we have $C \approx C^{\prime}$, and VII so $C=C^{\prime}$ by $\boxed{V}$.
Interestingly, a compact Hausdorff space is extremally disconnected iff each of VIII its open subsets is "measurable" in the sense of being almost clopen:
Proposition A compact Hausdorff space $X$ is extremally disconnected iff every open subset of $X$ is almost clopen.
Proof If $X$ is extremally disconnected, and $U$ is open subset of $X$, then $\bar{U}$ is clopen, and $\bar{U} \approx U$ by 52 III giving us that $U$ is almost clopen.

Conversely, suppose that each open subset of $X$ is almost clopen. To show that $X$ is extremally disconnected we must show that $\bar{U}$ is open given an open subset $U$ of $X$. Pick a clopen $C$ with $U \approx C$. Then $\bar{U} \approx U \approx C$ (by 52 III), and so $\bar{U}=C$ by $I \mathrm{~V}$.

Theorem Let $\mathscr{A}$ be a commutative von Neumann algebra $\mathscr{A}$. Recall that the XI Gelfand representation $\gamma_{\mathscr{A}}: \mathscr{A} \rightarrow C(\operatorname{sp}(\mathscr{A}))$ is a nmiu-isomorphism (by 53II), $C(\operatorname{sp}(\mathscr{A}))$ is a von Neumann algebra, and that the almost clopen subsets (see52 II) of $\operatorname{sp}(\mathscr{A})$ form a $\sigma$-algebra.

Given a faithful np-functional $\omega: \mathscr{A} \rightarrow \mathbb{C}$ there is a (unique) measure $\mu$ on the almost clopen subsets of $\operatorname{sp}(\mathscr{A})$ such that $\mu(A)=0$ iff $A$ is meagre, and $\mu(C)=\omega\left(\gamma_{\mathscr{A}}^{-1}\left(\mathbf{1}_{C}\right)\right)$ for every clopen subset $C$ of $\operatorname{sp}(\mathscr{A})$; and this turns $\operatorname{sp}(\mathscr{A})$ into a finite complete measure space.

With respect to this measure space a bounded function $f: \operatorname{sp}(\mathscr{A}) \rightarrow \mathbb{C}$ is measurable iff $f$ is continuous almost everywhere. Moreover, $f \mapsto f^{\circ}: C(\operatorname{sp}(\mathscr{A})) \rightarrow$ $L^{\infty}(\operatorname{sp}(\mathscr{A}))$ is a nmiu-isomorphism, and $\int f^{\circ}=\omega\left(\gamma_{\mathscr{A}}^{-1}(f)\right)$ for all $f \in C(X)$. All in all, we get the following commuting diagram.

..53, $54 .$.

XII Proof By VII we know that given an almost clopen subset $A$ of $\operatorname{sp}(\mathscr{A})$ there is a unique clopen $C_{A}$ with $A \approx C_{A}$, and so we may define $\mu(A):=\omega\left(\gamma_{\mathscr{A}}^{-1}\left(\mathbf{1}_{C_{A}}\right)\right)$. It is easily seen that $\mu$ is finitely additive. Further $\mu(A)=0$ for every meagre $A \subseteq$ $X$, and so $\mu(A)=\mu(B)$ when $A \approx B$. Conversely, an almost clopen subset $A$ of $\mathscr{A}$ with $\mu(A)=0$ is meagre, because for the unique clopen $C$ with $A \approx C$, we have $\omega\left(\gamma_{\mathscr{A}}^{-1}\left(\mathbf{1}_{C}\right)\right)=\mu(A)=0$, so that $\mathbf{1}_{C}=0$ and thus $C=\varnothing$-using here that $\omega$ is faithful.

To show that $\mu$ is a measure, it suffices to prove that $\bigwedge_{n} \mu\left(A_{n}\right)=0$ given $A_{1} \supseteq$ $A_{2} \supseteq \cdots$ with $\bigcap_{n} A_{n}=\varnothing$. To do this, pick clopen subsets $C_{1}, C_{2}, \ldots$ of $\operatorname{sp}(\mathscr{A})$ with $A_{n} \approx C_{n}$ for all $n$. Then $\bigwedge_{n} \mu\left(A_{n}\right)=\bigwedge_{n} \mu\left(C_{n}\right)=\omega\left(\gamma_{\mathscr{A}}^{-1}\left(\bigwedge_{n} \mathbf{1}_{C_{n}}\right)\right)$ using here that $\omega$ is normal. So to prove that $\bigwedge_{n} \mu\left(A_{n}\right)=0$ it suffices to show that $\bigwedge_{n} \mathbf{1}_{C_{n}}=0$, that is, given a lower bound $f$ of the $\mathbf{1}_{C_{n}}$ in $C(\operatorname{sp}(\mathscr{A}))_{\mathbb{R}}$ we must show that $f \leqslant 0$. Note that for such $f$ we have $f(x) \leqslant 0$ for all $x \in X \backslash \bigcap_{n} C_{n}$. Then $f(x) \leqslant 0$ for all $x \in X$ if we can show that $X \backslash \bigcap_{n} C_{n}$ is dense in $X$. But this indeed the case since $\bigcap_{n} C_{n} \approx \bigcap_{n} A_{n}=\varnothing$ is meagre, and therefore has empty interior (by III). Whence $\mu$ is a measure. Note that $\mu$ is finite, because $\mu(\operatorname{sp}(\mathscr{A}))=\omega(1)<\infty$, and complete, because a subset of a meagre set is meagre.

Let $h: \operatorname{sp}(\mathscr{A}) \rightarrow \mathbb{C}$ be a bounded function. We'll show that $h$ is continuous almost everywhere iff $h$ is measurable. Surely, if $h$ is continuous (everywhere), then $h$ is measurable (since every open subset $U$ of $\operatorname{sp}(\mathscr{A})$ is almost clopen, IX). So if $h$ is continuous almost everywhere, then $h$ is measurable too. For the converse, it suffices to show that $\varrho: h \mapsto h^{\circ}: C(\operatorname{sp}(\mathscr{A})) \rightarrow L^{\infty}(\operatorname{sp}(\mathscr{A}))$ is surjective. To this end, note first that $\varrho$ is injective, because a continuous function on $\operatorname{sp}(\mathscr{A})$ that is zero almost everywhere, is non-zero on a meagre set, and by $\prod_{\text {zero on }}$ a dense subset, and so is zero everywhere. Since the image of the injective miu-map $\varrho$ is norm closed in order to show that $\varrho$ is surjective it suffices to show that image of $\varrho$ is norm dense in $L^{\infty}(X)$. This is indeed the case since the elements of $L^{\infty}(\operatorname{sp}(\mathscr{A}))$ of the form $\sum_{n} \lambda_{n} \mathbf{1}_{A_{n}}^{\circ}$ where $\lambda_{1}, \ldots, \lambda_{N} \in \mathbb{C}$ and $A_{1}, \ldots, A_{N}$ are measurable (i.e. almost clopen) subsets of $\operatorname{sp}(\mathscr{A})$ are easily seen to be norm dense in $L^{\infty}(\operatorname{sp}(\mathscr{A}))$ (c.f. 243I of $[16]$ ), and are in the range of $\varrho$, because given an almost clopen $A \subseteq \operatorname{sp}(\mathscr{A})$ and a clopen $C$ with $A \approx C$ we have $\mathbf{1}_{A}^{\circ}=\mathbf{1}_{C}^{\circ}$ and $\mathbf{1}_{C} \in C(\operatorname{sp}(\mathscr{A}))$. Hence $\varrho$ is surjective.

It remains to be show that $\int f^{\circ}=\omega\left(\gamma_{\mathscr{A}}^{-1}(f)\right)$ for all $f \in C(\operatorname{sp}(\mathscr{A}))$, that is, $\int=\omega \circ \gamma_{\mathscr{A}}^{-1} \circ \varrho^{-1}$. By the previous discussion the linear span of the elements of $L^{\infty}(\operatorname{sp}(\mathscr{A}))$ of the form $\mathbf{1}_{C}^{\circ}$, where $C$ is (not just measurable but) clopen, is norm dense in $L^{\infty}(\operatorname{sp}(\mathscr{A}))$. Since $\int \mathbf{1}_{C}=\mu(C)=\omega\left(\gamma_{\mathscr{A}}^{-1}\left(\varrho^{-1}\left(\mathbf{1}_{C}^{\circ}\right)\right)\right.$ for all
clopen $C$, and both $\int$ and $\omega \circ \gamma_{\mathscr{A}}^{-1} \circ \varrho^{-1}$ are linear and bounded, we conclude that $\int=\omega \circ \gamma_{\mathscr{A}}^{-1} \circ \varrho^{-1}$, and so we are done.
To deduce from this that all commutative von Neumann algebras (and not just the ones admitting a faithful np-functional) are nmiu-isomorphic to direct sums of the form $\bigoplus_{i} L^{\infty}\left(X_{i}\right)$ where the $X_{i}$ are finite complete measure spaces we first need some basic facts concerning the projections of a commutative von Neumann algebra.

### 3.2 Projections

One pertinent feature of von Neumann algebras is an abundance of projections: above each effect $a$ there is a least projection $\lceil a\rceil$ we call the ceiling of $a$ (561); for every np-map $\omega: \mathscr{A} \rightarrow \mathscr{B}$ between von Neumann algebras there is a least projection $p$ with $\omega\left(p^{\perp}\right)=0$ called the carrier of $\omega$ (see 631; the directed supremum of projections is again a projection; the partial order of projections is complete (see 56XIII); and each element of a von Neumann algebra is the norm limit of linear combinations of projections (see 65 IV ). We'll prove all this and more in this section.

Definition An element $p$ of a $C^{*}$-algebra is a projection when $p^{*} p=p$.
Examples

1. The only projections in $\mathbb{C}$ are 0 and 1.
2. Given a measurable subset $A$ of a finite complete measure space $X$ the indicator function $\mathbf{1}_{A}$ is a projection in $L^{\infty}(X)$, and every projection in $L^{\infty}(X)$ is of this form.
3. Given a closed linear subspace $C$ of a Hilbert space $\mathscr{H}$ the inclusion $E: C \rightarrow \mathscr{H}$ is a bounded linear map, and $P_{C}:=E E^{*}: \mathscr{H} \rightarrow \mathscr{H}$ is a projection in $\mathscr{B}(\mathscr{H})$, and every projection in $\mathscr{B}(\mathscr{H})$ is of this form.

Exercise Show that in a $C^{*}$-algebra:

1. 0 and 1 are projections.
2. A projection $p$ is an effect, that is, $p^{*}=p$ and $0 \leqslant p \leqslant 1$.
3. The orthocomplement $p^{\perp} \equiv 1-p$ of a projection $p$ is a projection.
4. An effect $a$ is a projection iff $a a^{\perp}=0$.
$\vee \quad$ Lemma Let $a$ be an element of a $C^{*}$-algebra $\mathscr{A}$ with $\|a\| \leqslant 1$, and let $p$ and $q$ be projections on $\mathscr{A}$. Then $a^{*} p a \leqslant q^{\perp}$ iff $p a q=0$ iff $a q a^{*} \leqslant p^{\perp}$.
VI Proof Suppose that $a^{*} p a \leqslant q^{\perp}$. Then we have $q a^{*} p a q \leqslant q q^{\perp} q=0$ (see 25 II ) and so $p a q=0$, because $\|p a q\|^{2}=\left\|(p a q)^{*} p a q\right\|=0$ by the $C^{*}$-identity. Applying $(\cdot)^{*}$, we get $q a^{*} p=0$, and so both $q a^{*}=q a^{*} p^{\perp}$ and $a q=p^{\perp} a q$, giving us $a q a^{*}=p^{\perp} a q a^{*} p^{\perp} \leqslant p^{\perp}$, where we used that $a q a^{*} \leqslant a a^{*} \leqslant\left\|a a^{*}\right\|=\|a\|^{2} \leqslant 1$. By a similar reasoning, we get $a q a^{*} \leqslant p^{\perp} \Longrightarrow p a q=0 \Longrightarrow a^{*} p a \leqslant q^{\perp}$.
VII Exercise Let $a$ be an effect of a $C^{*}$-algebra $\mathscr{A}$, and $p$ be a projection from $\mathscr{A}$.
VIII Show that $a \leqslant p$ iff $p \sqrt{a}=\sqrt{a}$ iff $\sqrt{a} p=\sqrt{a}$ iff $p^{\perp} \sqrt{a}=0$ iff $\sqrt{a} p^{\perp}=0$ iff $a^{2} \leqslant p$ iff $p a=a$ iff $a p=a$ iff $p^{\perp} a=0$ iff $a p^{\perp}=0$ iff $\sqrt{a} \leqslant p$.
|X Show that $p \leqslant a$ iff $p \sqrt{a}=p$ iff $\sqrt{a} p=p$ iff $p \sqrt{a}^{\perp}=0$ iff $\sqrt{a}^{\perp} p=0$ iff $p \leqslant a^{2}$ iff $a p=p$ iff $p a=p$ iff $p a^{\perp}=0$ iff $a^{\perp} p=0$ iff $p \leqslant \sqrt{a}$.
X Lemma An effect $a$ of a $C^{*}$-algebra $\mathscr{A}$ is a projection iff the only effect below $a$ and $a^{\perp}$ is 0 .
XI Proof On the one hand, if $a$ is a projection, and $b$ is an effect with $b \leqslant a$ and $b \leqslant a^{\perp}$, then $a^{\perp} b=0$ and $a b=0$ by VIII, and so $b=a b+a^{\perp} b=0$. On the other hand, if 0 is the only effect below both $a$ and $a^{\perp}$, then $a a^{\perp} \equiv \sqrt{a} a^{\perp} \sqrt{a}$ being an effect below $a$, and below $a^{\perp}$, is zero, and so $a$ is projection, by IV. $\square$
XII Definition We say that projections $p$ and $q$ from a $C^{*}$-algebra $\mathscr{A}$ are orthogonal when $p q=0$, and we say that a subset of projections from $\mathscr{A}$ is orthogonal (and its elements are pairwise orthogonal) when all $p$ and $q$ from $E$ are either equal or orthogonal.
XIII Exercise Let $\mathscr{A}$ be a $C^{*}$-algebra.
5. Show that projections $p$ and $q$ from $\mathscr{A}$ are orthogonal iff $p q=0$ iff $q p=0$ iff $p q p=0$ iff $p+q \leqslant 1$ iff $p \leqslant q^{\perp}$ iff $p+q$ is a projection.
6. Show that a finite set of projections $p_{1}, \ldots, p_{n}$ from $\mathscr{A}$ is orthogonal iff $\sum_{i} p_{i} \leqslant 1$ iff $\sum_{i} p_{i}$ is a projection.
Show that, in that case, $\sum_{i} p_{i}$ is the least projection above $p_{1}, \ldots, p_{n}$.

XIV Exercise Let $p$ and $q$ be projections from a $C^{*}$-algebra with $p \leqslant q$.
Show that $q-p$ is a projection (either directly, or using XIII).

### 3.2.1 Ceiling and Floor

Proposition Above every effect $b$ of a von Neumann algebra $\mathscr{A}$, there is a smallest projection, $\lceil b\rceil$, we call the ceiling of $b$, given by $\lceil b\rceil=\bigvee_{n=0}^{\infty} b^{1 / 2^{n}}$. Moreover, if $a \in \mathscr{A}$ commutes with $b$, then $a$ commutes with $\lceil b\rceil$.
Proof Let $p$ denote the supremum of $0 \leqslant b \leqslant b^{1 / 2} \leqslant b^{1 / 4} \leqslant \cdots \leqslant 1$. |
To begin, note that if $a \in \mathscr{A}$ commutes with $b$, then $a$ commutes with $p$. Indeed, for such $a$ we have $a \sqrt{b}=\sqrt{b} a$ by 23 VIII , and so $a b^{1 / 2^{n}}=b^{1 / 2^{n}} a$ for each $n$ by induction. Thus $a p=p a$ by 44 XIII
Let us prove that $p$ is a projection, i.e. $p^{2}=p$. Since $p \leqslant 1$, we already have $p^{2} \equiv \sqrt{p} p \sqrt{p} \leqslant p$ by 25 II , and so we only need to show that $p \leqslant p^{2}$. We have:

$$
\begin{aligned}
p^{2} & =\bigvee_{m} \sqrt{p} b^{1 / 2^{m}} \sqrt{p} & & \text { by } 44 \mathrm{VIII} \\
& =\bigvee_{m} b^{1 / 2^{m+1}} p b^{1 / 2^{m+1}} & & \text { by } \text { TII and } 23 \mathrm{VII} \\
& =\bigvee_{m} \bigvee_{n} b^{1 / 2^{m+1}} b^{1 / 2^{n}} b^{1 / 2^{m+1}} & & \text { by } 44 \mathrm{VIII}
\end{aligned}
$$

Thus $p^{2} \geqslant b^{1 / 2^{k}}$ for each $k$ (taking $n=m=k+1$,) and so $p^{2} \geqslant p$.
It remains to be shown that $p$ is the least projection above $b$. Let $q$ be a projection in $\mathscr{A}$ with $b \leqslant q$; we must show that $q \leqslant p$. We have $b^{1 / 2} \leqslant q$ by 55 VIII , and so $b^{1 / 2^{n}} \leqslant q$ for each $n$ by induction. Hence $p \leqslant q$.
Proposition Below every effect $b$ of a von Neumann algebra $\mathscr{A}$, there is greatest projection, $\lfloor b\rfloor$, we call the floor of $b$, given by $\lfloor b\rfloor=\bigwedge_{n=0}^{\infty} b^{2^{n}}$.
Moreover, if $a \in \mathscr{A}$ commutes with $b$, then $b$ commutes with $\lfloor b\rfloor$.
Proof Let $p$ denote the infimum of $1 \geqslant b \geqslant b^{2} \geqslant b^{4} \geqslant \cdots \geqslant 0$.
If $a \in \mathscr{A}$ commutes with $b$, then $a$ commutes with $p$. Indeed, such $a$ commutes with $b^{2}$ (because $a b^{2}=b a b=b^{2} a$, ) and so $a$ commutes with $b^{2^{n}}$ for each $n$ by induction. Thus $a$ commutes with $p \equiv \bigwedge_{n} b^{2^{n}}$ (by a variation on 44 XII)
To see that $p$ is a projection, c.q. $p^{2}=p$, we only need to show that $p \leqslant p^{2}$, because we get $p^{2} \equiv \sqrt{p} p \sqrt{p} \leqslant p$ from $p \leqslant 1$ (using 25II) Now, since

$$
\begin{aligned}
p^{2} & =\bigwedge_{m} \sqrt{p} b^{2^{m}} \sqrt{p} & & \text { by a variation on } 44 \mathrm{VIII} \\
& =\bigwedge_{m} b^{2^{m-1}} p b^{2^{m-1}} & & \text { by } \text { VIII and } 23 \mathrm{VII} \\
& =\bigwedge_{m} \bigwedge_{n} b^{2^{m-1}} b^{2^{n}} b^{2^{m-1}} & & \text { by } 44 \mathrm{VIIII}
\end{aligned}
$$

and $p \leqslant b^{2^{m-1}} b^{2^{n}} b^{2^{m-1}}$ for all $n, m$, we get $p \leqslant p^{2}$.
$\times$ It remains to be shown that $p$ is the greatest projection below $b$. Let $q$ be a projection in $\mathscr{A}$ with $q \leqslant b$. We must show that $q \leqslant p$. Since $q \leqslant b$, we have $q \leqslant$ $b^{2}$ (by 55IX), and so $q \leqslant b^{2^{n}}$ for each $n$ by induction. Thus $q \leqslant p \equiv \bigwedge_{n} b^{2^{n}}$.

XI Exercise Show that given an effect $a$ and a projection $p$ in a von Neumann algebra $\mathscr{A}$ we have

1. $p a=a$ iff $a p=a$ iff $\lceil a\rceil \leqslant p$, and
2. $p a=p$ iff $a p=p$ iff $p \leqslant\lfloor a\rfloor$.

Conclude that $\lceil a\rceil$ is the least projection $p$ with $a=a p$ (or, equivalently, $a=p a$ ), and that $\lfloor a\rfloor$ is the greatest projection $p$ with $p=a p$ (or, equivalently, $p=p a$.)

In particular, $a=a\lceil a\rceil=\lceil a\rceil a$ and $\lfloor a\rfloor=a\lfloor a\rfloor=\lfloor a\rfloor a$.
XII Example Given a finite complete measure space $X$ we have

$$
\left\lceil f^{\circ}\right\rceil=\mathbf{1}_{\{x \in X: f(x)>0\}}^{\circ} \quad \text { and } \quad\left\lfloor f^{\circ}\right\rfloor=\mathbf{1}_{\{x \in X: f(x)=0\}}^{\circ}
$$

for every $f \in \mathcal{L}^{\infty}(X)$ with $0 \leqslant f^{\circ} \leqslant 1$.
XIII Exercise Let $a, b$ be effects of a von Neumann algebra $\mathscr{A}$, and let $\lambda \in[0,1]$.

1. Show that $\lceil a\rceil^{\perp}=\left\lfloor a^{\perp}\right\rfloor$ and $\lfloor a\rfloor^{\perp}=\left\lceil a^{\perp}\right\rceil$.
2. Show that $\lceil\lambda a\rceil=\lceil a\rceil$ when $\lambda \neq 0$.

Use this to prove that $\left\lceil\lambda a+\lambda^{\perp} b\right\rceil$ is the supremum of $\lceil a\rceil$ and $\lceil b\rceil$ in the poset of projections of $\mathscr{A}$ when $\lambda \neq 0$ and $\lambda \neq 1$.
3. Show that $\lfloor a\rfloor=\left\lfloor a^{2}\right\rfloor$ and $\lceil a\rceil=\left\lceil a^{2}\right\rceil$.

XIV Lemma The supremum of a directed set $D$ of projections from a von Neumann algebra $\mathscr{A}$ is a projection.
XV Proof Writing $p=\bigvee D$, we must show that $p^{2}=p$. Note that $d p=d$ for all $d \in D$ (by 55IX because $d \leqslant p$.) Now, on the one hand, $(d)_{d \in D}$ converges ultraweakly to $p$. On the other hand, $(d p)_{d \in D}$ converges ultraweakly to $p^{2}$ by 44 VII Hence $p=p^{2}$ by uniqueness of ultraweak limits.

XVI Exercise Deduce from this result that every set $A$ of projections from $\mathscr{A}$ has a supremum $\bigcup A$ and an infimum $\bigcap A$ in the poset of projections from $\mathscr{A}$.
(Hint: use XIII and the fact that $p \mapsto p^{\perp}$ is an order anti-isomorphism on the poset of projections on $\mathscr{A}$.)

Exercise Let $\mathscr{A}$ be a von Neumann algebra.

1. Show that $\lceil\bigvee D\rceil=\bigcup_{d \in D}\lceil d\rceil$ for every directed set $D$ of effects from $\mathscr{A}$.
2. Show that $\lfloor\wedge D\rfloor=\bigcap_{d \in D}\lfloor d\rfloor$ for every filtered set $D$ of effects from $\mathscr{A}$.
3. Show that $\lceil\cdot\rceil$ does not preserve filtered infima, and $\lfloor\cdot\rfloor$ does not preserve directed suprema. (Hint: $1, \frac{1}{2}, \frac{1}{3}, \ldots$.)
Conclude that $\lceil\cdot\rceil$ and $\lfloor\cdot\rfloor$ are neither ultraweakly, ultrastrongly nor norm continuous as maps from $[0,1]_{\mathscr{A}}$ to $[0,1]_{\mathscr{A}}$.

Exercise Show that for a family $\left(p_{i}\right)_{i \in I}$ of pairwise orthogonal projections XVIII (with $I$ potentially uncountable) the series $\sum_{i} p_{i}$ converges ultrastrongly to $\bigcup_{i} p_{i}$.
(Hint: use the fact that $\sum_{i \in F} p_{i}=\bigcup_{i \in F} p_{i}$ for finite subsets $F$ of $I$ by 55 XIII)
Lemma Let $a, b$ be effects of a von Neumann algebra $\mathscr{A}$. Then $\lfloor\sqrt{a} b \sqrt{a}\rfloor$ is the greatest projection below $a$ and $b$, that is, $\lfloor\sqrt{a} b \sqrt{a}\rfloor=\lfloor a\rfloor \cap\lfloor b\rfloor$.
Proof Surely, $\lfloor\sqrt{a} b \sqrt{a}\rfloor \leqslant \sqrt{a} b \sqrt{a} \leqslant a$. Let us prove that $\lfloor\sqrt{a} b \sqrt{a}\rfloor \leqslant b$. To this end, recall that (by 55IX) a projection $e$ is below an effect $c$ iff $e c=e$ iff $e \sqrt{c}=e$. In particular, since $\lfloor\sqrt{a} b \sqrt{a}\rfloor \leqslant \sqrt{a} b \sqrt{a}$ and $\lfloor\sqrt{a} b \sqrt{a}\rfloor \leqslant a$, we get

$$
\lfloor\sqrt{a} b \sqrt{a}\rfloor=\lfloor\sqrt{a} b \sqrt{a}\rfloor \sqrt{a} b \sqrt{a}\lfloor\sqrt{a} b \sqrt{a}\rfloor=\lfloor\sqrt{a} b \sqrt{a}\rfloor b\lfloor\sqrt{a} b \sqrt{a}\rfloor,
$$

and so $\lfloor\sqrt{a} b \sqrt{a}\rfloor b^{\perp}\lfloor\sqrt{a} b \sqrt{a}\rfloor=0$, which implies that $\lfloor\sqrt{a} b \sqrt{a}\rfloor \leqslant b$ by 55 V . Now, let $e$ be a projection below $a$ and $b$, that is, $e \sqrt{a}=e$ and $e b=e$. We must show that $e \leqslant\lfloor\sqrt{a} b \sqrt{a}\rfloor$, or equivalently, $e \leqslant \sqrt{a} b \sqrt{a}$, or put yet differently, $e \sqrt{a} b \sqrt{a}=e$. But this is obvious: $e=e \sqrt{a}=e b \sqrt{a}=e \sqrt{a} b \sqrt{a}$.

Having seen that $\lfloor\sqrt{a} b \sqrt{a}\rfloor=\lfloor a\rfloor \cap\lfloor b\rfloor$ in 57 one might wonder whether there is a similar expression for $\lceil\sqrt{a} b \sqrt{a}\rceil$, but this doesn't seem to exist. However, for projections $p$ and $q$ we have $\lceil p q p\rceil=p \cap\left(p^{\perp} \cup q\right)$ as we'll show below.
Lemma Let $p$ be a projection, and let $a$ be an effect of a von Neumann algebra with $a \leqslant p$. We have $p-\lceil a\rceil=\lfloor p-a\rfloor$.
Proof We must show that $p-\lceil a\rceil$ is the greatest projection below $p-a$. To begin, $p-\lceil a\rceil \leqslant p-a$, because $a \leqslant\lceil a\rceil$. Further, since $a \leqslant p$, we have $\lceil a\rceil \leqslant p$, and so $p-\lceil a\rceil$ is a projection (by 55 XIV).

Let $q$ be a projection below $p-a$. We must show that $q \leqslant p-\lceil a\rceil$. The trick is to note that $a \leqslant p-q$. Since $p-q$ is a projection (by 55XIV because $q \leqslant p-a \leqslant p$ ), we have $\lceil a\rceil \leqslant p-q$, and so $q \leqslant p-\lceil a\rceil$.

IV Proposition We have $\lceil p q p\rceil=p \cap\left(p^{\perp} \cup q\right)$ for all projections $p$ and $q$ from a von Neumann algebra.
$\vee$ Proof Observe that $\left(p \cap\left(p^{\perp} \cup q\right)\right)^{\perp}=p^{\perp} \cup\left(p \cap q^{\perp}\right)$. Since $p^{\perp}$ and $p \cap q^{\perp}$ are disjoint, we have $p^{\perp} \cup\left(p \cap q^{\perp}\right)=p^{\perp}+p \cap q^{\perp}$, and so $p \cap\left(p^{\perp} \cup q\right)=p-p \cap q^{\perp}$.

By point V, it suffices to show that $\lceil p q p\rceil=p-p \cap q^{\perp}$, that is, $p-\lceil p q p\rceil=$ $p \cap q^{\perp}$. Since $p-\lceil p q p\rceil=\lfloor p-p q p\rfloor$ by $\Pi$ and $\left\lfloor p q^{\perp} p\right\rfloor=p \cap q^{\perp}$ by 57 we are done.

### 3.2.2 Range and Support

59 Notation Let $\mathscr{A}$ be a von Neumann algebra. Because it will be very convenient we extend the definition of $\lceil b\rceil$ to all positive $b$ from $\mathscr{A}$ - contrary to what the notation suggests, $b \leqslant\lceil b\rceil$ — by defining $\lceil b\rceil=\left\lceil\|b\|^{-1} b\right\rceil$ when $b \nless 1$.

Now, given an arbitrary element $b$ of $\mathscr{A}$, we'll call $\lceil b):=\left\lceil b^{*} b\right\rceil$ the support (projection) of $b$, and ( $b\rceil:=\left\lceil b b^{*}\right\rceil$ the range (projection) of $b$.

II Remark Some explanation is in order here. We did not just introduce the range and support notation for its own sake, but will use it extensively in 83.4 thanks to calculation rules such as $a b=0 \Longleftrightarrow\lceil a)(b\rceil=0$ (see 60 VIII$)$. The notation was chosen such that $(b\rceil b=b=b\lceil b)$ (see VIl). Good examples are

$$
\lceil|x\rangle\langle y|)=|y\rangle\langle y| \quad \text { and } \quad(|x\rangle\langle y|\rceil=|x\rangle\langle x|
$$

for unit vectors $x$ and $y$ from a Hilbert space $\mathscr{H}$.
III Exercise Let $a$ and $b$ be positive elements of a von Neumann algebra $\mathscr{A}$.

1. Given a projection $p$ in $\mathscr{A}$ show that $p a=a$ iff $a p=a$ iff $\lceil a\rceil \leqslant p$.
(In particular, $\lceil a\rceil$ is the least projection $p$ of $\mathscr{A}$ with $a p=a$.)
2. Show that $\lceil a\rceil a=a\lceil a\rceil$, and if fact, if $b \in \mathscr{A}$ commutes with $a$ then $b$ commutes with $\lceil a\rceil$.
3. Show that $a=0$ iff $\lceil a\rceil=0$.
4. Show that $\lceil a\rceil=\lceil\lambda a\rceil$ for every $\lambda>0$.

Show that $\lceil a+b\rceil=\lceil a\rceil \cup\lceil b\rceil$.
5. Show that $\left\lceil a^{2}\right\rceil=\lceil a\rceil$.

Exercise Let $a$ be a self-adjoint element of a von Neumann algebra.

1. Show that $\left\lceil a_{+}\right\rceil\left\lceil a_{-}\right\rceil=0$. (Hint: recall from $24 \|$ that $a_{+} a_{-}=0$.)
2. Show that $\left\lceil a_{+}\right\rceil a=a\left\lceil a_{+}\right\rceil=a_{+}$and $\left\lceil a_{-}\right\rceil a=a\left\lceil a_{-}\right\rceil=-a_{-}$.

Exercise Show that $\lceil\bigvee D\rceil=\bigcup_{d \in D}\lceil d\rceil$ for every bounded directed set of positive elements of a von Neumann algebra $\mathscr{A}$.

Exercise Let $a$ and $b$ be elements of a von Neumann algebra.

1. Show that $\lceil a) \equiv\left\lceil a^{*} a\right\rceil$ is the least projection $p$ of $\mathscr{A}$ with $a p=a$.
2. Show that $(a\rceil \equiv\left\lceil a a^{*}\right\rceil$ is the least projection $p$ of $\mathscr{A}$ with $p a=a$.
3. Show that $\left\lceil a^{*}\right)=(a\rceil$ and $\left(a^{*}\right\rceil=\lceil a)$.
4. Show that $\lceil a b) \leqslant\lceil b)$ and $(a b\rceil \leqslant(a\rceil$.

Exercise Let $T$ be a bounded operator on a Hilbert space $\mathscr{H}$.

1. Show that $(T\rceil$ is the projection onto the closure $\overline{\operatorname{Ran}(T)}$ of the range of $T$.
2. Show that $\lceil T)$ is the projection onto the support of $T$, i.e. the orthocomplement $\operatorname{Ker}(T)^{\perp}$ of the kernel of $T$.
3. Show that $\lfloor T\rfloor$ is the projection on $\{x \in \mathscr{H}: T x=x\}$ when $T$ is an effect.

Lemma Given a positive element $a$ of a von Neumann algebra $\mathscr{A}$ and an npfunctional $\omega: \mathscr{A} \rightarrow \mathbb{C}$ we have $\omega(a)=0$ iff $\omega(\lceil a\rceil)=0$.
Proof Note that if $a=0$, the stated result is clearly correct, and the other case, when $\|a\| \neq 0$, the problem reduces to the case that $0 \leqslant a \leqslant 1$ by replacing $a$ by $\frac{a}{\|a\| \|}$. So let us just assume that $a \in[0,1]_{\mathscr{A}}$ to begin with. For similar reasons, we may assume that $\omega(1) \leqslant 1$.

Now, since $0 \leqslant a \leqslant\lceil a\rceil$ we have $0 \leqslant \omega(a) \leqslant \omega(\lceil a\rceil)$, so $\omega(\lceil a\rceil)=0 \Longrightarrow$ $\omega(a)=0$ is obvious. It remains to be shown that $\omega(\lceil a\rceil)=0$ given $\omega(a)=0$. Since $\lceil a\rceil=\bigvee_{n} a^{1 / 2^{n}}$ (by 561 and $\omega$ is normal, we have $\omega(\lceil a\rceil)=\bigvee_{n} \omega\left(a^{1 / 2^{n}}\right)$,
and so it suffices to show that $\omega\left(a^{1 / 2^{n}}\right)=0$ for each $n$. As a result of Kadison's inequality (see 30IV) we have $\omega(\sqrt{a})^{2} \leqslant \omega(a)=0$, and so $\omega(\sqrt{a})=0$. Since then $\omega(\sqrt{\sqrt{a}})=0$ by the same token, and so on, we get $\omega\left(a^{1 / 2^{n}}\right)=0$ for all $n$ by induction.

Proposition For positive elements $a$ and $b$ of a von Neumann algebra $\mathscr{A}$,

$$
\lceil a\rceil \leqslant\lceil b\rceil \quad \Longleftrightarrow \quad \forall \omega[\omega(b)=0 \quad \Longrightarrow \quad \omega(a)=0 \quad],
$$

where $\omega$ ranges over all np-functionals $\mathscr{A} \rightarrow \mathbb{C}$.
IV Proof When $\lceil a\rceil \leqslant\lceil b\rceil$ and $\omega$ is an np-functional on $\mathscr{A}$ with $\omega(b)=0$, then $0 \leqslant$ $\omega(\lceil a\rceil) \leqslant \omega(\lceil b\rceil)=0$ (by $\rrbracket$, and so $\omega(\lceil a\rceil)=0$, so that $\omega(a)=0$ (again by $\rrbracket$ ).

For the other direction, assume that $\omega(b)=0 \quad \Longrightarrow \quad \omega(a)=0$ for every np-functional $\omega$ on $\mathscr{A}$; we must show that $\lceil a\rceil \leqslant\lceil b\rceil$, or in other words, $\lceil b\rceil^{\perp}\lceil a\rceil\lceil b\rceil^{\perp}=0$. Let $\omega: \mathscr{A} \rightarrow \mathbb{C}$ be an arbitrary np-functional; it suffices to show that $\omega\left(\lceil b\rceil^{\perp}\lceil a\rceil\lceil b\rceil^{\perp}\right)=0$. Since $\lceil b\rceil^{\perp} b\lceil b\rceil^{\perp}=0$ we have $\omega\left(\lceil b\rceil^{\perp} b\lceil b\rceil^{\perp}\right)=$ 0 and so $\omega\left(\lceil b\rceil^{\perp} a\lceil b\rceil^{\perp}\right)=0$ (by assumption, because $\omega\left(\lceil b\rceil^{\perp}(\cdot)\lceil b\rceil^{\perp}\right)$ is an npfunctional on $\mathscr{A}$ as well), which implies that $\omega\left(\lceil b\rceil^{\perp}\lceil a\rceil\lceil b\rceil^{\perp}\right)=0$ by $\|$,
V Proposition Let $f: \mathscr{A} \rightarrow \mathscr{B}$ be an np-map between von Neumann algebras. Then $\lceil f(a)\rceil=\lceil f(\lceil a\rceil)\rceil$ for every $a \in \mathscr{A}_{+}$.
VI Proof By III it suffices to show that $\omega(f(a))=0$ iff $\omega(f(\lceil a\rceil))=0$ for every np-functional $\omega: \mathscr{B} \rightarrow \mathbb{C}$, and this is indeed the case by $\square$

VII Exercise Let $a$ and $b$ be elements of a von Neumann algebra $\mathscr{A}$.

1. Deduce from $\nabla$ that $\left\lceil a^{*} b a\right\rceil=\left\lceil a^{*}\lceil b\rceil a\right\rceil$ when $b \geqslant 0$.
2. Conclude that $\lceil a b)=\lceil\lceil a) b)$ and $(a b\rceil=(a(b\rceil\rceil$ (see 59 $)$.

VIII Exercise Let $a$ and $b$ be elements of a von Neumann algebra $\mathscr{A}$.

1. Show that $c b=0$ iff $\lceil c)(b\rceil=0$ iff $\lceil c) \leqslant(b\rceil^{\perp}$ for $c \in \mathscr{A}$.
(Hint: if $c b=0$, then $\left\lceil b^{*} c^{*} c b\right\rceil \equiv\left\lceil b^{*}\left\lceil c^{*} c\right\rceil b\right\rceil=0$ by VII).
2. Show that $c_{1} b=c_{2} b \Longrightarrow c_{1}=c_{2}$ for all $c_{1}, c_{2} \in \mathscr{A}$ with $\left\lceil c_{i}\right) \leqslant(b\rceil$.
3. Show that $b^{*} c_{1} b=b^{*} c_{2} b \Longrightarrow c_{1}=c_{2}$ for all $c_{1}, c_{2} \in(b\rceil \mathscr{A}(b\rceil$

Exercise Let $f: \mathscr{A} \rightarrow \mathscr{B}$ be an np-map between von Neumann algebras.

1. Show that $\lceil f(p \cup q)\rceil=\lceil f(p)\rceil \cup\lceil f(q)\rceil$ for all projections $p$ and $q$ in $\mathscr{A}$. (Hint: recall from 56 XIII that $p \cup q=\left\lceil\frac{1}{2} p+\frac{1}{2} q\right\rceil$.)
2. Deduce from this and $\bigvee$ that $\lceil f(\bigcup A)\rceil=\bigcup_{a \in A}\lceil f(a)\rceil$ for every set of projections $A$ from $\mathscr{A}$.
3. Show that there is a greatest projection $e$ in $\mathscr{A}$ with $f(e)=0$.

Given the rule $\lceil f(\lceil a\rceil)\rceil=\lceil f(a)\rceil$ for an np-map $f$ and self-adjoint $a$ one might surmise that the equation $\lceil f(\lceil a))\rceil=\lceil f(a))$ holds for arbitrary $a$; but one would be mistaken to do so. We can, however, recover an inequality by assuming that $f$ is completely positive, see II. One of its corollaries is that ncpsu-isomorphisms are in fact nmiu-isomorphisms (see 99 IX ).

Proposition Given an ncp-map $f: \mathscr{A} \rightarrow \mathscr{B}$ between von Neumann algebras we have, for all $a \in \mathscr{A}$,

$$
\lceil f(\lceil a))\rceil \leqslant\lceil f(a)) \quad \text { and } \quad\lceil f((a\rceil)\rceil \leqslant(f(a)\rceil
$$

Proof Since $f(a)^{*} f(a) \leqslant\|f(1)\|^{2} f\left(a^{*} a\right)$ by 34 XIV, we get $\lceil f(a)) \equiv\left\lceil f(a)^{*} f(a)\right\rceil \leqslant$ III $\left.\left.\left\lceil\|f(1)\|^{2} f\left(a^{*} a\right)\right\rceil \leqslant\left\lceil f\left(a^{*} a\right)\right\rceil=\left\lceil f\left(\left\lceil a^{*} a\right\rceil\right)\right\rceil \equiv \mid f(\mid a)\right)\right\rceil$.

One obtains $\lceil f((a\rceil)\rceil \leqslant(f(a)\rceil$ along similar lines.
Proposition Let $f: \mathscr{A} \rightarrow \mathscr{B}$ be a ncpsu-map between von Neumann algebras.
Then $\lfloor f(a)\rfloor=\lfloor f(\lfloor a\rfloor)\rfloor$ for every effect $a$ from $\mathscr{A}$.
Proof Since $\lfloor a\rfloor \leqslant a$, we have $\lfloor f(\lfloor a\rfloor)\rfloor \leqslant\lfloor f(a)\rfloor$. Thus we only need to show II that $\lfloor f(a)\rfloor \leqslant\lfloor f(\lfloor a\rfloor)\rfloor$, or equivalently, $\lfloor f(a)\rfloor \leqslant f(\lfloor a\rfloor)$. We have

$$
\lfloor f(a)\rfloor \stackrel{\text { 56XIII }}{=}\left\lfloor f(a)^{2}\right\rfloor \stackrel{\text { 4XV }}{\leqslant}\left\lfloor f\left(a^{2}\right)\right\rfloor \leqslant\lfloor f(a)\rfloor,
$$

and so $\lfloor f(a)\rfloor=\left\lfloor f\left(a^{2}\right)\right\rfloor$. By induction, and similar reasoning, we get $\lfloor f(a)\rfloor=$ $\left\lfloor f\left(a^{2^{n}}\right)\right\rfloor \leqslant f\left(a^{2^{n}}\right)$ for every $n$, and so $\lfloor f(a)\rfloor \leqslant \bigwedge_{n} f\left(a^{2^{n}}\right)=f\left(\bigwedge_{n} a^{2^{n}}\right)=$ $f(\lfloor a\rfloor)$, where we used that $f$ is normal, and $\lfloor a\rfloor=\bigwedge_{n} a^{2^{n}}$ (see 56 VI ).

### 3.2.3 Carrier and Commutant

63 Definition The carrier of an np-map $f: \mathscr{A} \rightarrow \mathscr{B}$ between von Neumann algebras (written $\lceil f\rceil$ ) is the least projection $p$ with $f\left(p^{\perp}\right)=0$ (which exists by 60 IX)
II Exercise Let $f, g: \mathscr{A} \rightarrow \mathscr{B}$ be np-maps between von Neumann algebras.

1. Show that $\lceil\lambda f\rceil=\lceil f\rceil$ for all $\lambda>0$.
2. Show that $\lceil f+g\rceil=\lceil f\rceil \cup\lceil g\rceil$.
3. Show that $\lceil f\rceil=1$ iff $f$ is faithful.
4. Assuming $f$ is multiplicative show that $\lceil f\rceil=1$ iff $f$ is injective. (There is more to be said about the carrier of a nmiu-map, see 69 IV .)

## III Exercise

1. Given an element $a$ of a von Neumann algebra $\mathscr{A}$ show that

$$
\left\lceil a^{*}(\cdot) a\right\rceil=\left\lceil a a^{*}\right\rceil \equiv(a\rceil
$$

where $a^{*}(\cdot) a$ is interpreted as an np-map $\mathscr{A} \rightarrow \mathscr{A}$.
2. Given a bounded operator $T: \mathscr{H} \rightarrow \mathscr{K}$ between Hilbert spaces show that $\left\lceil T^{*}(\cdot) T\right\rceil$ is the projection onto $\overline{\operatorname{Ran}(T)}$ when $T^{*}(\cdot) T$ is interpreted as a map $\mathscr{B}(\mathscr{K}) \rightarrow \mathscr{B}(\mathscr{H})$.
3. Show that $\lceil\langle x,(\cdot) x\rangle\rceil=|x\rangle\langle x|$ for any unit vector $x$ from a Hilbert space $\mathscr{H}$ when $\langle x,(\cdot) x\rangle$ is interpreted as a map $\mathscr{B}(\mathscr{H}) \rightarrow \mathbb{C}$.
(But be warned: when $\mathscr{A}$ is a von Neumann subalgebra of $\mathscr{B}(\mathscr{H})$ the carrier of the restriction $\langle x,(\cdot) x\rangle: \mathscr{A} \rightarrow \mathbb{C}$ might differ from $|x\rangle\langle x|$ because the former is in $\mathscr{A}$, while the latter may not be, see 88 IV .)

IV Lemma Let $f: \mathscr{A} \rightarrow \mathscr{B}$ be a p-map between $C^{*}$-algebras, and let $p$ be an effect of $\mathscr{A}$ with $f\left(p^{\perp}\right)=0$. Then $f(a)=f(p a)=f(a p)=f(p a p)$ for all $a \in \mathscr{A}$.

Proof Assume $\mathscr{B}=\mathbb{C}$ for now. Since $p^{\perp} \leqslant 1$, we have $\left(p^{\perp}\right)^{2}=\sqrt{p^{\perp}} p^{\perp} \sqrt{p^{\perp}} \leqslant$ $p^{\perp}$, and so $0 \leqslant f\left(\left(p^{\perp}\right)^{2}\right) \leqslant f\left(p^{\perp}\right)=0$, giving us $f\left(\left(p^{\perp}\right)^{2}\right)=0$. Since $\left|f\left(p^{\perp} a\right)\right|^{2} \leqslant f\left(\left(p^{\perp}\right)^{2}\right) f\left(a^{*} a\right)=0$ by Kadison's inequality, 30 IV we get $f\left(p^{\perp} a\right)=$ 0 , and so $f(p a)=f(a)$ for all $a \in \mathscr{A}$. In particular, $f(a p)=f\left(p a^{*}\right)^{*}=f\left(a^{*}\right)^{*}=$ $f(a)$ for all $a \in \mathscr{A}$, and so $f(p a p)=f(p a)=f(a)$ for all $a \in \mathscr{A}$.

Letting $\mathscr{B}$ be again arbitrary, and given $a \in \mathscr{A}$, note that since the states on $\mathscr{B}$ are separating (by 22 VIII$)$ it suffices to show that $\omega(f(a))=\omega(f(a p))=$ $\omega(f(p a))=\omega(f(p a p))$ for all states $\omega: \mathscr{B} \rightarrow \mathbb{C}$. But this follows from the previous paragraph since $\omega \circ f$ is a p-map into $\mathbb{C}$.
Corollary Given an np-map $f: \mathscr{A} \rightarrow \mathscr{B}$ between von Neumann algebras we have $f(a)=f(\lceil f\rceil a)=f(a\lceil f\rceil)=f(\lceil f\rceil a\lceil f\rceil)$ for all $a \in \mathscr{A}$.

We turn to the task of showing that every element of a von Neumann algebra is the norm limit of linear combinations of projections in 65 IV . We'll deal with the commutative case first (see III).

Proposition Every element $a$ of a commutative von Neumann algebra $\mathscr{A}$ is the norm limit of linear combinations of projections.
Proof By 53 II it suffices to show that the linear span of projections is norm dense in $C(\operatorname{sp}(\mathscr{A}))$. For this, in turn, it suffices by Stone-Weierstraß' theorem (see 27 XX ) to show that the projections in $C(\operatorname{sp}(\mathscr{A}))$ separate the points of $\operatorname{sp}(\mathscr{A})$ in the sense that given $x, y \in \operatorname{sp}(\mathscr{A})$ with $x \neq y$ there is a projection $f$ in $C(\operatorname{sp}(\mathscr{A}))$ with $f(x) \neq f(y)$. Since $\operatorname{sp}(\mathscr{A})$ is Hausdorff there are for such $x$ and $y$ disjoint open subsets $U$ and $V$ of $\operatorname{sp}(\mathscr{A})$ with $x \in U$ and $y \in V$.

Then $f:=\mathbf{1}_{\bar{U}}$ is a projection in $C(\operatorname{sp}(\mathscr{A}))$ (continuous because $\bar{U}$ is clopen by 53 III) with $f(x)=0 \neq 1=f(y)$ since $x \in \bar{U} \subseteq \operatorname{sp}(\mathscr{A}) \backslash V$, and so $y \notin \bar{U}$.

To reduce the general case to the commutative case we need the following tool (that will be useful later on too for different reasons).

Definition Given a subset $S$ of a von Neumann algebra $\mathscr{A}$ the commutant of $S$ is the set, denoted by $S^{\square}$, of all $a \in \mathscr{A}$ with $a s=s a$ for all $s \in S$.

The commutant of $\mathscr{A}$ itself is denoted by $Z(\mathscr{A}):=\mathscr{A}^{\square}$ and is called the centre of $\mathscr{A}$. (Its elements, called central, are the subjects of the next section.)
Exercise Let $S$ and $T$ be subsets of a von Neumann algebra $\mathscr{A}$.

1. Show that $S \subseteq T^{\square}$ iff $T \subseteq S^{\square}$.

Show that $S \subseteq T$ entails $T^{\square} \subseteq S^{\square}$.
Show that $S \subseteq S^{\square \square}$, and $S^{\square \square \square}=S^{\square}$.
2. Show that $S^{\square}$ is closed under addition, (scalar) multiplication, contains the unit of $\mathscr{A}$, and is ultraweakly closed.
3. Show that the commutant $S^{\square}$ need not be closed under involution.
(Hint: compute $\left\{\left(\begin{array}{ll}0 & 1 \\ 0 & 0\end{array}\right)\right\}^{\square}$ in $M_{2}$.)
Suppose $S$ is closed under involution.
Show $S^{\square}$ is closed under involution as well, and conclude that in that case $S^{\square}$ is a von Neumann subalgebra of $\mathscr{A}$.
Show that $Z(\mathscr{A})$ is a von Neumann subalgebra of $\mathscr{A}$.
Show that $S^{\square \square}$ is a von Neumann subalgebra of $\mathscr{A}$ with $S \subseteq S^{\square \square}$.
Show that if $S$ is commutative (i.e. $S \subseteq S^{\square}$ ), then so is $S^{\square \square}$.
4. In particular, if $\mathscr{B}$ is a von Neumann subalgebra of $\mathscr{A}$, then $\mathscr{B}^{\square \square}$ is a von Neumann subalgebra of $\mathscr{A}$ with $\mathscr{B} \subseteq \mathscr{B}^{\square \square}$.
Show that $(\mathscr{A} \cap \mathbb{C})^{\square}=\mathscr{A}$, and so $(\mathscr{A} \cap \mathbb{C})^{\square \square}=Z(\mathscr{A})$.
Nevertheless, we'll see in 88 V that $\mathscr{B}^{\square \square}=\mathscr{B}$ when $\mathscr{A}$ is of the form $\mathscr{A}=\mathscr{B}(\mathscr{H})$ for some Hilbert space $\mathscr{H}$.
5. Given a von Neumann subalgebra $\mathscr{B}$ of $\mathscr{A}$ verify that $Z(\mathscr{B})=\mathscr{B} \cap \mathscr{B} \square$.

IV Proposition Every self-adjoint element $a$ of a von Neumann algebra $\mathscr{A}$ is the norm limit of linear combinations of projections from $\{a\}^{\square \square}$.
$\checkmark$ Proof Since $a$ is an element of the by 111 commutative von Neumann subalgebra $\{a\}^{\square \square}$ of $\mathscr{A}, a$ is the norm limit of linear combinations of projections from $\{a\}^{\square \square}$ by 64 II

66 The carriers of np-functionals play such an important role in the theory that we decided to give them a name.

II Definition We call a projection $p$ of a von Neumann algebra $\mathscr{A}$ ultracyclic if $p=\lceil\omega\rceil$ for some np-map $\omega: \mathscr{A} \rightarrow \mathbb{C}$.
III Remark Some explanation of this terminology is in order. A projection $E$ in a von Neumann subalgebra $\mathscr{R}$ of $\mathscr{B}(\mathscr{H})$ is usually defined to be cyclic when $E$ is the projection onto $\mathscr{R}^{\square}$ for some $x \in \mathscr{H}$ (see Definition 5.5.8 [43]). With 88 IV and 88 VI we'll be able to see that this amount to requiring that $E$ be the carrier of the vector functional $\langle x,(\cdot) x\rangle: \mathscr{R} \rightarrow \mathbb{C}$. So, loosely speaking, a cyclic
projection is the carrier of a vector functional with respect to some fixed Hilbert space, while an ultracyclic projection is the carrier of a vector functional with respect to some arbitrary Hilbert space.

Exercise Let $\mathscr{A}$ be a von Neumann algebra. Verify the following facts.

1. If $p$ and $q$ are ultracyclic projections in $\mathscr{A}$, then $p \cup q$ is ultracyclic.
2. If $p \leqslant q$ are projections in $\mathscr{A}$, and $q$ is ultracyclic, then $p$ is ultracyclic.
3. Every projection $p$ in $\mathscr{A}$ is a directed supremum of ultracyclic projections. In fact, $p=\bigvee_{\omega}\lceil\omega\rceil$ where $\omega$ ranges over the np-functionals on $\mathscr{A}$ with $\omega\left(p^{\perp}\right)=0$. (Hint: first consider $p=1$.)
4. Every projection $p$ in $\mathscr{A}$ is the sum of ultracyclic projections: there are np-functionals $\left(\omega_{i}\right)_{i}$ on $\mathscr{A}$ with $p=\sum_{i}\left\lceil\omega_{i}\right\rceil$.

### 3.2.4 Central Support and Central Carrier

Definition An element $a$ of a von Neumann algebra $\mathscr{A}$ is called central when $a b=$ $b a$ for all $b \in \mathscr{A}$ (that is, when $a \in Z(\mathscr{A})$, see 65 III ).

## Examples

1. In a commutative von Neumann algebra every element is central.
2. An element $a$ of a direct sum $\bigoplus_{i} \mathscr{A}_{i}$ of von Neumann algebras is central iff $a_{i}$ is central for each $i$.
3. In $\mathscr{B}(\mathscr{H})$, where $\mathscr{H}$ is a Hilbert space, only the scalars are central. Indeed, given a positive central element $A$ of $\mathscr{B}(\mathscr{H})$, we have $\left\langle x, A\|y\|^{2} x\right\rangle=$ $\langle x, A \mid x\rangle\langle y \mid y\rangle=\langle x, \mid x\rangle\langle y \mid A y\rangle=\left\langle x,\|\sqrt{A} y\|^{2} x\right\rangle$ for all $x, y \in \mathscr{H}$, and so $A\|y\|^{2}=\|\sqrt{A} y\|^{2}$ for all $y \in \mathscr{H}$. Hence $A$ is (zero or) a scalar.

Remark A von Neumann algebra in which only the scalars are central - of which the $\mathscr{B}(\mathscr{H})$ are but the simplest examples - is called a factor. The classification of these factors is an important part of the theory of von Neumann algebras that we did not need in this thesis.

IV Exercise Note that if a von Neumann algebra $\mathscr{A}$ can be written as a direct sum $\mathscr{A} \cong \mathscr{B}_{1} \oplus \mathscr{B}_{2}$, then $(1,0) \in \mathscr{B}_{1} \oplus \mathscr{B}_{2}$ gives a central projection in $\mathscr{A}$. The converse also holds:

1. Given a central projection $c$ in $\mathscr{A}$, show that $c \mathscr{A} \equiv\{c a: a \in \mathscr{A}\}$ is a von Neumann subalgebra of $\mathscr{A}$ for all but the fact that 1 need not be in $c \mathscr{A}$. Show $c \mathscr{A}$ is a von Neumann algebra with $c$ as unit, and that $a \mapsto\left(c a, c^{\perp} a\right)$ gives a nmiu-isomorphism $\mathscr{A} \rightarrow c \mathscr{A} \oplus c^{\perp} \mathscr{A}$.
2. Given a family of central projections $\left(c_{i}\right)_{i}$ in $\mathscr{A}$ with $\sum_{i} c_{i}=1$ show that $a \mapsto\left(c_{i} a\right)_{i}$ gives a nmiu-isomorphism $\mathscr{A} \rightarrow \bigoplus_{i} c_{i} \mathscr{A}$.

68 Proposition Given a projection $e$ of a von Neumann algebra $\mathscr{A}$

$$
\llbracket e \rrbracket:=\bigcup_{a \in \mathscr{A}}\left\lceil a^{*} e a\right\rceil
$$

is the least central projection above $e$.
II Proof Let us first show that $\llbracket e \rrbracket$ is central. Given $b \in \mathscr{A}$ we have $\lceil\llbracket e \rrbracket b)=$ $\left\lceil b^{*} \llbracket e \rrbracket b\right\rceil=\bigcup_{a \in \mathscr{A}}\left\lceil b^{*}\left\lceil a^{*} e a\right\rceil b\right\rceil=\bigcup_{a \in \mathscr{A}}\left\lceil(a b)^{*} e a b\right\rceil \leqslant \llbracket e \rrbracket$ by 60 IX , which implies that $\llbracket e \rrbracket b \llbracket e \rrbracket=\llbracket e \rrbracket b$. Since similarly (or consequently) $\|e\| b \| e \rrbracket=b \llbracket e \rrbracket$ we get $b \llbracket e \rrbracket=\llbracket e \rrbracket b \llbracket e \rrbracket=\llbracket e \rrbracket b$, and so $\llbracket e \rrbracket$ is central.

Clearly $e \leqslant \llbracket e \rrbracket$. It remains to be shown that $\llbracket e \rrbracket \leqslant c$ given a central projection $c$ with $e \leqslant c$. For this it suffices to show that $\lceil e a) \equiv\left\lceil a^{*} e a\right\rceil \leqslant c$ given $a \in \mathscr{A}$. Now, since $e \leqslant c$ we have $e c=e$ and so $e a c=e c a=e a$ which implies that $\lceil e a) \leqslant c$. Thus $\llbracket e \rrbracket \leqslant c$.
III Definition Let $a$ be an element of a von Neumann algebra $\mathscr{A}$. Since given a central projection $c$ of $\mathscr{A}$ we have $\llbracket\lceil a) \rrbracket \leqslant c$ iff $\lceil a) \leqslant c$ iff $a c=a$ iff $c a=a$ iff $\llbracket(a\rceil \rrbracket \leqslant c$, we see that $\llbracket a \rrbracket:=\llbracket\lceil a) \rrbracket=\llbracket(a\rceil \rrbracket$ is the smallest central projection $p$ with $p a=a$, which we'll call the central support of $a$.

IV Exercise Let $\mathscr{A}$ be a von Neumann algebra.

1. Show that $\llbracket a \rrbracket=\llbracket a^{*} \rrbracket=\llbracket a^{*} a \rrbracket=\llbracket a a^{*} \rrbracket$ for all $a \in \mathscr{A}$.
2. Show that $\llbracket \bigvee D \rrbracket=\bigcup_{d \in D} \llbracket d \rrbracket$ for any bounded directed subset of $\mathscr{A}$.

Show that $\llbracket \cup E \rrbracket=\bigcup_{e \in E} \llbracket e \rrbracket$ for any collection of projections from $\mathscr{A}$.
Show that $\llbracket a+b \rrbracket=\llbracket\lceil a\rceil \cup\lceil b\rceil\rceil=\llbracket a\rceil \cup \llbracket b\rceil$ for all $a, b \in \mathscr{A}$.
3. Given $a \in \mathscr{A}$ and a central projection $c$ of $\mathscr{A}$ show that $\llbracket a \rrbracket c=\llbracket a c \rrbracket$.

Conclude that $\llbracket a \rrbracket \llbracket b \rrbracket=\llbracket a \llbracket b \rrbracket \rrbracket=\llbracket \llbracket a \rrbracket b \rrbracket=\llbracket a \rrbracket \cap \llbracket b \rrbracket$ for all $a, b \in \mathscr{A}$.

Definition Let $f: \mathscr{A} \rightarrow \mathscr{B}$ be an np-map between von Neumann algebras. Show that given a central effect $c$ of $\mathscr{A}$ we have $f\left(c^{\perp}\right)=0$ iff $\lceil f\rceil \leqslant c$ iff $\llbracket\lceil f\rceil \rrbracket \leqslant$ $c$, and so $\llbracket f \rrbracket:=\llbracket\lceil f\rceil \rrbracket$ is the least central effect (and central projection) $p$ with $f\left(p^{\perp}\right)=0$, which we'll call the central carrier of $f$.

Proposition Every two-sided ideal $\mathscr{D}$ of a von Neumann algebra $\mathscr{A}$ that is closed under bounded directed suprema of self-adjoint elements - for example when $\mathscr{A}$ is ultrastrongly closed - is of the form $c \mathscr{A}$ for some unique central projection $c$ of $\mathscr{A}$. Moreover, $c$ is the greatest projection in $\mathscr{D}$.
Proof We'll obtain $c$ as the supremum over all effects in $\mathscr{D}$, and to this end we'll show first that $\mathscr{D} \cap[0,1]_{\mathscr{A}}$ is directed. Since $\lceil a\rceil \cup\lceil b\rceil \equiv\left\lceil\frac{1}{2} a+\frac{1}{2} b\right\rceil$ (see 56 XIII) is an upper bound for $a, b \in \mathscr{D} \cap[0,1]_{\mathscr{A}}$ it suffices to show that $\lceil a\rceil \in \mathscr{D}$ for all $a \in \mathscr{D} \cap[0,1]_{\mathscr{A}}$, which, in turn, follows from $\lceil a\rceil=\bigvee_{n} a^{1 / 2^{n}}$, see 56 I .

Hence $\mathscr{D} \cap[0,1]_{\mathscr{A}}$ is directed, and so we may define $c:=\bigvee \mathscr{D} \cap[0,1]_{\mathscr{A}}$. Since $\mathscr{D}$ is a von Neumann subalgebra of $\mathscr{A}$, we'll have $c \in \mathscr{D} \cap[0,1]_{\mathscr{A}}$, and so $c$ is the greatest element of $\mathscr{D} \cap[0,1]_{\mathscr{A}}$. In particular, $c$ will be above $\lceil c\rceil$ implying $\lceil c\rceil=c$ and making $c$ a projection-the greatest projection in $\mathscr{D}$.

Given $a \in \mathscr{A}$ we claim that $a \in \mathscr{D}$ iff $c a=a$. Surely, if $a=c a$, then $a=$ $c a \in \mathscr{D}$, because $\mathscr{D}$ is a two-sided ideal of $\mathscr{A}$. Concerning the other direction, note that given $a \in \mathscr{D}$ the equality $a c=a$ holds when $a$ is an effect by 55 VIII (because $a \leqslant c$ ), and thus when $a$ is self-adjoint too (by scaling), and hence for arbitrary $a \in \mathscr{D}$ by writing $a \equiv a_{\mathbb{R}}+i a_{\mathbb{I}}$ where $a_{\mathbb{R}}$ and $a_{\mathbb{I}}$ are self-adjoint.

Note that this claim entails that $\mathscr{D} \subseteq c \mathscr{A}$. Since $\mathscr{D}$ is an ideal we also have $c \mathscr{A} \subseteq \mathscr{D}$, and so $\mathscr{D}=c \mathscr{A}$. The claim also entails that $c$ is central. Indeed, given $a \in \mathscr{A}$ we have $a c \in \mathscr{D}$ (because $\mathscr{D}$ is an ideal) and so $c(a c)=a c$ by the claim. Since similarly $(c a) c=c a$, we get $a c=c a$.

The only thing that remains to be shown is that $c$ is unique. To this end let $c$ and $c^{\prime}$ be central projections with $c \mathscr{A}=\mathscr{D}=c^{\prime} \mathscr{A}$. As $c^{\prime} \in \mathscr{D}=c \mathscr{A}$, there is $a \in \mathscr{A}$ with $c^{\prime}=c a$. Then $c^{\prime}=c^{\prime}\left(c^{\prime}\right)^{*}=c a a^{*} c^{*} \leqslant c c^{*}\left\|a a^{*}\right\|=c\|a\|^{2}$, and so $c^{\prime} \leqslant c$. Since similarly $c \leqslant c^{\prime}$, we get $c=c^{\prime}$.

Corollary Given a nmiu-map $f: \mathscr{A} \rightarrow \mathscr{B}$ between von Neumann algebras we have $\lceil f\rceil=\llbracket f\rceil$ and $\operatorname{ker}(f)=\llbracket f \rrbracket^{\perp} \mathscr{A}$.
$V$ Lemma We have $\llbracket \omega\rceil=\left\lceil\varrho_{\omega}\right\rceil$ for every np-functional $\omega: \mathscr{A} \rightarrow \mathbb{C}$ on a von Neumann algebra $\mathscr{A}$, where $\varrho_{\omega}$ is as in 30 VI .
VI Proof Let $e$ be a projection in $\mathscr{A}$. Note that $0=\left\|\varrho_{\omega}(e)\left(\eta_{\omega}(a)\right)\right\|^{2} \equiv \omega\left(a^{*}\right.$ ea) iff $\left\lceil a^{*} e a\right\rceil \leqslant\lceil\omega\rceil^{\perp}$ iff $\left\lceil a\lceil\omega\rceil a^{*}\right\rceil \leqslant e^{\perp}$ for all $a \in \mathscr{A}$. So since the $\eta_{\omega}(a)$ 's lie dense in $\mathscr{H}_{\omega}$, we have $\varrho_{\omega}(e)=0$ iff $\varrho_{\omega}(e)\left(\eta_{\omega}(a)\right)=0$ for all $a \in \mathscr{A}$ iff $\bigcup_{a \in \mathscr{A}}\left\lceil a\lceil\omega\rceil a^{*}\right\rceil \leqslant e^{\perp}$. Hence $\left\lceil\varrho_{\omega}\right\rceil=\bigcup_{a \in \mathscr{A}}\left\lceil a\lceil\omega\rceil a^{*}\right\rceil \equiv \bigcup_{a \in \mathscr{A}}\left\lceil a^{*}\lceil\omega\rceil a\right\rceil=$ $\llbracket\lceil\omega\rceil\rceil=\llbracket \omega\rceil$ by 681
VII Proposition Given a collection of np-functionals $\Omega$ on a von Neumann algebra $\mathscr{A}$ we have $\left.\left\lceil\varrho_{\Omega}\right\rceil=\bigcup_{\omega \in \Omega} \llbracket \omega\right\rceil$ for $\varrho_{\Omega}: \mathscr{A} \rightarrow \mathscr{B}\left(\mathscr{H}_{\Omega}\right)$ from 30 VI .
VIII Proof Let $e$ be a projection of $\mathscr{A}$. Since $\varrho_{\Omega}(e)(x)=\sum_{\omega \in \Omega} \varrho_{\omega}\left(x_{\omega}\right)$ by 30 VI for all $x \in \mathscr{H}_{\Omega} \equiv \bigoplus_{\omega \in \Omega} \mathscr{H}_{\omega}$, we have $\varrho_{\Omega}(e)=0$ iff $\varrho_{\omega}(e)=0$ for all $\omega \in \Omega$ iff $e \leqslant$ $\left\lceil\varrho_{\omega}\right\rceil^{\perp} \equiv \llbracket \omega \rrbracket^{\perp}$ iff $\left.e \leqslant \bigcap_{\omega \in \Omega} \llbracket \omega \rrbracket^{\perp} \equiv\left(\bigcup_{\omega \in \Omega} \llbracket \omega\right\rceil\right)^{\perp}$. Hence $\left\lceil\varrho_{\Omega}\right\rceil=\bigcup_{\omega \in \Omega} \llbracket \omega \rrbracket$.

IX Corollary For a collection $\Omega$ of np-functionals on a von Neumann algebra, the following are equivalent.

1. $\Omega$ is centre separating (see 21 II ).
2. A central projection $z$ of $\mathscr{A}$ is zero when $\omega(z)=0$ for all $\omega \in \Omega$.
3. The map $\varrho_{\Omega}: \mathscr{A} \rightarrow \mathscr{B}\left(\mathscr{H}_{\Omega}\right)$ from 30 VI is injective.
$x$ Proof We've seen in $30 X$ that $1 \Longleftrightarrow 3$ and $1 \Rightarrow 2$ is trivial, which leaves us with $2 \Rightarrow 3$. So assume that $\forall \omega \in \Omega[\omega(z)=0] \Longrightarrow z=0$ for every central projection $z$ of $\mathscr{A}$. Then since $\left\lceil\varrho_{\Omega}\right\rceil^{\perp}$ is a central projection by VII with $\left\lceil\varrho_{\Omega}\right\rceil^{\perp}=$ $\left.\left(\bigcup_{\omega \in \Omega} \llbracket \omega \rrbracket\right)^{\perp}=\bigcap_{\omega \in \Omega} \llbracket \omega\right\rceil^{\perp} \leqslant \llbracket \omega \rrbracket^{\perp} \leqslant\lceil\omega\rceil^{\perp}$ and thus $\omega\left(\left\lceil\varrho_{\Omega}\right\rceil^{\perp}\right) \leqslant \omega\left(\lceil\omega\rceil^{\perp}\right)=$ 0 for all $\omega \in \Omega$ we get $\left\lceil\varrho_{\Omega}\right\rceil^{\perp}=0$, and so $\varrho_{\Omega}$ is injective by 63 II .

70 With our new-found knowledge on central elements we can complete the classification of commutative von Neumann algebras we started in 52
II Exercise Show that every central projection $c$ of a von Neumann algebra is of the form $c \equiv \sum_{i} \llbracket \omega_{i} \rrbracket$ for some family of np-functionals $\left(\omega_{i}\right)_{i}$ on $\mathscr{A}$.
III Theorem Every commutative von Neumann algebra is nmiu-isomorphic to a direct sum of the form $\bigoplus_{i} L^{\infty}\left(X_{i}\right)$ where $X_{i}$ are finite complete measure spaces.
IV Proof By $\Pi$ we have $1 \equiv \sum_{i} \llbracket \omega_{i} \rrbracket$ for some np-functionals $\omega_{i}: \mathscr{A} \rightarrow \mathbb{C}$, and so $\left.\mathscr{A} \cong \bigoplus_{i} \llbracket \omega_{i}\right\rceil \mathscr{A}$ by 67 IV . Since $\mathscr{A}$ is commutative, and so $\llbracket \omega_{i} \rrbracket=\left\lceil\omega_{i}\right\rceil$, we
see that restricting $\omega_{i}$ gives a faithful functional on $\left.\llbracket \omega_{i}\right\rceil \mathscr{A}$, which is therefore by 54 XI nmiu-isomorphic to $L^{\infty}\left(X_{i}\right)$ for some finite complete measure space $X_{i}$. From this the stated result follows.

### 3.3 Completeness

We set to work on the ultrastrong and bounded ultraweak completeness of von Neumann algebras (see 771) and their precursors:

1. A linear (not necessarily positive) functional on a von Neumann algebra is ultraweakly continuous iff it is ultrastrongly continuous (see 72 XI).
2. A convex subset of a von Neumann algebra is ultraweakly closed iff it is ultrastrongly closed (see 73 VIII ).
3. (Kaplansky's density theorem) The unit ball $(\mathscr{A})_{1}$ of a $C^{*}$-subalgebra $\mathscr{A}$ of a von Neumann algebra $\mathscr{B}$ is ultrastrongly dense in $(\overline{\mathscr{A}})_{1}$ where $\overline{\mathscr{A}}$ is the ultrastrong (=ultraweak, 73 VIII ) closure of $\mathscr{A}$ (see 74 IV ).
4. Any von Neumann subalgebra $\mathscr{A}$ of $\mathscr{B}$ is ultraweakly and ultrastrongly closed in $\mathscr{B}$ (see 75 VIIII ).
5. The von Neumann algebra $\mathscr{B}(\mathscr{H})$ of bounded operators on a Hilbert space $\mathscr{H}$ is ultrastrongly (761) and bounded ultraweakly complete 76III).

### 3.3.1 Closure of a Convex Subset

We saw in 46 III that a positive linear functional $f$ on a von Neumann algebra is ultrastrongly continuous iff it is ultraweakly continuous. In this section, we'll show that the same result holds for an arbitrary linear functional $f$. Note that if $f$ is ultraweakly continuous, then $f$ is automatically ultrastrongly continuous (because ultrastrong convergence implies ultraweak convergence). For the other direction, we'll show that if $f$ is ultrastrongly continuous, then $f$ can be written as a linear combination $f \equiv \sum_{k=0}^{3} i^{k} f_{k}$ of np-maps $f_{0}, \ldots, f_{3}$, and must therefore be ultraweakly continuous. We'll need the following tool.
Definition Let $\mathscr{A}$ be a von Neumann algebra. Given an np-map $\omega: \mathscr{A} \rightarrow \mathbb{C}$, II and $b \in \mathscr{A}$, define $b * \omega: \mathscr{A} \rightarrow \mathbb{C}$ by $(b * \omega)(a)=\omega\left(b^{*} a b\right)$ for all $a \in \mathscr{A}$.

III Exercise Let $\omega: \mathscr{A} \rightarrow \mathbb{C}$ be an np-map on a von Neumann algebra $\mathscr{A}$.

1. Note that $b * \omega: \mathscr{A} \rightarrow \mathbb{C}$ is an np-map for all $b \in \mathscr{A}$.

Show that $\left|\omega\left(a^{*} b c\right)\right| \leqslant\|\omega\|\|a\|_{\omega}\|b\|\|c\|_{\omega}$ for all $a, b, c \in \mathscr{A}$.
Deduce that $\left\|b * \omega-b^{\prime} * \omega\right\| \leqslant\|\omega\|\left\|b-b^{\prime}\right\|_{\omega}\left(\|b\|_{\omega}+\left\|b^{\prime}\right\|_{\omega}\right)$ for all $b, b^{\prime} \in \mathscr{A}$.
2. Let $b_{1}, b_{2}, \ldots$ be a sequence in $\mathscr{A}$ which is Cauchy with respect to $\|\cdot\|_{\omega}$. Show that the sequence $b_{1} * \omega, b_{2} * \omega, \ldots$ is Cauchy (in the operator norm on bounded linear functionals $\mathscr{A} \rightarrow \mathbb{C}$ ), and converges to a bounded linear map $f: \mathscr{A} \rightarrow \mathbb{C}$. Show that $f$ is an np-map.

IV Exercise Let $f: \mathscr{A} \rightarrow \mathbb{C}$ be an ultrastrongly continuous linear functional on a von Neumann algebra $\mathscr{A}$. Show that there are an np-map $\omega: \mathscr{A} \rightarrow \mathbb{C}$ and $\delta>0$ with $|f(a)| \leqslant 1$ for all $a \in \mathscr{A}$ with $\|a\|_{\omega} \leqslant \delta$.
(Keep this in mind when reading the following lemma.)
V Lemma Let $\omega: \mathscr{A} \rightarrow \mathbb{C}$ be an np-map, and let $f: \mathscr{A} \rightarrow \mathbb{C}$ be a linear map. The following are equivalent.

1. $|f(a)| \leqslant B$ for all $a \in \mathscr{A}$ with $\|a\|_{\omega} \leqslant \delta$, for some $\delta, B>0$;
2. $|f(a)| \leqslant B\|a\|_{\omega}$ for all $a \in \mathscr{A}$, for some $B>0$;
3. $f(a)=[b, a]_{\omega}$ for all $a \in \mathscr{A}$, for some $b \in \mathscr{H}_{\omega}$ (where $\mathscr{H}_{\omega}$ is the Hilbert space completion of $\mathscr{A}$ with respect to the inner-product $\left.[\cdot, \cdot]_{\omega}\right)$.
4. $f \equiv f_{0}+i f_{1}-f_{2}-i f_{3}$ where $f_{0}, \ldots, f_{3}: \mathscr{A} \rightarrow \mathbb{C}$ are np-maps for which there is $B>0$ such that $f_{k}(a) \leqslant B \omega(a)$ for all $a \in \mathscr{A}_{+}$and $k$.

VI Proof We make a circle.
VII $4 \sqrt{1}$ For $a \in \mathscr{A}$ and $k$, we have $\left|f_{k}(a)\right|^{2} \leqslant f_{k}(1) f_{k}\left(a^{*} a\right) \leqslant f_{k}(1) B \omega\left(a^{*} a\right)$, giving $\left|f_{k}(a)\right| \leqslant\left(f_{k}(1) B\right)^{1 / 2}\|a\|_{\omega}$, and so $|f(a)| \leqslant \tilde{B}\|a\|_{\omega}$, where

$$
\tilde{B}=B^{1 / 2} \sum_{k=0}^{3} f_{k}(1)^{1 / 2} .
$$

Hence $|f(a)| \leqslant \tilde{B}$ for all $a \in \mathscr{A}$ with $\|a\|_{\omega} \leqslant 1$.
VIII 1.2 Let $a \in \mathscr{A}$, and $\varepsilon>0$ be given. Then for $\tilde{a}:=\delta\left(\varepsilon+\|a\|_{\omega}\right)^{-1} a$, we have $\|\tilde{a}\|_{\omega} \leqslant \delta$, and so $|f(\tilde{a})| \equiv \delta\left(\varepsilon+\|a\|_{\omega}\right)^{-1}|f(a)| \leqslant B$, which entails $|f(a)| \leqslant B \delta^{-1}\left(\varepsilon+\|a\|_{\omega}\right)$. Since $\varepsilon>0$ was arbitrary, we get $|f(a)| \leqslant B \delta^{-1}\|a\|_{\omega}$.
(2) 3 Since $|f(a)| \leqslant B\|a\|_{\omega}$ for all $a \in \mathscr{A}$, the map $f$ can be extended to a bounded linear map $\tilde{f}: \mathscr{H}_{\omega} \rightarrow \mathbb{C}$. Then by Riesz' representation theorem, 5IX there is $b \in \mathscr{H}_{\omega}$ with $\tilde{f}(x)=[b, x]_{\omega}$ for all $x \in \mathscr{H}_{\omega}$. In particular, $f(a)=[b, a]_{\omega}$ for all $a \in \mathscr{A}$.
34 We know that $f(a) \equiv[b, a]_{\omega}$ for all $a \in \mathscr{A}$, for some $b \in \mathscr{H}_{\omega}$. Then, by definition of $\mathscr{H}_{\omega}$, there is a sequence $b_{1}, b_{2}, \ldots$ in $\mathscr{A}$ which converges to $b$ in $\mathscr{H}_{\omega}$. Then the maps $\left[b_{n}, \cdot\right]_{\omega}: \mathscr{A} \rightarrow \mathbb{C}$ approximate $f=[b, \cdot]_{\omega}$ in the sense that $\left|f(a)-\left[b_{n}, a\right]_{\omega}\right|=\left|\left[b-b_{n}, a\right]_{\omega}\right| \leqslant\left\|b-b_{n}\right\|_{\omega}\|a\|_{\omega} \leqslant\left\|b-b_{n}\right\|_{\omega}\|\omega\|^{1 / 2}\|a\|$ for all $a \in \mathscr{A}$. In particular, $\left[b_{1}, \cdot\right]_{\omega},\left[b_{2}, \cdot\right]_{\omega}, \ldots$ converges to $f$ (in the operator norm). By "polarisation" (c.f. 44 II ), we have $\left[b_{n}, a\right]_{\omega}=\frac{1}{4} \sum_{k=0}^{3} i^{k} f_{k, n}(a)$, where $f_{k, n}:=\left(i^{k} b_{n}+1\right) * \omega$ is an np-map. Since $\left(i^{k} b_{n}+1\right)_{n}$ is Cauchy with respect to $\|\cdot\|_{\omega}$, we see by $\Pi$ III that $\left(f_{k, n}\right)_{n}$ converges to an np-map $f_{k}: \mathscr{A} \rightarrow \mathbb{C}$ (with respect to the operator norm). It follows that $f=\frac{1}{4} \sum_{k=0}^{3} i^{k} f_{k}$.

It remains to be shown that there is $B>0$ with $f_{k}(a) \leqslant B \omega(a)$ for all $k$ and $a \in \mathscr{A}_{+}$. Note that since $f_{k, n}(a) \leqslant\left\|i^{k} b_{n}+1\right\|_{\omega} \omega(a) \leqslant\left(\left\|b_{n}\right\|_{\omega}+1\right) \omega(a)$, for all $n, k$, and $a \in \mathscr{A}_{+}$, the number $B:=\lim _{n}\left\|b_{n}\right\|_{\omega}+1$ will do.
Corollary For a linear map $f: \mathscr{A} \rightarrow \mathbb{C}$ on a von Neumann algebra $\mathscr{A}$ the following are equivalent.

1. $f$ is ultrastrongly continuous;
2. $f$ is ultraweakly continuous;
3. $f \equiv f_{0}+i f_{1}-f_{2}-i f_{3}$ for some np-maps $f_{0}, \ldots, f_{3}: \mathscr{A} \rightarrow \mathbb{C}$;
4. " $f$ is bounded on some $\|\cdot\|_{\omega}$-ball," that is,

$$
\sup \left\{|f(a)|: a \in \mathscr{A}:\|a\|_{\omega} \leqslant \delta\right\}<\infty
$$

for some $\delta>0$ and np-map $\omega: \mathscr{A} \rightarrow \mathbb{C}$;
5. $|f(a)| \leqslant\|a\|_{\omega}$ for all $a \in \mathscr{A}$, for some np-map $\omega: \mathscr{A} \rightarrow \mathbb{C}$.

We'll show that the ultrastrong and ultraweak closure of a convex set agree. For this we need the following proto-Hahn-Banach separation theorem, which concerns the following notion of openness.
Definition A subset $A$ of a real vector space $V$ is called radially open if for all $a \in A$ and $v \in V$ there is $t \in(0, \infty)$ with $a+s v \in A$ for all $s \in[0, t)$.

Exercise Let $V$ be a vector space.

1. Show that the radially open subsets of $V$ form a topology.
2. Show that with respect to this topology, scalar multiplication and translations $x \mapsto x+a$ by a fixed vector $a \in V$ are continuous.
3. Show that $\{(0,0)\} \cup B_{1}(-1) \cup B_{1}(1) \cup B_{2}(-2)^{c} \cup B_{2}(2)^{c}$ is a radially open subset of $\mathbb{R}^{2}$, which is not open in the usual topology.
4. Show that addition on $\mathbb{R}^{2}$ is not jointly radially continuous.
5. Show that nevertheless $\{s \in \mathbb{R}: s x+(1-s) y \in A\}$ is open for every radially open $A \subseteq V$, and $x, y \in V$.
6. Show that $A+B$ is radially open when $A, B \subseteq V$ are radially open.

Show that $\{\lambda a: a \in A, \lambda>0\}$ is radially open when $A$ is radially open.

IV Theorem For every radially open convex subset $K$ of a real vector space $V$ with $0 \notin K$ there is a linear map $f: V \rightarrow \mathbb{R}$ with $f(x)>0$ for all $x \in K$.
$\checkmark$ Proof (Based on Theorem 1.1.2 of 43].)
By Zorn's Lemma we may assume without loss of generality that $K$ is maximal among radially open convex subsets of $V$ that do not contain 0 .

We also assume that $K$ is non-empty, because if $K=\varnothing$, the result is trivial.
We will show in a moment that $H:=\{x \in V:-x, x \notin K\}$ is a linear subspace and $V / H$ is one-dimensional. From this we see that there is a linear map $f: V \rightarrow \mathbb{R}$ with $\operatorname{ker}(f)=H$. Since $f(K)$ is a convex subset which does not contain 0 (because $H \cap K=\varnothing$ ) we either have $f(K) \subseteq(0, \infty)$ or $f(K) \subseteq$ $(-\infty, 0)$. Thus, by replacing $f$ by $-f$ if necessary, we see that there is a linear map $f: V \rightarrow \mathbb{R}$ with $f(x)>0$ for all $x \in K$.
VI ( $H$ is a linear subspace) Note that $x \in K, \lambda>0 \Longrightarrow \lambda x \in K$, because the subset $\{\lambda x: x \in K, \lambda \in(0, \infty)\} \supseteq K$ is radially open, convex, doesn't contain 0 , and is thus $K$ itself. Furthermore, $x, y \in K \Longrightarrow x+y \in K$, because $x+y=2\left(\frac{1}{2} x+\frac{1}{2} y\right)$, and $K$ is convex.

Let $\bar{K}$ be the set of all $x \in V$ with $x+y \in K$ for all $y \in K$. Then it is not difficult to check that $\bar{K}$ is a cone: $0 \in \bar{K}$, and $x \in \bar{K}, \lambda \geqslant 0 \Longrightarrow \lambda x \in \bar{K}$, and $x, y \in \bar{K} \Longrightarrow x+y \in \bar{K}$.

We claim that $x \in \bar{K}$ iff $-x \notin K$. Indeed, if $x \in \bar{K}$, then $-x \notin K$, because otherwise $-x \in K$ and so $0=x+(-x) \in K$, which is absurd. For the other
direction, suppose that $-x \notin K$. Then $x+y \in K$ for all $y \in K$, because $\{\lambda x+y: y \in K, \lambda \geqslant 0\} \supseteq K$ is radially open, convex, doesn't contain 0 , and is thus $K$.

It follows that $H=\bar{K} \cap-\bar{K}$. Since $\bar{K}$ is a cone, $-\bar{K}$ is a cone, and thus $H$ is a cone. But then $-H=H$ is a cone too, and thus $H$ is a linear subspace.
( $V / H$ is one-dimensional) Note that $H \neq V$, because $K \cap H=\varnothing$ and $K$ is (assumed to be) non-empty. So to show that $V / H$ is one-dimensional, it suffices to show that any $x, y \in V$ are linearly dependent in $V / H$. We may assume that $x \in K$ and $y \in-K$. It suffices to find $s \in[0,1]$ with $0=s x+s^{\perp} y$. The trick is to consider the sets $S_{0}=\left\{s \in[0,1]: s x+s^{\perp} y \in-K\right\}$ and $S_{1}=$ $\left\{s \in[0,1]: s x+s^{\perp} y \in K\right\}$, which are open (because $K$ and $-K$ are radially open), non-empty (because $0 \in S_{0}$ and $1 \in S_{1}$ ), and therefore cannot cover [ 0,1 ] (because $[0,1]$ is connected). So there must be $s \in(0,1)$ such that $s x+s^{\perp} y$ is neither in $K$ nor in $-K$, and thus $s x+s^{\perp} y \in H$ (by definition of $H$ ). Whence $x$ and $y$ are linearly dependent in $V / H$ (since $s \neq 0)$.
Exercise We will use $\boxed{\mathrm{V}}$ to prove that an ultrastrongly closed convex subset $K$ of a von Neumann algebra $\mathscr{A}$ is ultraweakly closed as well.

Let us first simplify the problem a bit. If $K$ is empty, the result is trivial, so we may as well assume that $K \neq \varnothing$. Note that we must show that no net in $K$ converges ultraweakly to any element $a_{0} \in \mathscr{A}$ outside $K$, but by considering $K-a_{0}$ instead of $K$, we see that it suffices to show that no net in $K$ converges ultraweakly to 0 under the assumption that $0 \notin K$. To this end we'll find an ultraweakly continuous linear map $g: \mathscr{A} \rightarrow \mathbb{C}$ and $\delta>0$ with $g(k)_{\mathbb{R}} \geqslant \delta$ for all $k \in K$-if a net $\left(k_{\alpha}\right)_{\alpha}$ in $K$ were to converge ultraweakly to 0 , then $g\left(k_{\alpha}\right)_{\mathbb{R}}$ would converge to 0 as well, which is impossible.

1. Show that there is an np-map $\omega: \mathscr{A} \rightarrow \mathbb{C}$ and $\varepsilon>0$ with $\|k\|_{\omega} \geqslant \varepsilon$ for all $k \in K$. (Hint: use that $K$ is ultrastrongly closed).
2. Show that $B:=\left\{b \in \mathscr{A}:\|b\|_{\omega}<\varepsilon\right\}$ is convex, radially open, $B \cap K=\varnothing$. Show that $B-K$ is convex, radially open, and $0 \notin B-K$.
3. Use $\triangle$ to show that there is an $\mathbb{R}$-linear map $f: \mathscr{A} \rightarrow \mathbb{R}$ with $f(b)<f(k)$ for all $b \in B$ and $k \in K$. Show that $f$ can be extended to a $\mathbb{C}$-linear map $g: \mathscr{A} \rightarrow \mathbb{C}$ by $g(a)=f(a)-i f(i a)$ for all $a \in \mathscr{A}$.
4. Show that $|f(b)| \leqslant f(k)$ and $|g(b)| \leqslant 2 f(k)$ for all $b \in B$ and $k \in K$.
(Hint: $b \in B \Longrightarrow-b \in B$.)
Conclude that $g$ is ultraweakly continuous (using 72XI and $K \neq \varnothing$ ).
5. It remains to be shown that there is $\delta>0$ with $f(k) \equiv g(k)_{\mathbb{R}} \geqslant \delta$ for all $k \in K$. Show that in fact there is $b_{0} \in B$ with $f\left(b_{0}\right)>0$, and that $f(k) \geqslant f\left(b_{0}\right)>0$ for all $k \in K$.

### 3.3.2 Kaplansky's Density Theorem

74 Proposition Let $\mathscr{A}$ be a von Neumann algebra, and let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a continuous map with $f(t)=\mathcal{O}(t)$, that is, there are $n \in \mathbb{N}$ and $b \in[0, \infty)$ such that $|f(t)| \leqslant b|t|$ for all $t \in \mathbb{R}$ with $|t| \geqslant n$.

Then the map $a \mapsto f(a), \mathscr{A}_{\mathbb{R}} \rightarrow \mathscr{A}_{\mathbb{R}}$, see 28 II is ultrastrongly continuous.
II Proof (An adaptation of Lemma 44.2 from 12 .)
Let $S$ denote the set of all continuous $g: \mathbb{R} \rightarrow \mathbb{R}$ such that $a \mapsto g(a), \mathscr{A}_{\mathbb{R}} \rightarrow$ $\mathscr{A}_{\mathbb{R}}$ is ultrastrongly continuous. We must show that $f \in S$.

Let us first make some general observations. The identity map $t \mapsto t$ is in $S$, any constant function is in $S$, and $S$ is closed under addition, and scalar multiplication. In particular, any affine transformation $(t \mapsto a t+b)$ is in $S$. Moreover, we have $g \circ h \in S$ when $g, h \in S$, and also $g h \in S$ provided that $g$ is bounded. Finally, $S$ is closed with respect to uniform convergence.

Now, as $f(t)=f(t) \frac{1}{1+t^{2}}+f(t) \frac{t^{2}}{1+t^{2}}$ one can see from the remarks above that it suffices to show that $t \mapsto f(t) \frac{1}{1+t^{2}}$ is in $S$ - here we use that $t \mapsto$ $f(t) \frac{t}{1+t^{2}}$ is bounded. In other words, we may assume without loss of generality, that $f$ vanishes at infinity, i.e. $\lim _{t \rightarrow \infty} f(t)=0$.

Suppose for the moment that there is $e \in S, e \neq 0$, which vanishes at infinity. Let $a, b \in \mathbb{R}$. Then $e_{a, b}: \mathbb{R} \rightarrow \mathbb{R}, t \mapsto e(a t+b)$ - an affine transformation followed by $e$ - is also in $S$, vanishes at infinity, and can be extended to a continuous real-valued function on the one-point compactification $\mathbb{R} \cup\{\infty\}$ of $\mathbb{R}$ (by defining $\left.e_{a, b}(\infty)=0\right)$. It is easy to see that the $C^{*}$-subalgebra of $C(\mathbb{R} \cup\{\infty\}$ ) generated by these extended $e_{a, b}$ 's separates the points of $\mathbb{R} \cup\{\infty\}$, and is thus $C(\mathbb{R} \cup\{\infty\})$ itself by the Stone-Weierstraßtheorem (see $27 \times X$. Since $f$ vanishes at infinity, $f$ can be extended to an element of $C(\mathbb{R} \cup\{\infty\})$, and can thus be obtained (by taking real parts if necessary) from the extended $e_{a, b}$ 's and real constants via uniform limits, addition and (real scalar) multiplication. Since $S$ contains the $e_{a, b}$ 's and constants and is closed under these operations (acting on bounded functions), we see that $f \in S$.

To complete the proof, we show that such $e$ indeed exists. Let $e, s: \mathbb{R} \rightarrow \mathbb{R}$ be given by $e(t)=t s(t)$ and $s(t)=\frac{1}{1+t^{2}}$. Clearly $e$ and $s$ are continuous and
vanish at infinity. To see that $e$ is ultrastrongly continuous, let $\left(b_{\alpha}\right)_{\alpha}$ be a net of self-adjoint elements of $\mathscr{A}$ which converges ultrastrongly to $a \in \mathscr{A}_{\mathbb{R}}$, and let $\omega: \mathscr{A} \rightarrow \mathbb{C}$ be an npu-map. Unfolding the definitions of $e$ and $s$ yields the following equality.

$$
e\left(b_{\alpha}\right)-e(a)=s\left(b_{\alpha}\right)\left(b_{\alpha}-a\right) s(a)-e\left(b_{\alpha}\right)\left(b_{\alpha}-a\right) e(a) .
$$

Since $\left\|s\left(b_{\alpha}\right)\right\| \leqslant 1$, we have $\left\|s\left(b_{\alpha}\right)\left(b_{\alpha}-a\right) s(a)\right\|_{\omega} \leqslant\left\|\left(b_{\alpha}-a\right) s(a)\right\|_{\omega} \equiv \| b_{\alpha}-$ $a \|_{s(a) * \omega}$. Similarly, since $\left\|e\left(b_{\alpha}\right)\right\| \leqslant 1$, we get

$$
\left\|e\left(b_{\alpha}\right)-e(a)\right\|_{\omega} \leqslant\left\|b_{\alpha}-a\right\|_{s(a) * \omega}+\left\|b_{\alpha}-a\right\|_{e(a) * \omega} .
$$

Thus $e\left(b_{\alpha}\right)$ converges ultrastrongly to $e(a)$, and so $e$ is ultrastrongly continuous.

Corollary Given a von Neumann algebra $\mathscr{A}$ the map $a \mapsto|a|: \mathscr{A}_{\mathbb{R}} \rightarrow \mathscr{A}_{\mathbb{R}}$ is III ultrastrongly continuous.

Kaplansky Density Theorem Let $b$ be an element of a von Neumann algebra $\mathscr{B}$ IV which is the ultrastrong limit of a net of elements from a $C^{*}$-subalgebra $\mathscr{A}$ of $\mathscr{B}$. Then $b$ is the ultrastrong limit of a net $\left(a_{\alpha}\right)_{\alpha}$ in $\mathscr{A}$ with $\left\|a_{\alpha}\right\| \leqslant\|b\|$ for all $\alpha$. Moreover,

1. if $b$ is self-adjoint, then the $a_{\alpha}$ can be chosen to be self-adjoint as well;
2. if $b$ is positive, then the $a_{\alpha}$ can be chosen to be positive as well, and
3. if $b$ is an effect, then the $a_{\alpha}$ can be chosen to be effects as well.

Proof Let $\left(a_{\alpha}\right)_{\alpha}$ be a net in $\mathscr{A}$ that converges ultrastrongly to $b$.
Assume for the moment that $b$ is self-adjoint. Then $\left(a_{\alpha}\right)_{\mathbb{R}}$ converges ultraweakly (but perhaps not ultrastrongly) to $b_{\mathbb{R}}=b$ as $\alpha \rightarrow \infty$, and so $b$ is in the ultraweak closure of the convex set $\mathscr{A}_{\mathbb{R}}$. Since the ultraweak and ultrastrong closure of convex subsets of $\mathscr{A}$ coincide (by 73 VIII ), we see that $b$ is also the ultrastrong limit of some net $\left(a_{\alpha}^{\prime}\right)_{\alpha}$ in $\mathscr{A}_{\mathbb{R}}$. Since the map $-\|b\| \vee(\cdot) \wedge\|b\|: \mathscr{B}_{\mathbb{R}} \rightarrow \mathscr{B}_{\mathbb{R}}$ is ultrastrongly continuous by \} we see that a _ { \alpha } ^ { \prime \prime } : = - \| b \| \vee a _ { \alpha } ^ { \prime } \wedge \| b \| gives a net $\left(a_{\alpha}^{\prime \prime}\right)_{\alpha}$ in $[-\|b\|,\|b\|]_{\mathscr{A}}$ that converges ultrastrongly to $b$.

If we assume in addition that $b$ is positive, then $a_{\alpha}^{\prime \prime \prime}:=\left(a_{\alpha}^{\prime \prime}\right)_{+}$gives a net $\left(a_{\alpha}^{\prime \prime \prime}\right)_{\alpha}$ in $[0,\|b\|]_{\mathscr{A}}$ that converges ultrastrongly to $b_{+}=b$, because the map
$(\cdot)_{+}: \mathscr{B}_{\mathbb{R}} \rightarrow \mathscr{B}_{\mathbb{R}}$ is ultrastrongly continuous by $\|$. Note that if $b$ is an effect, then so are the $a_{\alpha}^{\prime \prime \prime}$.

This takes care of all the special cases. The general case in which $b$ is an arbitrary element of $\mathscr{B}$ requires a trick: since the element $B:=\left(\begin{array}{cc}0 & b \\ b^{*} & 0\end{array}\right)$ of the von Neumann algebra $M_{2}(\mathscr{B})$ is self-adjoint, and the ultrastrong limit of the net $\left(\begin{array}{cc}0 & a_{\alpha} \\ a_{\alpha}^{*} & 0\end{array}\right)$ from the $C^{*}$-subalgebra $M_{2}(\mathscr{A})$ of $M_{2}(\mathscr{B})$, there is, as we've just seen, a net $\left(A_{\alpha}\right)_{\alpha}$ in $M_{2}(\mathscr{A})$ that converges ultrastrongly to $B$ with $\left\|A_{\alpha}\right\| \leqslant\|B\| \equiv\|b\|$ for all $\alpha$. Since the upper-right entries $\left(A_{\alpha}\right)_{12}$ will then converge ultrastrongly to $B_{12} \equiv b$ as $\alpha \rightarrow \infty$, and $\left\|\left(A_{\alpha}\right)_{12}\right\| \leqslant\left\|A_{\alpha}\right\| \leqslant\|b\|$ for all $\alpha$, we are done.

VI Corollary Given $\varepsilon>0$ and an ultraweakly dense $*$-subalgebra $\mathscr{S}$ of a von Neumann algebra $\mathscr{A}$ each element $a$ of $\mathscr{A}$ is the ultrastrong limit of a net $\left(s_{\alpha}\right)_{\alpha}$ from $\mathscr{S}$ with $\left\|s_{\alpha}\right\| \leqslant\|a\|(1+\varepsilon)$ for all $\alpha$.
VII Proof As the norm closure $\mathscr{C}$ of $\mathscr{S}$ in $\mathscr{A}$ is an ultraweakly (and thus by 73 VIII ultrastrongly) dense $C^{*}$-subalgebra of $\mathscr{A}$, the element $a$ of $\mathscr{A}$ is by IV the ultrastrong limit of net $\left(c_{\alpha}\right)_{\alpha \in D}$ in $\mathscr{C}$ with $\left\|c_{\alpha}\right\| \leqslant\|a\|$ for all $\alpha$. Each element $c_{\alpha}$ is in its turn the norm (and thus ultrastrong) limit of a sequence $s_{\alpha 1}, s_{\alpha 2}, \ldots$ in $\mathscr{S}$, and if we choose the $s_{\alpha n}$ such that $\left\|c_{\alpha}-s_{\alpha n}\right\| \leqslant 2^{-n}$, then $s_{\alpha n}$ converge ultrastrongly to $b$ as $D \times \mathbb{N} \ni(\alpha, n) \rightarrow \infty$. Finally, since $\lim _{n}\left\|s_{\alpha n}\right\|=\left\|c_{\alpha}\right\| \leqslant$ $\|c\| \leqslant(1+\varepsilon)\|c\|$ we have $\left\|s_{\alpha n}\right\| \leqslant(1+\varepsilon)\|c\|$ for sufficiently large $n$, and thus for all $n$ if we replace $\left(s_{\alpha n}\right)_{n}$ by the appropriate subsequence.

### 3.3.3 Closedness of Subalgebras

75 Recall that according to our definition 42 V a von Neumann subalgebra $\mathscr{B}$ of a von Neumann algebra $\mathscr{A}$ is a $C^{*}$-subalgebra of $\mathscr{A}$ which is closed under suprema of bounded directed sets of self-adjoint elements. We will show that such $\mathscr{B}$ is ultrastrongly closed in $\mathscr{A}$.

II Lemma Let $\mathscr{B}$ be a von Neumann subalgebra of a von Neumann algebra $\mathscr{A}$. Let $\omega_{0}, \omega_{1}: \mathscr{A} \rightarrow \mathbb{C}$ be npu-maps, which are separated by a net $\left(b_{\alpha}\right)_{\alpha}$ of effects of $\mathscr{B}$ in the sense that $\lim _{\alpha} \omega_{0}\left(b_{\alpha}\right)=0$ and $\lim _{\alpha} \omega_{1}\left(b_{\alpha}^{\perp}\right)=0$. Then $\omega_{0}$ and $\omega_{1}$ are separated by a projection $q$ of $\mathscr{B}$ in the sense that $\omega_{0}(q)=0=\omega_{1}\left(q^{\perp}\right)$.
III Proof (Based on Lemma 45.3 and Theorem 45.6 of 12 .)
Note that it suffices to find an effect $a$ in $\mathscr{B}$ with $\omega_{0}(a)=0=\omega_{1}\left(a^{\perp}\right)$, because then $\omega_{0}(\lceil a\rceil)=0=\omega_{1}\left(\lceil a\rceil^{\perp}\right)$ by 60 I and $\lceil a\rceil \in \mathscr{B}$.

Note that we can find a subsequence $\left(b_{n}\right)_{n}$ of $\left(b_{\alpha}\right)_{\alpha}$ such that $\omega_{0}\left(b_{n}\right) \leqslant$
$n^{-1} 2^{-n}$ and $\omega_{1}\left(b_{n}^{\perp}\right) \leqslant n^{-1}$ for all $n$. For $n<m$, define

$$
a_{n m}=\left(1+\sum_{k=n}^{m} k b_{k}\right)^{-1} \sum_{k=n}^{m} k b_{k} .
$$

Since we have seen in 25 II that the map $d \mapsto(1+d)^{-1} d$ is order preserving (on $\mathscr{B}_{+}$), we have $0 \leqslant a_{n m} \leqslant \frac{1}{2}$ and we get the formation

where $a_{n}:=\bigvee_{m \geqslant n} a_{n m}$ and $a:=\bigwedge_{n} a_{n}$. We'll prove that $\omega_{0}(a)=0=\omega_{1}\left(a^{\perp}\right)$. $\left(\omega_{0}(a)=0\right)$ Since $\omega_{0}\left(b_{n}\right) \leqslant n^{-1} 2^{-n}$ and $a_{n m} \leqslant \sum_{k=n}^{m} k b_{k}$, we get $\omega_{0}\left(a_{n m}\right) \leqslant$ $\sum_{k=n}^{m} k \omega_{0}\left(b_{k}\right) \leqslant 2^{1-n}$, and so $\omega_{0}(a)=\bigwedge_{n} \bigvee_{m \geqslant n} \omega_{0}\left(a_{n m}\right) \leqslant \bigwedge_{n} 2^{1-n}=0$.
$\left(\omega_{1}\left(a^{\perp}\right)=0\right)$ Let $m>n$ be given. Since $\sum_{k=n}^{m} k b_{k} \geqslant m b_{m}$ and $d \mapsto(1+d)^{-1} d$ is monotone on $\mathscr{B}_{+}$we get $a_{n m} \geqslant\left(1+m b_{m}\right)^{-1} m b_{m}$, and so $a_{n m}^{\perp} \leqslant\left(1+m b_{m}\right)^{-1}$.

Observe that for a real number $t \in[0,1]$, we have $t t^{\perp} \geqslant 0$, and so ( $1+$ $m t)\left(1+m t^{\perp}\right)=1+m+m^{2} t t^{\perp} \geqslant 1+m$. This yields the inequality $(1+$ $m t)^{-1} \leqslant(1+m)^{-1}\left(1+m t^{\perp}\right)$ for real numbers $t \in[0,1]$. The corresponding inequality for effects of a $C^{*}$-algebra (obtained via Gelfand's representation theorem, 27 XXVII gives us $\omega_{1}\left(a_{n m}^{\perp}\right) \leqslant \omega_{1}\left(\left(1+m b_{m}\right)^{-1}\right) \leqslant(1+m)^{-1}(1+$ $\left.m \omega_{1}\left(b_{m}^{\perp}\right)\right) \leqslant \frac{2}{1+m}$, where we have used that $\omega_{1}\left(b_{m}^{\perp}\right) \leqslant \frac{1}{m}$. Hence $\omega_{1}\left(a_{n}^{\perp}\right)=$ $\bigwedge_{m \geqslant n} \omega_{1}\left(a_{n m}^{\perp}\right) \leqslant \bigwedge_{m \geqslant n} \frac{2}{1+m}=0$ for all $n$, and so $\omega_{1}\left(a^{\perp}\right)=\bigvee_{n} \omega_{1}\left(a_{n}^{\perp}\right)=0$.

Lemma Let $\mathscr{B}$ be a von Neumann subalgebra of a von Neumann algebra $\mathscr{A}$. Let $p$ be a projection of $\mathscr{A}$, which is the ultrastrong limit of a net in $\mathscr{B}$.

For all npu-maps $\omega_{0}, \omega_{1}: \mathscr{A} \rightarrow \mathbb{C}$ with $\omega_{0}(p)=0=\omega_{1}\left(p^{\perp}\right)$ there is a projection $q$ of $\mathscr{B}$ with $\omega_{0}(q)=0=\omega_{1}\left(q^{\perp}\right)$.

VII Proof Let $\left(b_{\alpha}\right)_{\alpha}$ be a net in $\mathscr{B}$ which converges ultrastrongly to $p$. We may assume that all $b_{\alpha}$ are effects by Kaplansky's density theorem (74IV). Note that $\left(\omega_{0}\left(b_{\alpha}\right)\right)_{\alpha}$ converges to $\omega_{0}(p) \equiv 0$, and $\left(\omega_{1}\left(b_{\alpha}^{\perp}\right)\right)_{\alpha}$ converges to $\omega_{1}\left(p^{\perp}\right) \equiv 0$. Now apply II.

VIII Theorem A von Neumann subalgebra $\mathscr{B}$ of a von Neumann algebra $\mathscr{A}$ is ultrastrongly and ultraweakly closed.
IX Proof It suffices to show that $\mathscr{B}$ is ultrastrongly closed, because then, by 73 VIII . $\mathscr{B}$ will be ultraweakly closed as well.

Let $p$ be a projection of $\mathscr{A}$ which is the ultrastrong limit of a net from $\mathscr{B}$. It suffices to show that $p \in \mathscr{B}$, because the ultrastrong closure of $\mathscr{B}$ being a von Neumann subalgebra of $\mathscr{A}$ is generated by its projections, see 65 IV . Note that given an np-map $\omega: \mathscr{A} \rightarrow \mathbb{C}$, the carrier $\lceil\omega\rceil$ of $\omega$ need not be equal to the carrier of $\omega$ restricted to $\mathscr{B}$, which we'll therefore denote by $\lceil\omega\rceil_{\mathscr{B}}$; but we do have $\lceil\omega\rceil \leqslant\lceil\omega\rceil_{\mathscr{B}}$. Then by 66 IV

$$
\begin{equation*}
\bigvee_{\omega_{1}}\left\lceil\omega_{1}\right\rceil_{\mathscr{B}} \geqslant \bigvee_{\omega_{1}}\left\lceil\omega_{1}\right\rceil=p=\bigwedge_{\omega_{0}}\left\lceil\omega_{0}\right\rceil^{\perp} \geqslant \bigwedge_{\omega_{0}}\left\lceil\omega_{0}\right\rceil_{\mathscr{B}}^{\perp}, \tag{3.1}
\end{equation*}
$$

where $\omega_{0}$ ranges over np-maps $\omega_{0}: \mathscr{A} \rightarrow \mathbb{C}$ with $\omega_{0}(p)=0$, and $\omega_{1}$ ranges over np-maps $\omega_{1}: \mathscr{A} \rightarrow \mathbb{C}$ with $\omega_{1}\left(p^{\perp}\right)=0$. Since for such $\omega_{0}$ and $\omega_{1}$ there is by VI a projection $q$ in $\mathscr{B}$ with $\omega_{0}(q)=0=\omega_{1}\left(q^{\perp}\right)$, we get $\left\lceil\omega_{1}\right\rceil_{\mathscr{B}} \leqslant q \leqslant\left\lceil\omega_{0}\right\rceil_{\mathscr{B}}^{\perp}$, and so $\bigvee_{\omega_{1}}\left\lceil\omega_{1}\right\rceil_{\mathscr{B}} \leqslant \bigwedge_{\omega_{0}}\left\lceil\omega_{0}\right\rceil_{\mathscr{B}}^{\perp}$. It follows that the inequalities in (3.1) are in fact equalities, and so $p=\bigvee_{\omega_{1}}\left\lceil\omega_{1}\right\rceil_{\mathscr{B}} \in \mathscr{B}$.

### 3.3.4 Completeness

76 Proposition The von Neumann algebra $\mathscr{B}(\mathscr{H})$ of bounded operators on a Hilbert space $\mathscr{H}$ is ultrastrongly complete.
II Proof Let $\left(T_{\alpha}\right)_{\alpha}$ be an ultrastrongly Cauchy net in $\mathscr{B}(\mathscr{H})$ (which must be shown to converge ultrastrongly to some operator $T$ in $\mathscr{B}(\mathscr{H})$ ).

Note that given $x \in \mathscr{H}$, the net $\left(T_{\alpha} x\right)_{\alpha}$ in $\mathscr{H}$ is norm Cauchy, because $\left\|\left(T_{\alpha}-T_{\beta}\right) x\right\|=\left\|T_{\alpha}-T_{\beta}\right\|_{\langle x,(\cdot) x\rangle}$ vanishes for sufficiently large $\alpha, \beta$, and so we may define $T x:=\lim _{\alpha} T_{\alpha} x$, giving a map $T: \mathscr{H} \rightarrow \mathscr{H}$.

It is clear that $T$ will be linear, but the question is whether $T$ is bounded, and whether in that case $\left(T_{\alpha}\right)_{\alpha}$ converges ultrastrongly to $T$.

Suppose towards a contradiction that $T$ is not bounded. Then we can find $x_{1}, x_{2}, \ldots \in \mathscr{H}$ with $\left\|x_{n}\right\|^{2} \leqslant 2^{-n}$ and $\left\|T x_{n}\right\|^{2} \geqslant 1$ for all $n$. Since $\omega:=$ $\sum_{n}\left\langle x_{n},(\cdot) x_{n}\right\rangle: \mathscr{B}(\mathscr{H}) \rightarrow \mathbb{C}$ is an np-map by 38 IV , it follows that $\left\|T_{\alpha}\right\|_{\omega}^{2} \equiv$
$\sum_{n=1}^{\infty}\left\|T_{\alpha} x_{n}\right\|^{2}$ converges to some positive number $R$. Since any partial sum $\sum_{n=1}^{N}\left\|T_{\alpha} x_{n}\right\|^{2} \leqslant\left\|T_{\alpha}\right\|_{\omega}^{2}$ converges to $\sum_{n=1}^{N}\left\|T x_{n}\right\|^{2} \geqslant N$, we must conclude that $R \geqslant N$, for all natural numbers $N$, which is absurd. Hence $T$ is bounded.

It remains to be shown that $\left(T_{\alpha}\right)_{\alpha}$ converges ultrastrongly to $T$. So let $\omega: \mathscr{B}(\mathscr{H}) \rightarrow \mathbb{C}$ be an arbitrary np-map, being of the form $\omega \equiv \sum_{n}\left\langle x_{n},(\cdot) x_{n}\right\rangle$ for some $x_{1}, x_{2}, \ldots \in \mathscr{H}$ with $\sum_{n}\left\|x_{n}\right\|^{2}<\infty$ by 39 IX We must show that $\| T-$ $T_{\alpha} \|_{\omega} \equiv\left(\sum_{n}\left\|\left(T-T_{\alpha}\right) x_{n}\right\|^{2}\right)^{1 / 2}$ converges to 0 as $\alpha \rightarrow 0$.

Let $\varepsilon>0$ be given, and pick $\alpha_{0}$ such that $\left\|T_{\alpha}-T_{\beta}\right\|_{\omega} \leqslant \frac{1}{2 \sqrt{2}} \varepsilon$ for all $\alpha, \beta \geqslant$ $\alpha_{0}$ - this is possible because $\left(T_{\alpha}\right)_{\alpha}$ is ultrastrongly Cauchy. We claim that $\left\|T-T_{\alpha}\right\|_{\omega} \leqslant \varepsilon$ for any $\alpha \geqslant \alpha_{0}$. Since for such $\alpha$ the sum

$$
\sum_{n=1}^{\infty}\left\|\left(T-T_{\alpha}\right) x_{n}\right\|^{2}=\sum_{n=1}^{N-1}\left\|\left(T-T_{\alpha}\right) x_{n}\right\|^{2}+\sum_{n=N}^{\infty}\left\|\left(T-T_{\alpha}\right) x_{n}\right\|^{2}
$$

converges ( to $\left\|T-T_{\alpha}\right\|_{\omega}^{2}$ ), we can find $N$ such that the second term in the bound above is below $\frac{1}{2} \varepsilon^{2}$. The first term will also be below $\frac{1}{2} \varepsilon^{2}$, because

$$
\left(\sum_{n=1}^{N-1}\left\|\left(T-T_{\alpha}\right) x_{n}\right\|^{2}\right)^{1 / 2} \leqslant\left(\sum_{n=1}^{N-1}\left\|\left(T-T_{\beta}\right) x_{n}\right\|^{2}\right)^{1 / 2}+\left(\sum_{n=1}^{N-1}\left\|\left(T_{\beta}-T_{\alpha}\right) x_{n}\right\|^{2}\right)^{1 / 2}
$$

for any $\beta$, and in particular for $\beta$ large enough that the first term on the righthand side above is below $\frac{1}{2 \sqrt{2}} \varepsilon$. If we choose $\beta \geqslant \alpha_{0}$ the second term will be below $\frac{1}{2 \sqrt{2}} \varepsilon$ too, and we get $\left\|T-T_{\alpha}\right\|_{\omega}^{2} \leqslant \frac{1}{2} \varepsilon^{2}+\left(\frac{1}{2 \sqrt{2}} \varepsilon+\frac{1}{2 \sqrt{2}} \varepsilon\right)^{2} \equiv \varepsilon^{2}$ all in all. (This reasoning is very similar to that in 6 II )

Hence $\mathscr{B}(\mathscr{H})$ is ultrastrongly complete.
Proposition The von Neumann algebra $\mathscr{B}(\mathscr{H})$ of bounded operators on a III Hilbert space $\mathscr{H}$ is bounded ultraweakly complete.
Proof Let $\left(T_{\alpha}\right)_{\alpha}$ be a norm-bounded ultraweakly Cauchy net in $\mathscr{B}(\mathscr{H})$. We must show that $\left(T_{\alpha}\right)_{\alpha}$ converges ultraweakly to some bounded operator $T$ on $\mathscr{H}$.

Note that given $x, y \in \mathscr{H}$ the net $\left(\left\langle x, T_{\alpha} y\right\rangle\right)_{\alpha}$ is Cauchy (because $\langle x,(\cdot) y\rangle \equiv$ $\frac{1}{4} \sum_{k=0}^{3} i^{k}\left\langle i^{k} x+y,(\cdot)\left(i^{k} x+y\right)\right\rangle$ is ultraweakly continuous), and so we may define $[x, y]=\lim _{\alpha}\left\langle x, T_{\alpha} y\right\rangle$. The resulting 'form' [ $\left.\cdot, \cdot\right]: \mathscr{H} \times \mathscr{H} \rightarrow \mathbb{C}$ (see 36 IV) is bounded, because $\|[x, y]\| \leqslant\left(\sup _{\alpha}\left\|T_{\alpha}\right\|\right)\|x\|\|y\|$ for all $x, y \in \mathscr{H}$ and $\sup _{\alpha}\left\|T_{\alpha}\right\|<$ $\infty$ since $\left(T_{\alpha}\right)_{\alpha}$ is norm bounded. By 36 V , there is a unique bounded operator $T$ with $\langle x, T y\rangle=[x, y]$ for all $x, y \in \mathscr{H}$.

By definition of $T$ it is clear that $\lim _{\alpha}\left\langle x,\left(T-T_{\alpha}\right) x\right\rangle=0$ for any $x \in \mathscr{H}$, but it is not yet clear that $\left(T_{\alpha}\right)_{\alpha}$ converges ultraweakly to $T$. For this we must show that $\lim _{\alpha} \omega\left(T-T_{\alpha}\right)=0$ for any np-map $\omega: \mathscr{B}(\mathscr{H}) \rightarrow \mathbb{C}$. By 39IX, we
know that such $\omega$ is of the form $\omega=\sum_{n}\left\langle x_{n},(\cdot) x_{n}\right\rangle$ for some $x_{1}, x_{2}, \ldots \in \mathscr{H}$ with $\sum_{n}\left\|x_{n}\right\|^{2}<\infty$. Now, given $N$ and $\alpha$ we easily obtain the following bound.

$$
\left|\omega\left(T-T_{\alpha}\right)\right| \leqslant \sum_{n=1}^{N-1}\left|\left\langle x_{n}\left(T-T_{\alpha}\right), x_{n}\right\rangle\right|+\left(\|T\|+\sup _{\alpha}\left\|T_{\alpha}\right\|\right) \sum_{n=N}^{\infty}\left\|x_{n}\right\|^{2}
$$

Since the first term of this bound converges to 0 as $\alpha \rightarrow \infty$, we get, for all $N$,

$$
\limsup _{\alpha}\left|\omega\left(T-T_{\alpha}\right)\right| \leqslant\left(\|T\|+\sup _{\alpha}\left\|T_{\alpha}\right\|\right) \sum_{n=N}^{\infty}\left\|x_{n}\right\|^{2}
$$

Since the tail $\sum_{n=N}^{\infty}\left\|x_{n}\right\|^{2}$ converges to 0 as $N \rightarrow \infty, \lim \sup _{\alpha}\left|\omega\left(T-T_{\alpha}\right)\right|=0$. Hence $\omega(T)=\lim _{\alpha} \omega\left(T_{\alpha}\right)$, and so $\left(T_{\alpha}\right)_{\alpha}$ converges ultraweakly to $T$.

77 Theorem A von Neumann algebra $\mathscr{A}$ is ultrastrongly complete and bounded ultraweakly complete.
II Proof Let $\Omega$ be the set of all np-functionals on $\mathscr{A}$. Recall from 48 IX that $\varrho_{\Omega}$ gives a nmiu-isomorphism onto the von Neumann algebra $\mathscr{R}:=\varrho_{\Omega}(\mathscr{A})$ of operators on the Hilbert space $\mathscr{H}_{\Omega}$. Since $\mathscr{B}\left(\mathscr{H}_{\Omega}\right)$ is ultrastrongly complete (761), and $\mathscr{R}$ is ultrastrongly closed in $\mathscr{B}\left(\mathscr{H}_{\Omega}\right)$ (see 75 VIII ), we see that $\mathscr{R}$ is complete with respect to the ultrastrong topology of $\mathscr{B}\left(\mathscr{H}_{\Omega}\right)$, but since any np-functional $\omega: \mathscr{R} \rightarrow \mathbb{C}$ is of the form $\omega \equiv\langle x,(\cdot) x\rangle$ for some $x \in \mathscr{H}_{\Omega}$, and therefore the ultrastrong topology on $\mathscr{B}\left(\mathscr{H}_{\Omega}\right)$ coincides on $\mathscr{R}$ with the ultrastrong topology of $\mathscr{R}$, we see that $\mathscr{R}$ (and therefore $\mathscr{A}$ ) is complete with respect to its own ultrastrong topology. Since similarly $\mathscr{B}\left(\mathscr{H}_{\Omega}\right)$ is bounded ultraweakly complete 76 III ), the ultraweak topology on $\mathscr{B}\left(\mathscr{H}_{\Omega}\right)$ coincides on $\mathscr{R}$ with the ultraweak topology on $\mathscr{R}$, and $\mathscr{R}$ is ultraweakly closed in $\mathscr{B}\left(\mathscr{H}_{\Omega}\right)$ (by 75 VIII$)$, we see that $\mathscr{R}$ is bounded ultraweakly complete.
III Theorem The ball $(\mathscr{A})_{1}$ of a von Neumann algebra $\mathscr{A}$ is ultraweakly compact.
IV Proof Writing $\Omega$ for the set of npu-maps $\omega: \mathscr{A} \rightarrow \mathbb{C}$, the map $\kappa: \mathscr{A} \rightarrow \mathbb{C}^{\Omega}$ given by $\kappa(a)=(\omega(a))_{\omega}$ for all $a \in \mathscr{A}$ is clearly a linear homeomorphism from $\mathscr{A}$ with the ultraweak topology onto $\kappa(\mathscr{A}) \subseteq \mathbb{C}^{\Omega}$ endowed with the product topology. Since $\kappa$ restricts to an isomorphism of uniform spaces $(\mathscr{A})_{1} \rightarrow \kappa\left((\mathscr{A})_{1}\right)$, and $(\mathscr{A})_{1}$ is ultraweakly complete (being a norm-bounded ultraweakly closed subset of the bounded ultraweakly complete space $\mathscr{A}$, see $\rrbracket$, we see that $\kappa\left((\mathscr{A})_{1}\right)$ is complete, and thus closed in $\mathbb{C}^{\Omega}$. Now note that $\kappa\left((\mathscr{A})_{1}\right)$ is a closed subset of the (by Tychonoff's theorem) compact space $\left((\mathbb{C})_{1}\right)^{\Omega}$, because $|\omega(a)| \leqslant 1$ for all $a \in(\mathscr{A})_{1}$ and $\omega \in \Omega$. But then $\kappa\left((\mathscr{A})_{1}\right)$, being a closed subset of a
compact Hausdorff space, is compact, and so $(\mathscr{A})_{1}$ (being homeomorphic to it) is compact too.

Proposition Given an ultraweakly dense $*$-subalgebra $\mathscr{S}$ of a von Neumann algebra $\mathscr{A}$, any ultraweakly continuous and bounded linear map $f: \mathscr{S} \rightarrow \mathscr{B}$ can be extended uniquely to an ultraweakly continuous map $g: \mathscr{A} \rightarrow \mathscr{B}$.

Moreover, $g$ is bounded, and in fact, $\|g\|=\|f\|$.
Proof As the uniqueness of $g$ is rather obvious we concern ourselves only with its existence. Let $a \in \mathscr{A}$ be given in order to define $g(a)$. Let also $\varepsilon>0$ be given. Note that by 74 VI there is a net $\left(s_{\alpha}\right)_{\alpha}$ in $\mathscr{S}$ that converges ultrastrongly (and so ultraweakly too) to $a$ with $\left\|s_{\alpha}\right\| \leqslant(1+\varepsilon)\|a\|$ for all $\alpha$. Now, since the net $\left(s_{\alpha}\right)_{\alpha}$ is bounded an ultraweakly Cauchy, and $f$ is bounded and (uniformly) ultraweakly continuous, the net $\left(f\left(s_{\alpha}\right)\right)_{\alpha}$ is bounded and ultraweakly Cauchy too, and thus converges (by $\rrbracket$ ) to some element $u \lim _{\alpha} f\left(s_{\alpha}\right)$ of $\mathscr{B}$.

Of course we'd like to define $g(a):=\operatorname{uwlim}_{\alpha} f\left(s_{\alpha}\right)$, but must first check that uwlim ${ }_{\alpha} f\left(s_{\alpha}^{\prime}\right)=\operatorname{uwlim}_{\alpha} f\left(s_{\alpha}\right)$ when $\left(s_{\alpha}^{\prime}\right)_{\alpha}$ is a second net with the same properties as $\left(s_{\alpha}\right)_{\alpha}$. Let us for simplicity's sake assume that $\left(s_{\alpha}^{\prime}\right)_{\alpha}$ and $\left(s_{\alpha}\right)_{\alpha}$ have the same index set - matters can always be arranged this way. Then as the difference $s_{\alpha}-s_{\alpha}^{\prime}$ converges ultraweakly to 0 in $\mathscr{A}$ as $\alpha \rightarrow \infty$, $\operatorname{uwlim}_{\alpha} f\left(s_{\alpha}-s_{\alpha}^{\prime}\right)=0$, implying that uwlim ${ }_{\alpha} f\left(s_{\alpha}\right)=\operatorname{uwlim}_{\alpha} f\left(s_{\alpha}^{\prime}\right)$.
In this way we obtain a map $g: \mathscr{A} \rightarrow \mathscr{B}$ - which is clearly linear. The map $g$ is also bounded, because since $\left\|s_{\alpha}\right\| \leqslant(1+\varepsilon)\|a\|$ for all $\alpha$, where $\left(s_{\alpha}\right)_{\alpha}$ and $t$ are as before, we have $\left\|f\left(s_{\alpha}\right)\right\| \leqslant(1+\varepsilon)\|f\|\|a\|$ for all $\alpha$, and so $\|g(a)\|=$ $\|$ uwlim $_{\alpha} f\left(s_{\alpha}\right)\|\leqslant(1+\varepsilon)\| f\| \| a \|$. More precisely, $\|g\| \leqslant(1+\varepsilon)\|f\|$, andas $\varepsilon>0$ was arbitrary-in fact $\|g\| \leqslant\|f\|$, and so $\|g\|=\|f\|$.

That, finally, $g$ is ultraweakly continuous follows by a standard but abstract argument from the fact that $f$ is uniformly ultraweakly continuous. We'll give a concrete version of this argument here. To begin, note that it suffices to show that $\omega \circ g$ is ultraweakly continuous at 0 where $\omega: \mathscr{B} \rightarrow \mathbb{C}$ is an np-functional. Let $\varepsilon>0$ be given. Since $f$ is ultraweakly continuous, and thus $\omega \circ f$ is too, there is $\delta>0$ and an np-functional $\nu: \mathscr{A} \rightarrow \mathbb{C}$ such that $|\nu(s)| \leqslant \delta \Longrightarrow|\omega(f(s))| \leqslant \varepsilon$ for all $s \in \mathscr{S}$. We claim that $|\nu(a)| \leqslant \delta / 2 \Longrightarrow|\omega(g(a))| \leqslant 2 \varepsilon$ for all $a \in \mathscr{A}$, which implies, of course, that $\omega \circ g$ is ultraweakly continuous on 0 . So let $a \in \mathscr{A}$ with $|\nu(a)| \leqslant \delta / 2$ be given. Pick (as before) a bounded net $\left(s_{\alpha}\right)_{\alpha}$ in $\mathscr{S}$ such that $f\left(s_{\alpha}\right)$ converges to $a$ as $\alpha \rightarrow \infty$, and observe that, for all $\alpha$,

$$
|\omega(g(a))| \leqslant\left|\omega\left(g(a)-f\left(s_{\alpha}\right)\right)\right|+\left|\omega\left(f\left(s_{\alpha}\right)\right)\right| .
$$

The first term on the right-hand side above will vanish as $\alpha \rightarrow \infty$ (since
$g(a)=u^{\prime} \lim _{\alpha} f\left(s_{\alpha}\right)$ ), and will thus be smaller than $\varepsilon$ for sufficiently large $\alpha$. Since $\lim _{\alpha}\left|\nu\left(s_{\alpha}\right)\right|=|\nu(a)| \leqslant \delta / 2<\delta$ we see that for sufficiently large $\alpha$ we'll have $\left|\nu\left(s_{\alpha}\right)\right| \leqslant \delta$ and with it $|\omega(f(s))| \leqslant \varepsilon$. Combined, we get $|\omega(g(a))| \leqslant 2 \varepsilon$, and so $g$ is ultraweakly continuous.

### 3.4 Division

78 Using the ultrastrong completeness of von Neumann algebras (see 771) we'll address the question of division: given elements $a$ and $b$ of a von Neumann algebra $\mathscr{A}$, when is there an element $c \in \mathscr{A}$ with $a=c b$ ? Surely, such $c$ can not always exist, because its presence implies

$$
\begin{equation*}
a^{*} a \leqslant B b^{*} b, \tag{3.2}
\end{equation*}
$$

where $B=\|c\|^{2}$; but this turns out to be the only restriction: we'll see in 81 V that if (3.2) holds for some $B \in[0, \infty)$, then $a=c b$ for some unique $c \in \mathscr{A}$ with $\lceil c) \leqslant(b\rceil$, which we'll denote by $a / b$.

The main application of this division in our work is a universal property for the map $b \mapsto \sqrt{a} b \sqrt{a}: \mathscr{A} \rightarrow\lceil a\rceil \mathscr{A}\lceil a\rceil$ where $a$ is a positive element of a von Neumann algebra $\mathscr{A}$. Indeed, we'll show that for every np-map $f: \mathscr{B} \rightarrow \mathscr{A}$ with $f(1) \leqslant a$ there is a (unique) np-map $g: \mathscr{B} \rightarrow\lceil a\rceil \mathscr{A}\lceil a\rceil$ with $f(b)=\sqrt{a} g(b) \sqrt{a}$ for all $b \in \mathscr{B}$ - by taking $g(b)=\sqrt{a} \backslash(f(b) / \sqrt{a})$, see 96 V . This does not give a complete description of the map $b \mapsto \sqrt{a} b \sqrt{a}$, though, since it shares its universal property with all the maps $b \mapsto c^{*} b c, \mathscr{A} \rightarrow\lceil a\rceil \mathscr{A}\lceil a\rceil$ where $c \in \mathscr{A}$ with $c^{*} c=a$, but that is a challenge for the next chapter.

Returning to division again, another application is the polar decomposition of an element $a$ of a von Neumann algebra $\mathscr{A}$, see 821, which is simply

$$
a=\left(a / \sqrt{a^{*} a}\right) \sqrt{a^{*} a} .
$$

Before we get down to business, let us indicate the difficulty in defining $a / b$ for $a$ and $b$ that obey (3.2). Surely, if $b$ is invertible, then we could simply put $a / b:=a b^{-1}$; and also if $b$ is just pseudoinvertible in the sense that $b^{\sim 1} b=\lceil b)$ and $b b^{\sim 1}=(b\rceil$ for some $b^{\sim 1}$ the formula $a / b:=a b^{\sim 1}$ would work. But, of course, $b$ need not be pseudoinvertible. The ideal of $b^{\sim 1}$ can however be approximated in an appropriate sense by a formal series $\sum_{n} t_{n}$ (which we call an approximate pseudoinverse) so that we can take $a / b:=\sum_{n} a t_{n}$ (using ultrastrong completeness to see that the series converges.)

### 3.4.1 (Approximate) Pseudoinverses

Definition Let $a$ be an element of a von Neumann algebra $\mathscr{A}$. We'll say that $a$ is pseudoinvertible if it has a pseudoinverse, that is, an element $t$ of $\mathscr{A}$ with $t a=\lceil a)=(t\rceil$ and $a t=\lceil t)=(a\rceil$. When such $t$ exists, it is unique (by 60 VIII$)$, and we'll denote it by $a^{\sim 1}$. If $a^{\sim 1}=a^{*}$, we say that $a$ is a partial isometry (see IV).
Lemma For elements $a, t$ of a von Neumann algebra the following are equivalent.

1. $t a$ is a projection, and $\lceil t)=(a\rceil$.
2. ata $=a$, and $\lceil t) \leqslant(a\rceil$ and $(t\rceil \leqslant\lceil a)$.
3. at is a projection, and $\lceil a)=(t\rceil$.
4. tat $=t$, and $\lceil a) \leqslant(t\rceil$ and $(a\rceil \leqslant\lceil t)$.
5. $t$ is a pseudoinverse of $a$.
6. $a$ is a pseudoinverse of $t$.

Proof $\sqrt{5} \Longleftrightarrow 6$ is clear. For the remainder we make two loops. $1 \Rightarrow 2$ We have $\lceil t) \leqslant(a\rceil$ by assumption, and $(t\rceil=(t\lceil t)\rceil=(t(a\rceil\rceil=(t a\rceil=t a=\mid t a) \leqslant$ $\lceil a)$. Further, ata $=a$ by 60 VIII , because tata $=t a$ (since $t a$ is a projection) and $(a t a\rceil \leqslant(a\rceil \leqslant\lceil t)$. (3 $\Rightarrow 4)$ follows along the same lines. $2 \Longrightarrow 5)$ We have $t a=\lceil a)$ by 60 VIII , because $a t a=a=a\lceil a)$, and $(t a\rceil \leqslant(t\rceil \leqslant\lceil a)$. Also, $a t=(a\rceil$, (because ata $=a=(a\rceil a$, and $\lceil a t) \leqslant\lceil t) \leqslant(a\rceil)$. Further, $\lceil t)=(a\rceil$, because $(a\rceil=a t=\lceil a t) \leqslant\lceil t) \leqslant(a\rceil$; and, similarly, $\lceil a)=(t\rceil$. $4 \square 5)$ is proven by the same principles, and 513 ) is rather obvious.
Exercise Show that an element $u$ of a von Neumann algebra is a partial isometry iff $u^{*} u$ is a projection iff $u u^{*} u=u$ iff $u u^{*}$ is a projection iff $u^{*} u u^{*}=u^{*}$ iff $u^{*}$ is the pseudoinverse of $u$. (Hint: use II, or give a direct proof.)
Exercise Let $a$ and $b$ be a elements of a von Neumann algebra $\mathscr{A}$.

1. Show that $a$ is pseudoinvertible iff $a^{*}$ is pseudoinvertible, and, in that case, $\left(a^{*}\right)^{\sim 1}=\left(a^{\sim 1}\right)^{*}$.
2. Assuming that $a$ and $b$ are pseudoinvertible, and $(b\rceil=\lceil a)$, show that $a b$ is pseudoinvertible, and $(a b)^{\sim 1}=b^{\sim 1} a^{\sim 1}$.
3. Show that $a$ is pseudoinvertible iff $a^{*} a$ is pseudoinvertible, and, in that case, $a^{\sim 1}=\left(a^{*} a\right)^{\sim 1} a^{*}$ and $\left(a^{*} a\right)^{\sim 1}=a^{\sim 1}\left(a^{\sim 1}\right)^{*}$.

VI Exercise Let $a$ be a positive element of a von Neumann algebra $\mathscr{A}$.

1. Show that $a$ is pseudoinvertible iff $a$ is invertible in $\lceil a\rceil \mathscr{A}\lceil a\rceil$ iff at $=\lceil a\rceil$ for some $t \in \mathscr{A}_{+}$. Show, moreover, that at $=t a$ for such $t$.
2. Show that $a$ is pseudoinvertible iff there is $\lambda>0$ with $\lambda\lceil a\rceil \leqslant a$.
3. Assume that $a$ is pseudoinvertible.

Show that $\left\lceil a^{\sim 1}\right\rceil=\lceil a\rceil$.
Show that if $b \in \mathscr{A}$ commutes with $a$, then $b$ commutes with $a^{\sim 1}$.
(In other words, $a^{\sim 1} \in\{a\}^{\square \square}$.)
4. Show that $c^{\sim 1} \leqslant b^{\sim 1}$ when $b \leqslant c$ are pseudoinvertible positive commuting elements of $\mathscr{A}$. (The statement is still true without the requirement that $b$ and $c$ commute, but also much harder to prove.)
5. Show that $\left(0,0,1, \frac{1}{2}, \frac{1}{3}, \ldots\right)$ is not pseudoinvertible in $\ell_{\infty}$.

80 Remark Note that the obvious candidate for the pseudoinverse of ( $0,0,1, \frac{1}{2}, \frac{1}{3}, \ldots$ ) from $\ell^{\infty}$ being $(0,0,1,2,3, \ldots)$ is not bounded, and therefore not an element of $\ell^{\infty}$. We can nevertheless approximate $(0,0,1,2,3, \ldots)$ by the elements

$$
(0,0,1,0,0, \ldots),(0,0,1,2,0, \ldots), \ldots
$$

of $\ell^{\infty}$ forming what we will call "approximate pseudoinverse" for $\left(0,0,1, \frac{1}{2}, \frac{1}{3}, \ldots\right)$. That this can also be done for an arbitrary element of a von Neumann algebra is what we'll see next.

II Definition An approximate pseudoinverse of an element $a$ of a von Neumann algebra $\mathscr{A}$ is a sequence $t_{1}, t_{2}, \ldots$ of elements of $\mathscr{A}$ such that $t_{1} a, t_{2} a, \ldots, a t_{1}, a t_{2}, \ldots$ are projections with $\sum_{n} t_{n} a=\lceil a)=\sum_{n}\left(t_{n}\right\rceil$ and $\sum_{n} a t_{n}=(a\rceil=\sum_{n}\left\lceil t_{n}\right)$.
III Exercise Let $b$ be an element of a von Neumann algebra $\mathscr{A}$, and let $t_{1}, t_{2}, \ldots$ be an approximate pseudoinverse of $b^{*} b$. Show that $t_{1} b^{*}, t_{2} b^{*}, \ldots$ is an approximate pseudoinverse of $b$.

Theorem Every element $a$ of a von Neumann algebra $\mathscr{A}$ has an approximate pseudoinverse.
Proof By III, it suffices to consider the case that $a$ is positive. When $a=0$ the sequence $0,0,0, \ldots$ clearly yields an approximate pseudoinverse for $a$, so let us disregard this case, and assume that $a$ is positive and non-zero.

Note that $a-1 \leqslant a-\frac{1}{2} \leqslant a-\frac{1}{3} \leqslant \cdots$ converges in the norm to $a \equiv a_{+}$, and so does $(a-1)_{+} \leqslant\left(a-\frac{1}{2}\right)_{+} \leqslant \ldots$, which converges also ultraweakly to $\bigvee_{n}\left(a-\frac{1}{n}\right)$, so that $a=\bigvee_{n}\left(a-\frac{1}{n}\right)_{+}$, and thus $\lceil a\rceil=\bigcup_{n}\left\lceil\left(a-\frac{1}{n}\right)_{+}\right\rceil$by 56 XVII

Writing $q_{n}=\left\lceil\left(a-\frac{1}{n}\right)_{+}\right\rceil$- and picturing it as the places where $a \geqslant \frac{1}{n}$ we have $\left(a-\frac{1}{n}\right) q_{n}=\left(a-\frac{1}{n}\right)_{+} \geqslant 0$ (because $b\left\lceil b_{+}\right\rceil=b_{+}$for a positive element $b$ of a von Neumann algebra, by 59 IV, and so $\frac{1}{n} q_{n} \leqslant a q_{n}$ for all $n>0$.

Writing $e_{n}=q_{n+1}-q_{n}$ for all $n$ (taking $q_{0}:=0$ ) - and thinking of it as the places where $\frac{1}{n+1} \leqslant a<\frac{1}{n}$ - we get a sequence of (pairwise orthogonal) projections $e_{1}, e_{2}, \ldots$ in $\{a\}^{\square \square}$ with $\sum_{n} e_{n}=\lceil a\rceil$. By an easy computation involving the facts that $\frac{1}{n+1} \leqslant \frac{1}{n}$ and $a q_{n} \leqslant a q_{n+1}$, we get $\frac{1}{n+1} e_{n} \leqslant a e_{n} \leqslant \frac{1}{n} e_{n}$.

We claim that $\left\lceil a e_{n}\right\rceil=\left\lceil e_{n}\right\rceil$ for any $n$. Indeed, on the one hand $a e_{n}=$ $e_{n} a e_{n} \leqslant\|a\| e_{n}$ (as $e_{n} \in\{a\}^{\square \square}$ ) and so $\left\lceil a e_{n}\right\rceil \leqslant\left\lceil\|a\| e_{n}\right\rceil=e_{n}$ (using here that $\|a\| \neq 0$ ), while on the other hand, $\frac{1}{n+1} e_{n} \leqslant a e_{n}$ gives $e_{n} \equiv\left\lceil\frac{1}{n+1} e_{n}\right\rceil \leqslant\left\lceil a e_{n}\right\rceil$. In particular, $\frac{1}{n+1}\left\lceil a e_{n}\right\rceil=\frac{1}{n+1} e_{n} \leqslant a e_{n}$, so that $a e_{n}$ is pseudoinvertible (by 79 Vl ).

Writing $t_{n}:=\left(a e_{n}\right)^{\sim 1}$, we have $\left\lceil t_{n}\right\rceil=e_{n}$ (since $\left\lceil a e_{n}\right\rceil=e_{n}$ ). Then $t_{n} a=t_{n}\left\lceil t_{n}\right\rceil a=t_{n} e_{n} a=\left\lceil a e_{n}\right\rceil=e_{n}$, and similarly, $a t_{n}=e_{n}$, so that $\sum_{n} a t_{n}=\sum_{n} t_{n} a=\sum_{n} e_{n}=\lceil a\rceil=\sum_{n}\left\lceil t_{n}\right\rceil$, making $t_{1}, t_{2}, \ldots$ an approximate pseudoinverse of $a$.

### 3.4.2 Division

Definition Let $b$ be an element of a von Neumann algebra $\mathscr{A}$, and let $a$ be an element of $\mathscr{A} b$ - so $a \equiv c b$ for some $c \in \mathscr{A}$. We denote by $a / b$ the (by 60 VIIII ) unique element $c$ of $\mathscr{A}(b\rceil$ with $a=c b$, and, dually, given an element $a$ of $b \mathscr{A}$ we denote by $b \backslash a$ the unique element $c$ of $\lceil b) \mathscr{A}$ with $a=b c$.

Exercise Let $a$ and $b$ be elements of a von Neumann algebra $\mathscr{A}$.

1. Show that $c / b$ is an element of $(c\rceil \mathscr{A}(b\rceil$ for every element $c$ of $b \mathscr{A}$.
2. Show that $(a b) / b=a(b\rceil$ and $b \backslash(b a)=\lceil b) a$.
3. Let $c$ be an element of $a \mathscr{A} b$. Show that $a \backslash c \in \mathscr{A} b$, and $c / b \in a \mathscr{A}$, and

$$
(a \backslash c) / b=a \backslash(c / b) \quad=: a \backslash c / b
$$

Show that $a \backslash c / b$ is the unique element $d$ of $\lceil a) \mathscr{A}(b\rceil$ with $c=a d b$.
4. Let $c$ be an element of $\mathscr{A} b$ and let $d$ be an element of $a \mathscr{A}$.

Show that $d c \in a \mathscr{A} b$, and $a \backslash(d c) / b=(a \backslash d)(c / b)$.
5. Let $c$ be an element of $\mathscr{A} b$. Show that $c^{*} \in b^{*} \mathscr{A}$ and $b^{*} \backslash c^{*}=(c / b)^{*}$.

III Lemma Given elements $a$ and $b$ of a von Neumann algebra $\mathscr{A}$ with $a^{*} a \leqslant b^{*} b$ we have $a \in \mathscr{A} b$. Moreover, given an approximate pseudoinverse $t_{1}, t_{2}, \ldots$ of $b$, the series $\sum_{n} a t_{n}$ converges ultrastrongly to $a / b$, and uniformly so in $a$.
IV Proof To show that $\sum_{n=0}^{N} a t_{n}$ converges ultrastrongly as $N \rightarrow \infty$ it suffices to show that $\left(\sum_{n=0}^{N} a t_{n}\right)_{N}$ is ultrastrongly Cauchy (because $\mathscr{A}$ is ultrastrongly complete, by 771). To this end, note that

$$
\begin{aligned}
\left(\sum_{n=M}^{N} a t_{n}\right)^{*} \sum_{n=M}^{N} a t_{n} & =\left(\sum_{n=M}^{N} t_{n}^{*}\right) a^{*} a\left(\sum_{n=M}^{N} t_{n}\right) \\
& \leqslant\left(\sum_{n=M}^{N} t_{n}^{*}\right) b^{*} b\left(\sum_{n=M}^{N} t_{n}\right) \\
& =\sum_{n, m=M}^{N} t_{n}^{*} b^{*} b t_{m} \\
& =\sum_{m=M}^{N} b t_{m},
\end{aligned}
$$

where we've used that $b t_{1}, b t_{2}, \ldots$ are pairwise orthogonal projections - but then the series $\sum_{n=0}^{\infty} b t_{m}$ converges ultraweakly by 56 XVIII . This, coupled with the inequality above, gives us that $\sum_{n=0}^{N} a t_{n}$ is ultrastrongly Cauchy, and therefore converges ultrastrongly - and even uniformly so in $a$, because " $a$ " does not appear in the expression " $\sum_{m=M}^{N} b t_{m}$ " that gave the bound.

Define $c:=\sum_{n=0}^{\infty} a t_{n}$. Since $a^{*} a \leqslant b^{*} b$, we have $\lceil a) \leqslant\lceil b$ ), and so $a=$ $a\lceil b)=a \sum_{n} t_{n} b=\sum_{n} a t_{n} b=c b$. So to get $c=a / b$ we only need to prove that $\lceil c) \leqslant(b\rceil$, that is, $c(b\rceil=c$. To this end, recall that $\sum_{n}\left\lceil t_{n}\right)=(b\rceil$, so that $\left\lceil t_{n}\right) \leqslant(b\rceil$, and $t_{n}(b\rceil=t_{n}$, which implies that $a t_{n}(b\rceil=a t_{n}$, and so $c(b\rceil=\sum_{n} a t_{n}(b\rceil=\sum_{n} a t_{n}=c$.

V Exercise Let $a$ and $b$ be elements of a von Neumann algebra $\mathscr{A}$.

1. Let $\lambda \geqslant 0$ be given, and recall that $(\mathscr{A})_{\lambda}=\{c \in \mathscr{A}:\|c\| \leqslant \lambda\}$.

Show that $a$ is in $(\mathscr{A})_{\lambda} b$ iff $a^{*} a \leqslant \lambda^{2} b^{*} b$, and then $\|a / b\| \leqslant \lambda$.
(Compare this with "Douglas' Lemma" from 15.)
2. Show that $a \in \mathscr{A}(b\rceil$ need not entail that $a \in \mathscr{A} b$.

Exercise Let $b$ be an element of a von Neumann algebra $\mathscr{A}$.

1. Let $a$ be a positive element of $\mathscr{A}$, and let $\lambda \geqslant 0$.

Show that $a \in b^{*}(\mathscr{A})_{\lambda} b$ iff $a \leqslant \lambda b^{*} b$, and then $\left\|b^{*} \backslash a / b\right\| \leqslant \lambda$.
2. Show that $b^{*} \backslash a / b$ is positive for every positive element $a$ of $b^{*} \mathscr{A} b$.
(Hint: prove that $\left(b^{*} \backslash \sqrt{a}\right)(\sqrt{a} / b)=b^{*} \backslash a / b$.)

Exercise Given elements $b$ and $c$ of a von Neumann algebra $\mathscr{A}$, an approximate pseudoinverse $t_{1}, t_{2}, \ldots$ of $b$, and an approximate pseudoinverse of $s_{1}, s_{2}, \ldots$ of $c$, show that $\left(\sum_{n=1}^{N} s_{n}\right) a\left(\sum_{m=1}^{N} t_{m}\right)$, converges ultrastrongly to $c \backslash a / b$ as $N \rightarrow \infty$ (and uniformly so) for $a \in c(\mathscr{A})_{1} b$.
Exercise Show that for positive elements $a$ and $b$ of a von Neumann algebra $\mathscr{A}$, the following are equivalent.

1. $a \leqslant \lambda b$ for some $\lambda \geqslant 0$;
2. $a=\sqrt{b} c \sqrt{b}$ for some positive $c \in \mathscr{A}$.

In that case, there is a unique $c \in \mathscr{A}_{+}$with $a=\sqrt{b} c \sqrt{b}$ and $\lceil c\rceil \leqslant\lceil b\rceil$. Moreover, if $t_{1}, t_{2}, \ldots$ is an approximate pseudoinverse of $\sqrt{b}$, then $\sum_{m, n} t_{m} a t_{n}$ converges ultraweakly to such $c$.
Lemma Given elements $b$ and $c$ of a von Neumann algebra $\mathscr{A}$ the maps

$$
a \mapsto a / b: \quad(\mathscr{A})_{1} b \rightarrow \mathscr{A} \quad \text { and } \quad a \mapsto c \backslash a / b: \quad c(\mathscr{A})_{1} b \rightarrow \mathscr{A}
$$

are ultrastrongly continuous (where $(\mathscr{A})_{1}$ is the unit ball).
Proof By III the series $\sum_{n} a t_{n}$ converges ultraweakly to $a / b$, where $t_{1}, t_{2}, \ldots$ is an approximate pseudoinverse of $b$, and in fact uniformly so for $a \in(\mathscr{A})_{1} b$ (because $a^{*} a \leqslant b^{*} b$ for such $a$ ). Since $a \mapsto \sum_{n=1}^{N} a t_{n},(\mathscr{A})_{1} b \rightarrow \mathscr{A}$ is ultrastrongly continuous (by 45 IV ) - and the uniform limit of continuous functions is continuous - we see that $a \mapsto a / b,(\mathscr{A})_{1} b \rightarrow \mathscr{A}$ is ultrastrongly continuous. It follows that $(\cdot) / b: c(\mathscr{A})_{1} b \rightarrow c(\mathscr{A})_{1}$ and $c \backslash(\cdot): c(\mathscr{A})_{1} \rightarrow \mathscr{A}$ are ultrastrongly continuous; as must be their composition $c \backslash \cdot / b: c(\mathscr{A})_{1} b \rightarrow \mathscr{A}$.

XI Remark The map $a \mapsto a / b$ might not give an ultrastrongly continuous map on the larger domain $\mathscr{A} b$, because, for example, upon applying $(\cdot) /\left(1, \frac{1}{2}, \frac{1}{3}, \ldots\right)$ to the ultrastrongly Cauchy sequence $(1,0,0, \ldots),(1,1,0, \ldots), \ldots$ in $\ell^{\infty}$ we get the sequence $(1,0,0, \ldots),(1,2,0, \ldots), \ldots$, which is not ultrastrongly Cauchy.

### 3.4.3 Polar Decomposition

82 Proposition (Polar Decomposition) Any element $a$ of a von Neumann algebra $\mathscr{A}$ can be uniquely written as $a=[a] \sqrt{a^{*} a}$, where $[a]$ is an element of $\mathscr{A}\lceil a)$. Moreover,

1. $[a]$ is partial isometry with $[a]^{*}[a]=\left\lceil a^{*} a\right\rceil \equiv\lceil a)$ and $[a\rceil[a]^{*}=\left\lceil a a^{*}\right\rceil \equiv(a\rceil$,
2. and $\left[a^{*}\right]=[a]^{*}$, so that $\sqrt{a a^{*}}[a]=a=[a] \sqrt{a^{*} a}$.

II Proof Since $a^{*} a \leqslant \sqrt{a^{*} a} \sqrt{a^{*} a}$, the existence and uniqueness of an element [a] of $\mathscr{A}$ with $a=[a] \sqrt{a^{*} a}$ and $\lceil[a]) \leqslant\lceil a) \equiv\left(\sqrt{a^{*} a}\right\rceil$ is provided by 81 V , and we get $([a]\rceil \leqslant(a\rceil$ to boot! Note that $[a]^{*}[a]=\left\lceil a^{*} a\right\rceil$, by 60 VIII , because

$$
\sqrt{a^{*} a}[a]^{*}[a] \sqrt{a^{*} a}=a^{*} a=\sqrt{a^{*} a}\left\lceil a^{*} a\right\rceil \sqrt{a^{*} a},
$$

and $\left\lceil[a]^{*}[a]\right\rceil \leqslant\lceil a)=\left\lceil\sqrt{a^{*} a}\right\rceil$. In particular, $[a]$ is a partial isometry (by 79 IV ).
Let us prove that $[a][a]^{*}=(a\rceil$. Note that $[a][a]^{*}$ is a projection (because $[a]$ is a partial isometry, by 79 IV$)$. We already know that $[a][a]^{*}=([a]\rceil \leqslant(a\rceil$. Concerning the other direction, $a a^{*}=[a] \sqrt{a^{*} a} \sqrt{a^{*} a}[a]^{*}=[a] a^{*} a[a]^{*}$, so that $(a\rceil=\left\lceil a a^{*}\right\rceil=\left\lceil[a] a^{*} a[a]^{*}\right\rceil \leqslant\left\lceil\|a\|^{2}[a][a]^{*}\right\rceil=\left\lceil[a][a]^{*}\right\rceil \leqslant[a][a]^{*}$.

To prove that $a=\sqrt{a a^{*}}[a]$, we'll first show that $\sqrt{a a^{*}}=[a] \sqrt{a^{*} a}[a]^{*}$. Indeed, since $[a]^{*}[a]=\left\lceil\sqrt{a^{*} a}\right\rceil$, we have $[a] \sqrt{a^{*} a}[a]^{*}[a] \sqrt{a^{*} a}[a]^{*}=[a] \sqrt{a^{*} a} \sqrt{a^{*} a}[a]^{*}=$ $a a^{*}$ - now take the square root. It follows that $\sqrt{a a^{*}}[a]=[a] \sqrt{a^{*} a}[a]^{*}[a]=$ $[a] \sqrt{a^{*} a}=a$. Finally, upon applying $(\cdot)^{*}$, we see that $a^{*}=[a]^{*} \sqrt{a a^{*}}$, and thus $\left[a^{*}\right]=[a]^{*}$, by uniqueness of $\left[a^{*}\right]$, because $\left\lceil[a]^{*}\right)=([a]\rceil=(a\rceil=\left\lceil a^{*}\right)$.

83 Recall from 681 that the least central projection $\llbracket e \rrbracket$ above a projection $e$ of a von Neumann algebra $\mathscr{A}$ is given by $\llbracket e \rrbracket=\bigcup_{a \in \mathscr{A}}\left\lceil a^{*} e a\right\rceil$. Using the polar decomposition we can give a more economical description of $\llbracket e \rrbracket$, see $\nabla$.
II Proposition Given projections $e^{\prime}$ and $e$ of a von Neumann algebra $\mathscr{A}$, the following are equivalent.

1. $e^{\prime}=\left\lceil a^{*} e a\right\rceil$ for some $a \in \mathscr{A}$;
2. $e^{\prime}=\lceil a)$ and $(a\rceil \leqslant e$ for some $a \in \mathscr{A}$;
3. $e^{\prime}=u^{*} u$ and $u u^{*} \leqslant e$ for some partial isometry $u$.

In that case we write $e^{\prime} \lesssim e$ (and say $e^{\prime}$ is Murray-von Neumann below $e$ ).
Proof That 3 implies 2 is clear. $2 \geqslant 1\}$ Since $(a\rceil \leqslant e$, we have $e a=a$, and so $\left\lceil a^{*} e a\right\rceil=\left\lceil a^{*} a\right\rceil=\lceil a)=e^{\prime}$. $1=3$ ) By the polar decomposition (see 82I) we get a partial isometry $u:=[e a]$ for which $u^{*} u=[e a]^{*}[e a]=\left\lceil(e a)^{*} e a\right\rceil=e^{\prime}$ and $u u^{*}=\left\lceil e a a^{*} e\right\rceil \leqslant e$.

Exercise Show that $\lesssim$ preorders the projections of a von Neumann algebra.
Lemma Given a projection $e$ of a von Neumann algebra $\mathscr{A}$ there is a family $\left(e_{i}\right)_{i}$ of non-zero projections with $\llbracket e \rrbracket=\sum_{i} e_{i}$, and $e_{i} \lesssim e$ for all $i$.
Proof Let $\left(e_{i}\right)_{i}$ be a maximal set of non-zero pairwise orthogonal projections in $\mathscr{A}$ with $e_{i} \lesssim e$ for all $i$. Our goal is to show that $\sum_{i} e_{i} \equiv \bigcup_{i} e_{i}=\llbracket e \rrbracket$.

Let $u_{i}$ be a partial isometry with $u_{i}^{*} u_{i}=e_{i}$ and $u_{i} u_{i}^{*} \leqslant e$. Since $e_{i}=u_{i}^{*} u_{i}=$ $u_{i}^{*} u_{i} u_{i}^{*} u_{i} \leqslant u_{i}^{*} e u_{i} \leqslant \bigcup_{a \in \mathscr{A}}\left\lceil a^{*} e a\right\rceil=\llbracket e \rrbracket$, we have $\bigcup_{i} e_{i} \leqslant \llbracket e \rrbracket$.

Suppose that $\bigcup_{i} e_{i}<\llbracket e \rrbracket$ (towards a contradiction). Then since $p:=$ $\llbracket e \rrbracket-\bigcup_{i} e_{i}$ is a non-zero projection, and $p=p \llbracket e \rrbracket p=\bigcup_{a \in \mathscr{A}}\left\lceil p\left\lceil a^{*} e a\right\rceil p\right\rceil=$ $\bigcup_{a \in \mathscr{A}}\left\lceil(e a p)^{*} e a p\right\rceil$, there must be $a \in \mathscr{A}$ with (eap) ${ }^{*} e a p \neq 0$. The polar decomposition (see 821) of eap gives us a partial isometry $u:=[e a p]$ with $u u^{*}=\left\lceil e a p(e a p)^{*}\right\rceil=\lceil$ eapa* $e\rceil \leqslant e$ and $u^{*} u=\left\lceil(e a p)^{*} e a p\right\rceil \leqslant p$, so that $u^{*} u$ is a non-zero projection, orthogonal to all $e_{i}$ with $u^{*} u \lesssim e$. In other words, $e$ could have been added to $\left(e_{i}\right)_{i}$, contradicting its maximality. Hence $\bigcup_{i} e_{i}=\llbracket e \rrbracket$.

Using 8311 we can classify all finite-dimensional $C^{*}$-algebras.
Theorem Any finite-dimensional $C^{*}$-algebra $\mathscr{A}$ is a direct sum of full matrix algebras, that is, $\mathscr{A} \cong \bigoplus_{m} M_{N_{m}}$ for some $N_{1}, \ldots, N_{M} \in \mathbb{N}$.
Proof Let $e_{1}, \ldots, e_{N}$ be a basis for $\mathscr{A}$. We'll first show that $\mathscr{A}$ is a von Neumann algebra, and for this we'll need the fact that the unit ball $(\mathscr{A})_{1}$ is compact with respect to the norm on $\mathscr{A}$. For this it suffices to show that $\|\cdot\|$ is equivalent to the norm $\|\cdot\|^{\prime}$ on $\mathscr{A}$ given by $\|a\|^{\prime}=\sum_{n}\left|z_{n}\right|$ for all $a \equiv \sum_{n} z_{n} e_{n}$ where $z_{1}, \ldots, z_{N} \in \mathbb{C}$, (because the unit $\|\cdot\|^{\prime}$-ball is clearly compact being homeomorphic to the unit ball of $\mathbb{C}^{N}$.) Since for such $a \equiv \sum_{n} z_{n} e_{n}$ we have

$$
\|a\| \leqslant \sum_{n}\left|z_{n}\right|\left\|e_{n}\right\| \leqslant \sum_{n}\left|z_{n}\right| \sup _{n}\left\|e_{n}\right\|=\|a\|^{\prime} \sup _{n}\left\|e_{n}\right\|
$$

we see that $a \mapsto a: \mathscr{A} \rightarrow \mathscr{A}$ is continuous from $\|\cdot\|^{\prime}$ to $\|\cdot\|$. For the converse it suffices to show that $f_{m}: a \equiv \sum_{n} z_{n} \mapsto z_{m}, \mathscr{A} \rightarrow \mathbb{C}$ is bounded with respect to $\|\cdot\|$, because then

$$
\|a\|^{\prime} \equiv\left\|\sum_{n} f_{n}(a) e_{n}\right\|^{\prime} \leqslant \sum_{n}\left|f_{n}(a)\right| \leqslant\left(\sum_{n}\left\|f_{n}\right\|\right)\|a\| .
$$

In fact, we'll show that any linear functional on $\mathscr{A}$ is bounded. Since the bounded linear functionals form a linear subspace $\mathscr{A}^{*}$ of $N$-dimensional vector space of all linear functionals on $\mathscr{A}$ it suffices to show that $\mathscr{A}^{*}$ has dimension $N$. So let $f_{1}, \ldots, f_{M}$ be a basis for $\mathscr{A}^{*}$; we must show that $N \leqslant M$. Since the states of $\mathscr{A}$ (see 22 VIIII ) and thus all linear functionals on $\mathscr{A}$ form a separating collection, the functionals $f_{1}, \ldots, f_{N}$ form a separating set too; since therefore

$$
a \mapsto\left(f_{1}(a), \ldots, f_{M}(a)\right): \mathscr{A} \rightarrow \mathbb{C}^{M}
$$

is a linear injection from the $N$-dimensional space $\mathscr{A}$ to the $M$-dimensional space $\mathbb{C}^{M}$ we get $N \leqslant M$. Whence all linear functionals on $\mathscr{A}$ are bounded, the norms $\|\cdot\|$ and $\|\cdot\|^{\prime}$ are equivalent, and $(\mathscr{A})_{1}$ is norm compact.
IV ( $\mathscr{A}$ is a von Neumann algebra) First we need to show that every bounded directed set $D$ of self-adjoint elements of $\mathscr{A}$ has a supremum (in $\mathscr{A}_{\mathbb{R}}$ ). We may assume without loss of generality that $\|d\| \leqslant 1$ for all $d \in D$, and so $D \subseteq$ $(\mathscr{A})_{1}$. Since $(\mathscr{A})_{1}$ is norm compact there is a cofinal subset $D^{\prime}$ of $D$ that norm converges to some $a \in \mathscr{A}$, and thus $D$ norm converges to $a$ itself. It's easily seen that $a$ is the supremum of $D$. Indeed, given $d_{0} \in D$ we have $d_{0} \leqslant d$ for all $d \geqslant d_{0}$, and so $d_{0} \leqslant \lim _{d \geqslant d_{0}} d=a$. Hence $a$ is an upper bound for $D$; and if $b$ is an upper bound for $D$, then $d \leqslant b$ for all $d \in D$, and so $a=\lim _{d} d \leqslant b$.

Since in this finite-dimensional setting $\bigvee D$ is apparently the norm limit of $(d)_{d \in D}$, any positive functional $f$ on $\mathscr{A}$ will map $\bigvee D$ to the limit of $(f(d))_{d \in D}$, which is $\bigvee_{d \in D} f(d)$, and so $f(\bigvee D)=\bigvee_{d \in D} f(d)$. Whence every positive functional on $\mathscr{A}$ is normal; and since the positive functionals on $\mathscr{A}$ form a separating collection, $\mathscr{A}$ is a von Neumann algebra.
$\checkmark$ (Reduction to a factor) Since pairwise orthogonal non-zero projections are easily seen to be linearly independent, and $\mathscr{A}$ is finite dimensional, every orthogonal set of projections in $\mathscr{A}$ is finite. In particular, any descending sequence of nonzero projections must eventually become constant. It follows that below every (central) projection $p$ in $\mathscr{A}$ there is a minimal (central) projection, and even that $p$ is the finite sum of minimal (central) projections. In particular, the unit 1 of $\mathscr{A}$ can be written as $1=\sum_{n} z_{n}$ where $z_{1}, \ldots, z_{M}$ are minimal central projections of $\mathscr{A}$. By 67 IV we know that $z_{m} \mathscr{A}$ is a von Neumann algebra for
each $m$, and that $\mathscr{A}$ is nmiu-isomorphic to the direct sum $\bigoplus_{m} z_{m} \mathscr{A}$ of these von Neumann algebras via $a \mapsto\left(z_{m} a\right)_{m}$. Since $z_{m}$ is a minimal central projection, the von Neumann algebra $z_{n} \mathscr{A}$ has no non-trivial central projections.
(When $\mathscr{A}$ is a factor) Let $e$ be a minimal projection of $\mathscr{A}$ (which exists by the previous discussion). Since $e \neq 0$, and $\mathscr{A}$ has no non-trivial central projections, we have $\llbracket e \rrbracket=1$. By 83 V we have $1 \equiv \llbracket e \rrbracket=\sum_{k} e_{k}$ for some nonzero projections $e_{1}, \ldots, e_{K}$ in $\mathscr{A}$ with $e_{k} \lesssim e$. So there are partial isometries $u_{1}, \ldots, u_{K} \in \mathscr{A}$ with $u_{k}^{*} u_{k}=e_{k}$ and $u_{k} u_{k}^{*} \leqslant e$ for all $k$. In fact, since $e$ is minimal, we have $u_{k} u_{k}^{*}=e$. Thinking of $u_{k}$ as $|0\rangle\langle k|$ define $u_{k \ell}=u_{k}^{*} u_{\ell}$; we'll show that $\varrho: A \mapsto \sum_{k \ell} A_{k \ell} u_{k \ell}: M_{K} \rightarrow \mathscr{A}$ is a miu-isomorphism. It's easy to see that $\varrho$ is linear, involution preserving and unital. To see that $\varrho$ is multiplicative, first note that $u_{j} u_{k}^{*}$ equals $e$ when $j=k$ and is zero otherwise. It follows that $u_{i j} u_{k \ell}$ equals $u_{i \ell}$ when $k=j$ and is zero otherwise. Whence

$$
\varrho(A) \varrho(B)=\sum_{i j k \ell} A_{i j} u_{i j} B_{k \ell} u_{k \ell}=\sum_{i \ell}\left(\sum_{k} A_{i k} B_{k \ell}\right) u_{i \ell}=\varrho(A B)
$$

for all matrices $A, B \in M_{K}$, and so $\varrho$ is multiplicative.
It remains to be shown that $\varrho$ is a bijection. To see that $\varrho$ is injective, first note that $\varrho$ is normal, because using the fact that $\varrho$ is positive and thus bounded, we can show that $\varrho$ preserves suprema of bounded directed sets in much the same way we showed that all np-functionals on $\mathscr{A}$ are bounded. We can thus speak of the central carrier $\llbracket \varrho \rrbracket$ of $\varrho$, and thus to show that $\varrho$ is injective it suffices to show that $\llbracket \varrho \rrbracket=1$. Since $M_{K}$ is a factor (see 67 II ) the only alternative is $\llbracket \varrho \rrbracket=0$ i.e. $\varrho=0$, which is clearly absurd unless $\mathscr{A}=\{0\}$ in which case we'd already be done. Hence $\varrho$ is injective.

To see that $\varrho$ is surjective let $a \in \mathscr{A}$ with $a \neq 0$ be given. Since $a \equiv$ $\sum_{k, \ell} e_{k} a e_{\ell}=\sum_{k, \ell} u_{k 1} u_{1 k} a u_{\ell 1} u_{1 \ell}$, and $u_{k 1}$ and $u_{1 \ell}$ are in the range of $\varrho$ it suffices to show that $u_{1 k} a u_{\ell 1}$ is in the range of $\varrho$ for all $k$ and $\ell$. In other words, we may assume without loss of generality that eae $=a$, where $e$ is the minimal projection in $\mathscr{A}$ we started with. Since $e\left(a_{\mathbb{R}}\right)_{+} e=\left(a_{\mathbb{R}}\right)_{+}$, and so on, we may assume that $a$ is positive. By scaling, we may also assume that $\|a\| \leqslant 1 / 3$. Since $\lceil\|a\| e-a\rceil \leqslant e$, and $e$ is minimal, we either have $\lceil\|a\| e-a\rceil=e$ or $\lceil\|a\| e-a\rceil=0$.

The former case is impossible: indeed, if $e=\lceil\|a\| e-a\rceil \equiv \bigvee_{n}(\|a\| e-a)^{1 / 2^{n}}$ (see 56|), then $(\|a\| e-a)^{1 / 2^{n}}$ norm converges to $\lceil\|a\| e-a\rceil=e$ (cf. |V|), and so $\|\|a\| e-a\|^{1 / 2^{n}}$ converges to $\|e\|=1$. Then $\|\|a\| e-a\|=1$, while $\|\|a\| e-a\| \leqslant$ $\|a\|\|e\|+\|a\| \leqslant \frac{2}{3}$, which is absurd.

Hence $\lceil\|a\| e-a\rceil=0$, and so $a=\|a\| e$. In particular, $a$ is in the range of $\varrho$. Whence $\varrho$ is surjective, and thus a miu-isomorphism $M_{N} \rightarrow \mathscr{A}$.

### 3.5 Normal Functionals

85 For our study of the category of von Neumann algebras we need two more technical results concerning the normal functionals on a von Neumann algebra.

The first one, that a net $\left(b_{\alpha}\right)_{\alpha}$ in a von Neumann algebra $\mathscr{A}$ is (norm) bounded provided that $\left(\omega\left(b_{\alpha}\right)\right)_{\alpha}$ is bounded for each np-functionals $\omega: \mathscr{A} \rightarrow \mathbb{C}$ (see 87 VIII ), ultimately follows from a type of polar decomposition for ultraweakly linear functionals (see 86 IX ).

The second one, that the ultraweak topology of a von Neumann subalgebra coincides with the ultraweak topology of the surrounding space (see 89 8 II ), is proven using the double commutant theorem ( 88 VI ) and requires a lot of hard work.

### 3.5.1 Ultraweak Boundedness

86 To get a better handle on the normal positive functionals on a von Neumann algebra, we first analyse the not-necessarily-positive normal functionals in greater detail.

II Lemma A linear map $f: \mathscr{A} \rightarrow \mathbb{C}$ on a $C^{*}$-algebra $\mathscr{A}$ is positive iff $\|f\| \leqslant f(1)$.
III Proof (Based on Theorem 4.3.2 of 43 .)
If $f(1)=0$, then $f=0$ in both cases (viz. $f$ is positive, and $\|f\| \leqslant f(1)$ ), so we may assume that $f(1) \neq 0$. The problem is easily reduced farther to the case that $f(1)=1$ by replacing $f$ by $f(1)^{-1} f$ (noting that $f(1) \geqslant 0$ in both cases), so we'll assume that $f(1)=1$.

IV ( $f$ positive $\Longrightarrow\|f\| \leqslant 1$ ) This follows immediately from 34 XVI and 34 IX , but here's a concrete proof: Let $a \in \mathscr{A}$ be given. Pick $\lambda \in \mathbb{C}$ with $|\lambda|=1$ and $\lambda f(a) \geqslant 0$. Then $|f(a)|=f(\lambda a)=f(\lambda a)_{\mathbb{R}}=f\left((\lambda a)_{\mathbb{R}}\right) \leqslant f(\|a\|)=\|a\|$, because $(\lambda a)_{\mathbb{R}} \leqslant\left\|(\lambda a)_{\mathbb{R}}\right\| \leqslant\|\lambda a\|=\|a\|$, and $f$ is positive. Hence $\|f\| \leqslant 1$.
$\vee \quad\left(\|f\| \leqslant 1 \Longrightarrow f\right.$ is positive) Let $a \in[0,1]_{\mathscr{A}}$ be given. To prove that $f$ is positive, it suffices to show that $f(a) \geqslant 0$. Since $\left(f(a)_{\mathbb{R}}\right)^{\perp}=\left(f(a)^{\perp}\right)_{\mathbb{R}} \leqslant\left|f(a)^{\perp}\right|=$ $\left|f\left(a^{\perp}\right)\right| \leqslant 1$, and therefore $f(a)_{\mathbb{R}} \geqslant 0$, we just need to show that $f(a)_{\mathbb{I}}=0$.

The trick is to consider $b_{n}:=\left(a-f(a)_{\mathbb{R}}\right)+n i f(a)_{\mathbb{I}}$. Indeed, since $(n+$ $1)^{2}\left(f(a)_{\mathbb{I}}\right)^{2}=\left|f\left(b_{n}\right)\right|^{2} \leqslant\left\|b_{n}\right\|^{2}=\left\|b_{n}^{*} b_{n}\right\| \leqslant\left\|a-f(a)_{\mathbb{R}}\right\|^{2}+n^{2}\left(f(a)_{\mathbb{I}}\right)^{2}$, one sees that $(2 n+1)\left(f(a)_{\mathbb{I}}\right)^{2} \leqslant\left\|a-f(a)_{\mathbb{R}}\right\|^{2}$ for all $n$, which is impossible unless $\left(f(a)_{\mathbb{I}}\right)^{2}=0$, that is, $f(a)_{\mathbb{I}}=0$.

Lemma An extreme point $u$ of the unit ball $(\mathscr{A})_{1}$ of a $C^{*}$-algebra $\mathscr{A}$ is a partial isometry with $\left(u u^{*}\right)^{\perp} \mathscr{A}\left(u^{*} u\right)^{\perp}=\{0\}$.
Remark The converse (viz. every such partial isometry is extreme in $\left.(\mathscr{A})_{1}\right)$ also holds, but we won't need it.

Proof (Based on Theorem 7.3.1 of 43].)
To show $u$ is a partial isometry it suffices to prove that $u^{*} u$ is a projection. Suppose towards a contradiction that $u^{*} u$ is not a projection. Then $u^{*} u$, represented as continuous function (on $\operatorname{sp}\left(u^{*} u\right)$ cf. 28 III ), takes neither the value 0 nor 1 on a neighbourhood of some point, and so by considering a positive continuous function, which is sufficiently small but non-zero on this neighbourhood and zero elsewhere, we can find a non-zero element $a$ of the (commutative) $C^{*}$-subalgebra generated by $u^{*} u$ with $0 \leqslant a \leqslant u^{*} u$ and $\left\|u^{*} u(1 \pm a)^{2}\right\| \leqslant 1$, so that $\|u(1 \pm a)\| \leqslant 1$. Since $u$ is extreme in $(\mathscr{A})_{1}$, and $u=\frac{1}{2} u(1+a)+\frac{1}{2} u(1-a)$, we get $u a=0$, and so $0 \leqslant a^{2} \leqslant \sqrt{a} u^{*} u \sqrt{a}=u^{*} u a=0$, which contradicts $a \neq 0$.

Let $a \in\left(u u^{*}\right)^{\perp} \mathscr{A}\left(u^{*} u\right)^{\perp}$ be given; we must show that $a=0$. Assume (without loss of generality) that $\|a\| \leqslant 1$. We'll show that $\|u \pm a\| \leqslant 1$, because, since $u$ is extreme in $(\mathscr{A})_{1}, u \equiv \frac{1}{2}(u+a)+\frac{1}{2}(u-a)$ implies that $u=u+a$, and so $a=0$. Note that $a^{*} a \leqslant\left(u^{*} u\right)^{\perp}$ (because $a\left(u^{*} u\right)^{\perp}=a$ ) and $u^{*} a=0$ (because $\left.\left(u u^{*}\right)^{\perp} a=a\right)$. Thus $(u \pm a)^{*}(u \pm a)=u^{*} u \pm u^{*} a \pm a^{*} u+a^{*} a=u^{*} u+a^{*} a \leqslant$ $u^{*} u+\left(u^{*} u\right)^{\perp}=1$, so $\|u \pm a\| \leqslant 1$.

Theorem (Polar decomposition of functionals) Every functional $f: \mathscr{A} \rightarrow \mathbb{C}$ on a von Neumann algebra $\mathscr{A}$ which is ultraweakly continuous on the unit ball $(\mathscr{A})_{1}$ is of the form $f \equiv f\left(u u^{*}(\cdot)\right)=f\left((\cdot) u^{*} u\right)$ for some partial isometry $u$ on $\mathscr{A}$ such that $f(u(\cdot))$ and $f((\cdot) u): \mathscr{A} \rightarrow \mathbb{C}$ are positive.
Proof (Based on Theorem 7.3.2 of 43 .)
We'll first show that $f$ takes the value $\|f\|$ at some extreme point $u$ of $(\mathscr{A})_{1}$. To begin, since $(\mathscr{A})_{1}$ is ultraweakly compact 77 III , and $f$ is ultraweakly continuous the subset $\left\{f(a): a \in(\mathscr{A})_{1}\right\}$ of $\mathbb{R}$ is compact, and therefore has a largest element, which must be $\|f\|$. Thus the convex set $F:=\left\{a \in(\mathscr{A})_{1}: f(a)=\|f\|\right\}$ is non-empty. Since $F$ is ultraweakly compact (being an ultraweakly closed subset of the ultraweakly compact $\left.(\mathscr{A})_{1}\right), F$ has at least one extreme point by the Krein-Milman Theorem (see e.g. Theorem V7.4 of (13]), say $u$. Note that $F$ is a face of $(\mathscr{A})_{1}$ : if $\frac{1}{2} a+\frac{1}{2} b \in F$ for some $a, b \in(\mathscr{A})_{1}$, then $\frac{1}{2} f(a)+\frac{1}{2} f(b)=\|f\|$, so $f(a)=f(b)=\|f\|$ (since $\|f\|$ is extreme in $\left.(\mathbb{C})_{\|f\|}\right)$ and thus $a, b \in F$. It follows that $u$ is not only extreme in $F$, but also in $(\mathscr{A})_{1}$, so that $u$ is an partial isometry with $\left(u u^{*}\right)^{\perp} \mathscr{A}\left(u^{*} u\right)^{\perp}=\{0\}$ by VI .

Note that $f(u(\cdot))$ is positive by $\Pi$, because $\|f(u(\cdot))\|\|\leqslant\| f\|\|u\| \leqslant\| f \|=$ 85, 86..
$f(u)=f(u(1))$. By a similar argument $f((\cdot) u)$ is positive.
Let $a \in \mathscr{A}$ be given. It remains to be shown that $f(a)=f\left(u u^{*} a\right)=f\left(a u^{*} u\right)$. First note that $u\left(u^{*} u\right)^{\perp}=0$ (since $u$ is an isometry) and so $f\left(u\left(u^{*} u\right)^{\perp}\right)=0$, that is, $u^{*} u \geqslant\lceil f(u(\cdot))\rceil$. This entails that $f\left(u b u^{*} u\right)=f(u b)$ for all $b \in \mathscr{A}$ by 63 VI , and in particular $f\left(u u^{*} a u^{*} u\right)=f\left(u u^{*} a\right)$.

Now, since $\left(u u^{*}\right)^{\perp} \mathscr{A}\left(u^{*} u\right)^{\perp}=\{0\}$, we have $u u^{*} a u^{*} u+a=u u^{*} a+a u^{*} u$, and thus $f(a)+f\left(u u^{*} a\right)=f(a)+f\left(u u^{*} a u^{*} u\right)=f\left(u u^{*} a\right)+f\left(a u^{*} u\right)$, which yields $f(a)=f\left(a u^{*} u\right)$. By a similar reasoning we get $f\left(u u^{*} a\right)=f(a)$.
XII Corollary A functional $f: \mathscr{A} \rightarrow \mathbb{C}$ on a von Neumann algebra $\mathscr{A}$ is ultraweakly continuous when it is ultraweakly continuous on the unit ball $(\mathscr{A})_{1}$.
XIII Proof By IX there is a partial isometry $u$ such that $f\left(u u^{*}(\cdot)\right)=f$ and $f(u(\cdot))$ is positive. Recall from 44 XV that such a positive functional $f(u(\cdot))$ is normal when it is ultraweakly continuous on $[0,1]_{\mathscr{A}}$; which it is, because $a \mapsto u a$ is ultraweakly continuous (see 45 IV ), maps $[0,1]_{\mathscr{A}}$ into $(\mathscr{A})_{1}$, and $f$ is ultraweakly continuous on $(\mathscr{A})_{1}$. But then $f \equiv f\left(u u^{*}(\cdot)\right)$ being the composition of the ultraweakly continuous maps $f(u(\cdot))$ and $a \mapsto u^{*} a$ is ultraweakly continuous on $\mathscr{A}$ too.

XIV Lemma Let $f: \mathscr{A} \rightarrow \mathbb{C}$ be a normal functional on a von Neumann algebra $\mathscr{A}$, and let $u$ be a partial isometry in $\mathscr{A}$ such that $f(u(\cdot))$ is positive, and $f=$ $f\left(u u^{*}(\cdot)\right)$. Then $\|f\|=f(u)$.
XV Proof Since $f(u(\cdot))$ is positive, we have $\|f(u(\cdot))\|=f(u)$ by 34 XVI hence $\|f\|=\left\|f\left(u u^{*}(\cdot)\right)\right\| \leqslant\|f(u(\cdot))\|\left\|u^{*}\right\| \equiv f(u) \leqslant\|f\|$, and thus $\|f\|=f(u)$.

87 Definition Given a von Neumann algebra $\mathscr{A}$, the vector space of ultraweakly continuous linear maps $f: \mathscr{A} \rightarrow \mathbb{C}$ endowed with the operator norm is denoted by $\mathscr{A}_{*}$, and called the predual of $\mathscr{A}$.
II Remark The reason that the space $\mathscr{A}_{*}$ is called the predual of $\mathscr{A}$ is the nontrivial fact due to Sakai 62 (which we don't need and therefore won't prove), that the obvious map $\mathscr{A} \rightarrow\left(\mathscr{A}_{*}\right)^{*}$, where $\left(\mathscr{A}_{*}\right)^{*}$ is the dual of $\mathscr{A}_{*}$ - the vector space of bounded linear maps $\mathscr{A}_{*} \rightarrow \mathbb{C}$ endowed with the operator norm - is a surjective isometry, and so $\mathscr{A}$ "is" the dual of $\mathscr{A}_{*}$, (albeit only as normed space, because $\left(\mathscr{A}_{*}\right)^{*}$ doesn't come equipped with a multiplication.)

We will need this:
III Proposition The predual $\mathscr{A}_{*}$ of a von Neumann algebra $\mathscr{A}$ is complete (with respect to the operator norm).

Proof Let $f_{1}, f_{2}, \ldots$ be a sequence in $\mathscr{A}_{*}$ which is Cauchy with respect to the operator norm. We already know (from 4 V ) that $f_{1}, f_{2}, \ldots$ converges to a bounded linear functional $f: \mathscr{A} \rightarrow \mathbb{C}$; so we only need to prove that $f$ is ultraweakly continuous to see that $\mathscr{A}_{*}$ is complete, and for this, we only need to show (by 86XII) that $f$ is ultraweakly continuous on the unit ball $(\mathscr{A})_{1}$ of $\mathscr{A}$. So let $\left(b_{\alpha}\right)_{\alpha}$ be a net in $(\mathscr{A})_{1}$ which converges ultraweakly to 0 ; we must show that $\lim _{\alpha} f\left(b_{\alpha}\right)=0$. Now, note that for every $n$ and $\alpha$ we have the bound

$$
\left|f\left(b_{\alpha}\right)\right| \leqslant\left|\left(f-f_{n}\right)\left(b_{\alpha}\right)\right|+\left|f_{n}\left(b_{\alpha}\right)\right| \leqslant\left\|f-f_{n}\right\|+\left|f_{n}\left(b_{\alpha}\right)\right| .
$$

From this, and $\lim _{n}\left\|f-f_{n}\right\|=0$, and $\lim _{\alpha} f_{n}\left(b_{\alpha}\right)=0$ for all $n$, one easily deduces that $\lim _{\alpha} f\left(b_{\alpha}\right)=0$. Thus $f$ is ultraweakly continuous, and so $\mathscr{A}_{*}$ is complete.

Note that for a self-adjoint element $a$ of a von Neumann algebra $\mathscr{A}$ we have $\|a\|=\sup _{\omega}|\omega(a)|$ where $\omega$ ranges over the npsu-functionals, but that the same identity does not need to hold for arbitrary (not necessarily self-adjoint) $a \in \mathscr{A}$. The following lemma shows that this restriction to self-adjoint elements can be lifted by letting $\omega$ range over all of $\mathscr{A}_{*}$.
Lemma We have $\|a\|=\sup _{f \in\left(\mathscr{A}_{*}\right)_{1}}|f(a)|$ for every element $a$ of a von Neumann algebra $\mathscr{A}$.
Proof It's clear that $\sup _{f \in\left(\mathscr{A}_{*}\right)_{1}}|f(a)| \leqslant\|a\|$.
For the other direction, write $a \equiv[a] \sqrt{a^{*} a}$ (see 821 ) and note that $\|a\|=$ $\left\|\sqrt{a^{*} a}\right\|=\sup _{\omega \in \Omega}\left|\omega\left(\sqrt{a^{*} a}\right)\right|$, where $\Omega$ is the set of npu-maps $\mathscr{A} \rightarrow \mathbb{C}$ (which is order separating). Let $\omega \in \Omega$ be given. Since $[a]^{*} a=\sqrt{a^{*} a}$ we have $\omega\left(\sqrt{a^{*} a}\right)=\omega\left([a]^{*} a\right)=f(a)$, where $f:=\omega\left([a]^{*}(\cdot)\right) \in\left(\mathscr{A}_{*}\right)_{1}$, and so $\|a\|=$ $\sup _{\omega \in \Omega} \omega\left(\sqrt{a^{*} a}\right) \leqslant \sup _{f \in\left(\mathscr{A}_{*}\right)_{1}}|f(a)|$.
Theorem A net $\left(b_{\alpha}\right)_{\alpha}$ in a von Neumann algebra $\mathscr{A}$ is norm bounded (that is, $\left.\sup _{\alpha}\left\|b_{\alpha}\right\|<\infty\right)$ provided it is ultraweakly bounded, i.e., $\sup _{\alpha}\left|\omega\left(b_{\alpha}\right)\right|<\infty$ for every $\mathrm{np}(\mathrm{u})$-map $\omega: \mathscr{A} \rightarrow \mathbb{C}$.
Proof Note that $f \mapsto f\left(b_{\alpha}\right)$ gives a linear map $(\cdot)\left(b_{\alpha}\right): \mathscr{A}_{*} \rightarrow \mathbb{C}$ with $\left\|(\cdot)\left(b_{\alpha}\right)\right\|=$ $\left\|b_{\alpha}\right\|$ by VI for each $\alpha$. So to prove that $\left(b_{\alpha}\right)_{\alpha}$ is norm bounded, viz. $\sup _{\alpha}\left\|b_{\alpha}\right\| \equiv$ $\sup _{\alpha}\left\|(\cdot)\left(b_{\alpha}\right)\right\|<\infty$, it suffices to show (by the principle of uniform boundedness, 35 II , using that $\mathscr{A}_{*}$ is complete, III), that $\sup _{\alpha}\left|f\left(b_{\alpha}\right)\right|<\infty$ for all $f \in \mathscr{A}_{*}$.

Since such $f \in \mathscr{A}_{*}$ can be written as $f \equiv \sum_{k=0}^{3} i^{k} \omega_{k}$ where $\omega_{k}: \mathscr{A} \rightarrow \mathbb{C}$ are np-maps (by 72 V ), we see that $\sup _{\alpha}\left|f\left(b_{\alpha}\right)\right| \leqslant \sum_{k=0}^{3} \sup _{\alpha}\left|\omega_{k}\left(b_{\alpha}\right)\right|<\infty$, because $\left(b_{\alpha}\right)_{\alpha}$ is ultraweakly bounded. Thus $\left(b_{\alpha}\right)_{\alpha}$ is norm bounded.

### 3.5.2 Ultraweak Permanence

88 We turn to a subtle, and surprisingly difficult matter: it is not immediately clear that the ultraweak topology on a von Neumann subalgebra $\mathscr{A}$ of a von Neumann algebra $\mathscr{B}$, coincides (on $\mathscr{A}$ ) with the ultraweak topology on $\mathscr{B}$. While it is easily seen that the former is finer (that is, a net in $\mathscr{A}$ which converges ultraweakly in $\mathscr{A}$, converges ultraweakly in $\mathscr{B}$ too, because any np-map $\omega: \mathscr{B} \rightarrow \mathbb{C}$ is also an np-map restricted to $\mathscr{A}$ ), it is not obvious that an np-map $\omega: \mathscr{A} \rightarrow \mathbb{C}$ can be extended to an np-map on $\mathscr{B}$ - but it can, as we'll see 89XI. We'll call this independence of the ultraweak topology from the surrounding space ultraweak permanence being not unlike the independence of the spectrum of an operator from the surrounding space known as spectral permanence 11 XXIII$)$.

It is tempting to think that the extension of an np-map $\omega: \mathscr{A} \rightarrow \mathbb{C}$ on a von Neumann subalgebra $\mathscr{A}$ of a von Neumann algebra $\mathscr{B}$ to $\mathscr{B}$ is simply a matter of applying Hahn-Banach to $\omega$, but this approach presents two problems: it yields a normal but not necessarily positive extension of $\omega$; and it not clear that $\omega$ is ultraweakly continuous on $\mathscr{A}$ (that is, whether Hahn-Banach applies).

Instead of applying general techniques we feel forced to delve deeper into the particular structure provided to us by von Neumann algebras (namely the commutant, 65 II$)$ to show that any np-map $\omega: \mathscr{A} \rightarrow \mathbb{C}$ on a von Neumann algebra $\mathscr{A}$ of bounded operators on a Hilbert space $\mathscr{H}$ can be extended to an np-map on $\mathscr{B}(\mathscr{H})$, and in fact, is of the form $\omega \equiv \sum_{n}\left\langle x_{n},(\cdot) x_{n}\right\rangle$ for some $x_{1}, x_{2}, \ldots \in \mathscr{H}$, see 89IX.

II Proposition Let $S$ be a subset of a von Neumann algebra $\mathscr{A}$ that is closed under multiplication, involution, and contains 1 . Let $e$ be a projection in $\mathscr{A}$. Then $\lceil e\rceil_{S \square}=\bigcup_{a \in S}\left\lceil a^{*} e a\right\rceil$ is the least projection in $S^{\square}$ above $e$.
(Compare this with the paragraph "Subspaces" of $\S 2.6$ of [43].)
III Proof Let us first show that $p:=\lceil e\rceil_{S \square}$ is in $S^{\square}$. Let $b \in S$ be given; we must show that $p b=b p$. We may may assume without loss of generality that $\|b\| \leqslant 1$. Since $b^{*}(\cdot) b: \mathscr{A} \rightarrow \mathscr{A}$ is normal and completely positive, and $p=\bigcup_{a \in S}\left\lceil a^{*} e a\right\rceil$, we have $b^{*} p b \leqslant\left\lceil b^{*} p b\right\rceil=\bigcup_{a \in S}\left\lceil b^{*}\left\lceil a^{*} e a\right\rceil b\right\rceil=\bigcup_{a \in S}\left\lceil(a b)^{*} e a b\right\rceil \leqslant p$ by 60 IX and 60 V . Applying $p^{\perp}(\cdot) p^{\perp}$, we get $p^{\perp} b^{*} p b p^{\perp} \leqslant p^{\perp} p p^{\perp}=0$, so that $p b p^{\perp}=0$, and thus $p b p=p b$. Since similarly $p b^{*}=p b^{*} p$, we get $b p=p b p=p b$ (upon applying (•)*) and so $p \in S^{\square}$.

Note that $e \leqslant\left\lceil 1^{*} e 1\right\rceil \leqslant p$, because $1 \in S$. It remains to be shown that $p$ is the least projection in $S^{\square}$ above $e$, so let $q$ be a projection in $S^{\square}$ above $e$. Since for $a \in S$, we have $a q^{\perp} a^{*}=q^{\perp} a a^{*} q^{\perp} \leqslant\|a\|^{2} q^{\perp} \leqslant\|a\|^{2} e^{\perp}$, and so $a^{*} e a \leqslant\|a\|^{2} q$
we get $\left\lceil a^{*} e a\right\rceil \leqslant q$ for all $a \in S$, and thus $p=\bigcup_{a \in S}\left\lceil a^{*} e a\right\rceil \leqslant q$.

Exercise Show that given a vector $x$ of Hilbert space $\mathscr{H}$, and a collection $S$ of bounded operators on $\mathscr{H}$ that is closed under addition, (scalar) multiplication, involution, and contains the identity operator, the following coincide.

1. $\lceil|x\rangle\langle x|\rceil_{S \square}$, the least projection in $S^{\square}$ above $\lceil|x\rangle\langle x|\rceil$;
2. $\left\lceil\langle x,(\cdot) x\rangle \mid S^{\square}\right\rceil$, the carrier of the vector functional on $S^{\square}$ given by $x$;
3. $\bigcup_{a \in S}\lceil|a x\rangle\langle a x|\rceil$; and
4. the projection on $\overline{S x}$.

Conclude that $\overline{S^{\square \square} x}=\overline{S x}$. (Hint: $S^{\square \square \square}=S^{\square}$.)
Now consider (instead of $x$ ) an np-map $\omega: \mathscr{B}(\mathscr{H}) \rightarrow \mathbb{C}$, which we know must be of the form $\omega \equiv \sum_{n}\left\langle x_{n},(\cdot) x_{n}\right\rangle$ (by 39IX) and is therefore given by an element $x^{\prime} \equiv\left(x_{1}, x_{2}, \ldots\right)$ of the $\mathbb{N}$-fold product $\mathscr{H}^{\prime}:=\bigoplus_{n} \mathscr{H}$ of $\mathscr{H}$.

1. Show that $\omega(t)=\left\langle x^{\prime}, \varrho^{\prime}(t) x^{\prime}\right\rangle$, where $\varrho^{\prime}: \mathscr{B}(\mathscr{H}) \rightarrow \mathscr{B}\left(\mathscr{H}^{\prime}\right)$ is the nmiumap given by $\varrho^{\prime}(t) y=\left(t y_{n}\right)_{n}$ for all $t \in \mathscr{B}(\mathscr{H})$ and $y \in \mathscr{H}^{\prime}$.
Prove that $\varrho^{\prime}(t)=\sum_{n} P_{n}^{*} t P_{n}$, where $P_{n}:=\pi_{n}: \mathscr{H}^{\prime} \equiv \bigoplus_{n} \mathscr{H} \rightarrow \mathscr{H}$ is the $n$-th projection.
2. Let $t \in S^{\square \square}$ be given (with $S$ as above). Show that $\varrho^{\prime}(t) \in \varrho^{\prime}(S)^{\square \square}$.
(Hint: first show $P_{n} a P_{m}^{*} \in S^{\square}$ for all $m, n$, and $a \in \varrho^{\prime}(S)^{\square}$.)
Conclude that $\varrho^{\prime}(t) x^{\prime} \in \overline{\varrho^{\prime}(S)^{\square \square} x^{\prime}} \equiv \overline{\varrho^{\prime}(S) x^{\prime}}$.
Whence for every $\varepsilon>0$ one can find $a \in S$ with $\|t-a\|_{\omega} \leqslant \varepsilon$.
3. Deduce that $S^{\square \square}$ is contained in the ultrastrong closure of $S$.

Double Commutant Theorem For a collection $S$ of bounded operators on a Hilbert space $\mathscr{H}$ that is closed under addition, (scalar) multiplication, involution, and contains the identity operator the following are the same.

1. $S^{\square \square}$, the "double commutant" of $S$ in $\mathscr{B}(\mathscr{H})$;
2. us-cl $(S)$, the ultrastrong closure of $S$ in $\mathscr{B}(\mathscr{H})$;
3. uw-cl $(S)$, the ultraweak closure of $S$ in $\mathscr{B}(\mathscr{H})$;
4. $W^{*}(S)$, the least von Neumann subalgebra of $\mathscr{B}(\mathscr{H})$ that contains $S$.

VII Proof (Based on Theorem 5.3.1 of 43].)
Note that: us-cl $(S) \subseteq$ uw-cl $(S)$, because ultrastrong convergence implies ultraweak convergence; and uw-cl $(S) \subseteq W^{*}(S)$, because $W^{*}(S)$ is ultraweakly closed in $\mathscr{B}(\mathscr{H})$ by 75 VIII and $W^{*}(S) \subseteq S^{\square \square}$, because $S^{\square \square}$ is a von Neumann subalgebra of $\mathscr{B}(\mathscr{H})$ by 65 III and, finally, $S^{\square \square} \subseteq$ us-cl $(S)$ by V .
VIII Exercise Show that central elements of a von Neumann algebra $\mathscr{A}$ of bounded operators on a Hilbert space $\mathscr{H}$ coincide with the central elements of the commutant $\mathscr{A}^{\square}$, that is, $Z(\mathscr{A})=Z\left(\mathscr{A}^{\square}\right)$. (Hint: $\mathscr{A}^{\square \square}=\mathscr{A}$ by V )
IX Deduce that $\llbracket f|\mathscr{A} \rrbracket=\llbracket f| \mathscr{A}^{\square} \rrbracket$ for every np-map $f: \mathscr{B}(\mathscr{H}) \rightarrow \mathscr{B}$ into a von Neumann algebra $\mathscr{B}$.

89 Lemma Let $\omega: \mathscr{A} \rightarrow \mathbb{C}$ be an np-map on a von Neumann algebra $\mathscr{A}$, which is represented by nmiu-maps $\varrho: \mathscr{A} \rightarrow \mathscr{B}(\mathscr{H})$ and $\pi: \mathscr{A} \rightarrow \mathscr{B}(\mathscr{K})$ on Hilbert spaces $\mathscr{H}$ and $\mathscr{K}$. If $\langle x, \varrho(\cdot) x\rangle=\omega=\langle y, \pi(\cdot) y\rangle$ for some $x \in \mathscr{H}$ and $y \in \mathscr{K}$, then there is a bounded operator $U: \mathscr{K} \rightarrow \mathscr{H}$ for which $U U^{*}$ is the projection on $\overline{\varrho(\mathscr{A}) x}, U^{*} U$ is the projection on $\overline{\pi(\mathscr{A}) y}$, and $U \pi(a)=\varrho(a) U$ for all $a \in \mathscr{A}$.
11 Proof (Compare this with Proposition 4.5.3 of 43.)
Since $\|\varrho(a) x\|^{2}=\left\langle x, \varrho\left(a^{*} a\right) x\right\rangle=\omega\left(a^{*} a\right)=\left\langle y, \pi\left(a^{*} a\right) y\right\rangle=\|\pi(a) y\|^{2}$ for all $a \in \mathscr{A}$, there is a unique bounded operator $V: \overline{\pi(\mathscr{A}) y} \rightarrow \overline{\varrho(\mathscr{A}) x}$ with $V \pi(a) y=\varrho(a) x$ for all $a \in \mathscr{A}$. A moment's thought reveals that $V$ is a unitary (and so $V^{*} V=1$ and $V V^{*}=1$.) Now, define $U:=E V F^{*}$ where $E: \overline{\varrho(\mathscr{A}) x} \rightarrow \mathscr{H}$ and $F: \overline{\pi(\mathscr{A}) y} \rightarrow \mathscr{K}$ are the inclusions (and so $E^{*} E=1$ and $F^{*} F=1$ ). Then $U U^{*}=E V F^{*} F V^{*} E^{*}=E V V^{*} E^{*}=E E^{*}$ is the projection onto $\overline{\varrho(\mathscr{A}) x}$, and $U U^{*}=F F^{*}$ is the projection onto $\overline{\pi(\mathscr{A}) y}$.

Let $a \in \mathscr{A}$ be given. It remains to be shown that $U \pi(a)=\varrho(a) U$. To this end, observe that $V F^{*} \pi(a) F=E^{*} \varrho(a) E V$ (because these two bounded linear maps are easily seen to agree on the dense subset $\pi(\mathscr{A}) y$ of $\overline{\pi(\mathscr{A}) y})$; and $\varrho(a) E=$ $E E^{*} \varrho(a) E$ (because $\varrho(a)$ maps $\varrho(\mathscr{A}) x$ into $\varrho(\mathscr{A}) x$; and similarly $\varrho\left(a^{*}\right) F=$ $F F^{*} \varrho\left(a^{*}\right) F$, so that $F^{*} \varrho(a)=F^{*} \varrho(a) F F^{*}$ (upon application of the $\left.(\cdot)^{*}\right)$. By these observations, $U \pi(a)=E V F^{*} \pi(a)=E V F^{*} \pi(a) F F^{*}=E E^{*} \varrho(a) E V F^{*}=$ $\varrho(a) E V F^{*}=\varrho(a) U$.
III Exercise It is not too difficult to see that the (ultraweak) sum $\sum_{i} u_{i}$ of a collection $\left(u_{i}\right)_{i}$ of partial isometries from some von Neumann algebra is again a partial isometry, provided that the initial projections $u_{i}^{*} u_{i}$ are pairwise orthogonal, and the final projections $u_{i} u_{i}^{*}$ are pairwise orthogonal. In this exercise,
you'll establish a similar result, but for partial isometries between two different Hilbert spaces, and avoiding the use of an analogue of the ultraweak topology for such operators.

Let $\mathscr{H}$ and $\mathscr{K}$ be Hilbert spaces, and let $U_{i}: \mathscr{H} \rightarrow \mathscr{K}$ be a bounded operator for every element $i$ from some set $I$. Assume that the operators $U_{i}^{*} U_{i}$ are pairwise orthogonal projections in $\mathscr{B}(\mathscr{K})$, and that $U_{i} U_{i}^{*}$ are pairwise orthogonal projections in $\mathscr{B}(\mathscr{H})$.

1. Let $x \in \mathscr{H}$ and $y \in \mathscr{K}$ be given.

Show that $\left|\left\langle x, U_{i} y\right\rangle\right| \leqslant\left\|U_{i}^{*} x\right\|\left\|U_{i} y\right\|$ for each $i$ (perhaps by first proving that $\left.U_{i}=U_{i} U_{i}^{*} U_{i}\right)$.
Show that $\sum_{i}\left\|U_{i} y\right\|^{2} \leqslant\|y\|^{2}$ and $\sum_{i}\left\|U_{i}^{*} x\right\|^{2} \leqslant\|x\|^{2}$, and deduce from this that $\sum_{i}\left|\left\langle x, U_{i} y\right\rangle\right| \leqslant\|x\|\|y\|$.
Now use 36 V to show that there is a bounded operator $U: \mathscr{K} \rightarrow \mathscr{H}$ with $\langle x, U y\rangle=\sum_{i}\left\langle x, U_{i} y\right\rangle$ for all $x \in \mathscr{H}$ and $y \in \mathscr{K}$.
2. Show that $U_{i}^{*} U_{j}=0$ when $i \neq j$. Deduce from this that $U^{*} U=\sum_{i} U_{i}^{*} U_{i}$. Prove that $U U^{*}=\sum_{i} U_{i} U_{i}^{*}$.

Lemma Let $\Omega$ be a collection of np-maps $\omega: \mathscr{A} \rightarrow \mathbb{C}$ on a von Neumann algebra $\mathscr{A}$ whose central carriers, $\llbracket \omega \rrbracket$, are pairwise orthogonal to one another, and let $\mathscr{H}$ and $\mathscr{K}$ be Hilbert spaces on which $\mathscr{A}$ is represented such that each $\omega \in \Omega$ is given by vectors $x_{\omega} \in \mathscr{H}$ and $y_{\omega} \in \mathscr{K}$, that is, $\left\langle x_{\omega}, \varrho(\cdot) x_{\omega}\right\rangle=$ $\omega=\left\langle y_{\omega}, \pi(\cdot) y_{\omega}\right\rangle$, where $\varrho: \mathscr{A} \rightarrow \mathscr{B}(\mathscr{H})$ and $\pi: \mathscr{A} \rightarrow \mathscr{B}(\mathscr{K})$ are nmiu-maps.

Then there is a bounded operator $U: \mathscr{K} \rightarrow \mathscr{H}$ which intertwines $\pi$ and $\varrho$ in the sense that $U \pi(a)=\varrho(a) U$ for all $a \in \mathscr{A}$ such that $U^{*} U$ is a projection in $\pi(\mathscr{A})^{\square}$ with $\llbracket U^{*} U \rrbracket_{\pi(\mathscr{A}) \square}=\pi\left(\sum_{\omega} \llbracket \omega \rrbracket\right)$, and $U U^{*}$ is projection in $\varrho(\mathscr{A})^{\square}$ with $\llbracket U U^{*} \rrbracket_{\varrho(\mathscr{A})} \square=\varrho\left(\sum_{\omega} \llbracket \omega \rrbracket\right)$.
Proof Given $\omega \in \Omega$, let $\sigma_{\omega}: \varrho(\mathscr{A}) \rightarrow \mathbb{C}$ and $\sigma_{\omega}^{\prime}: \varrho(\mathscr{A})^{\square} \rightarrow \mathbb{C}$ denote the restrictions of the vector functional $\left\langle x_{\omega},(\cdot) x_{\omega}\right\rangle: \mathscr{B}(\mathscr{H}) \rightarrow \mathbb{C}$, and let $\tau_{\omega}: \pi(\mathscr{A}) \rightarrow \mathbb{C}$ and $\tau_{\omega}^{\prime}: \pi(\mathscr{A})^{\square} \rightarrow \mathbb{C}$ be similar restrictions of $\left\langle y_{\omega},(\cdot) y_{\omega}\right\rangle$. We already know (by \and 88 IV ) that there is a bounded operator $U_{\omega}: \mathscr{K} \rightarrow \mathscr{H}$ with $U_{\omega}^{*} U_{\omega}=$ $\left\lceil\tau_{\omega}^{\prime}\right\rceil, U_{\omega} U_{\omega}^{*}=\left\lceil\sigma_{\omega}^{\prime}\right\rceil$, and $U_{\omega} \pi(a)=\varrho(a) U_{\omega}$ for all $a \in \mathscr{A}$.

We'll combine these $U_{\omega} \mathrm{s}$ into one operator $U$ using (1II, but for this we must verify that the projections $U_{\omega} U_{\omega}^{*}=\left\lceil\sigma_{\omega}^{\prime}\right\rceil$ are pairwise orthogonal, and
that the projections $U_{\omega}^{*} U_{\omega}$ are pairwise orthogonal too. To this end note that $\llbracket \sigma_{\omega} \rrbracket=\llbracket \sigma_{\omega}^{\prime} \rrbracket$ by 88 IX Thus, since the projections $\llbracket \omega \rrbracket$ are orthogonal to one another, and $\left.\left.\left\lceil\sigma_{\omega}^{\prime}\right\rceil \leqslant \llbracket \sigma_{\omega}^{\prime}\right\rceil=\llbracket \sigma_{\omega} \rrbracket=\varrho(\llbracket \omega\rceil\right)$, we see that the projections $U_{\omega} U_{\omega}^{*} \equiv\left\lceil\sigma_{\omega}^{\prime}\right\rceil$ are indeed pairwise orthogonal. Since for a similar reason the projections $U_{\omega}^{*} U_{\omega} \equiv\left\lceil\tau_{\omega}^{\prime}\right\rceil$ are pairwise orthogonal too, there is by III a bounded operator $U: \mathscr{K} \rightarrow \mathscr{H}$ with $U^{*} U=\sum_{\omega} U_{\omega}^{*} U_{\omega}, U U^{*}=\sum_{\omega} U_{\omega} U_{\omega}^{*}$, and $\langle x, U y\rangle=\sum_{\omega}\left\langle x, U_{\omega} y\right\rangle$ for all $x \in \mathscr{H}$ and $y \in \mathscr{K}$.

Let us check that $U$ has the desired properties. To begin, since the projections $\llbracket U_{\omega} U_{\omega}^{*} \rrbracket=\llbracket \sigma_{\omega}^{\prime} \rrbracket=\varrho(\llbracket \omega \rrbracket)$ are pairwise orthogonal, we have $\llbracket U U^{*} \rrbracket=$ $\sum_{\omega} \llbracket U_{\omega} U_{\omega}^{*} \rrbracket=\varrho\left(\sum_{\omega} \llbracket \omega \rrbracket\right)$ by 68 IV and 56 XVIIT Similarly, $\llbracket U^{*} U \rrbracket=\pi\left(\sum_{\omega} \llbracket \omega \rrbracket\right)$.

Finally, given $a \in \mathscr{A}$ we have $U \pi(a)=\varrho(a) U$, because $\langle x, U \pi(a) y\rangle=$ $\sum_{\omega}\left\langle x, U_{\omega} \pi(a) y\right\rangle=\sum_{\omega}\left\langle x, \varrho(a) U_{\omega} y\right\rangle=\sum_{\omega}\left\langle\varrho(a)^{*} x, U_{\omega} y\right\rangle=\left\langle\varrho(a)^{*} x, U y\right\rangle=$ $\langle x, \varrho(a) U y\rangle$ for all $x \in \mathscr{H}$ and $y \in \mathscr{K}$.

VII Corollary Let $\mathscr{A}$ be a von Neumann of bounded operators on some Hilbert space $\mathscr{H}$, and let $\varrho: \mathscr{A} \rightarrow \mathscr{B}(\mathscr{H})$ denote the inclusion. Let $\Omega$ be the collection of all np-maps $\mathscr{A} \rightarrow \mathbb{C}$, and let $\varrho_{\Omega}: \mathscr{A} \rightarrow \mathscr{B}\left(\mathscr{H}_{\Omega}\right)$ be as in 30 IX .

There is a bounded operator $U: \mathscr{H}_{\Omega} \rightarrow \mathscr{H}$ such that $U^{*} U$ is a projection in $\varrho_{\Omega}(\mathscr{A})^{\square}$ with $\llbracket U^{*} U \rrbracket_{\varrho_{\Omega}(\mathscr{A})}{ }^{\square}=1$ and $U \varrho_{\Omega}(a)=\varrho(a) U$ for all $a \in \mathscr{A}$.

VIII Proof Let $\left\{x_{i}\right\}_{i}$ be a maximal set of vectors in $\mathscr{H}$ such that the central carriers $\llbracket \omega_{i} \rrbracket$ of the corresponding vector functionals $\omega_{i}:=\left\langle x_{i}, \varrho(\cdot) x_{i}\right\rangle$ on $\mathscr{A}$ are pairwise orthogonal; so that we'll have $\sum_{i} \llbracket \omega_{i} \rrbracket=1$. Now, the point of $\mathscr{H}_{\Omega}$ is that there are vectors $y_{i} \in \mathscr{H} \Omega_{\Omega}$ with $\omega_{i}=\left\langle y_{i}, \varrho_{\Omega}(\cdot) y_{i}\right\rangle$ for each $i$. Now apply $\nabla$ to get a map $U: \mathscr{H}_{\Omega} \rightarrow \mathscr{H}$ with the desired properties.

IX Theorem Every np-map $\omega: \mathscr{A} \rightarrow \mathbb{C}$ on a von Neumann subalgebra $\mathscr{A}$ of $\mathscr{B}(\mathscr{H})$, where $\mathscr{H}$ is some Hilbert space, is of the form $\omega \equiv \sum_{n}\left\langle x_{n},(\cdot) x_{n}\right\rangle$ for some $x_{1}, x_{2}, \ldots \in \mathscr{H}$ (with $\sum_{n}\left\|x_{n}\right\|^{2}<\infty$ ).
$\times$ Proof (Based on Theorem 7.1.8 of 43].)
Let $\varrho: \mathscr{A} \rightarrow \mathscr{B}(\mathscr{H})$ denote the inclusion, and let $U: \mathscr{H}_{\Omega} \rightarrow \mathscr{H}$ be as in VII. Since $\omega \in \Omega$, there is $y \in \mathscr{H}_{\Omega}$ with $\omega=\left\langle y, \varrho_{\Omega}(\cdot) y\right\rangle$. We're going to 'transfer' $y$ from $\mathscr{H}_{\Omega}$ to $\mathscr{H}$ using the following device. Since $1=\llbracket U^{*} U \rrbracket_{\varrho_{\Omega}(\mathscr{A})}{ }^{\square}$, we can (by 83 V find partial isometries $\left(v_{i}\right)_{i}$ in $\varrho_{\Omega}(\mathscr{A})^{\square}$ with $1=\sum_{i} v_{i}^{*} v_{i}$ and
$v_{i} v_{i}^{*} \leqslant U^{*} U$ for all $i$. Then for every $a \in \mathscr{A}$,

$$
\begin{aligned}
\omega(a) & =\left\langle y, \varrho_{\Omega}(a) y\right\rangle & & \\
& =\sum_{i}\left\langle y, v_{i}^{*} v_{i} \varrho_{\Omega}(a) y\right\rangle & & \text { since } 1=\sum_{i} v_{i}^{*} v_{i} \\
& =\sum_{i}\left\langle y, v_{i}^{*} U^{*} U v_{i} \varrho_{\Omega}(a) y\right\rangle & & \text { since } v_{i} v_{i}^{*} \leqslant U^{*} U \\
& =\sum_{i}\left\langle U v_{i} y, U \varrho_{\Omega}(a) v_{i} y\right\rangle & & \text { since } v_{i} \in \varrho(\mathscr{A})^{\square} \\
& =\sum_{i}\left\langle U v_{i} y, \varrho(a) U v_{i} y\right\rangle & & \text { since } U \varrho_{\Omega}(a)=\varrho(a) U .
\end{aligned}
$$

In particular, $\omega(1)=\sum_{i}\left\|U v_{i} y\right\|^{2}$, so at most countably many $U v_{i} y$ 's are nonzero; and denoting those by $x_{1}, x_{2}, \ldots$, we get $\omega=\sum_{n}\left\langle x_{n},(\cdot) x_{n}\right\rangle$.
Corollary Let $\mathscr{A}$ be a von Neumann subalgebra of a von Neumann algebra $\mathscr{B}$.

1. For every np-map $\omega: \mathscr{A} \rightarrow \mathbb{C}$ there is an np-map $\xi: \mathscr{B} \rightarrow \mathbb{C}$ with $\xi \mid \mathscr{A}=\omega$.
2. Ultraweak permanence: the restriction of the ultraweak topology on $\mathscr{B}$ to $\mathscr{A}$ coincides with the ultraweak topology on $\mathscr{A}$.
3. Ultrastrong permanence: the restriction of the ultrastrong topology on $\mathscr{B}$ to $\mathscr{A}$ coincides with the ultrastrong topology on $\mathscr{A}$.

Exercise Let $\varrho: \mathscr{A} \rightarrow \mathscr{B}$ be an injective nmiu-map.
Show using 48 VI that any np-functional $\omega: \mathscr{A} \rightarrow \mathbb{C}$ can be extended along $\varrho$, that is, there is an np-functional $\omega^{\prime}: \mathscr{B} \rightarrow \mathbb{C}$ with $\varrho \circ \omega^{\prime}=\omega$.

We end the chapter with another corollary to 89 IX that the np-functionals on a von Neumann algebra are generated (in a certain sense) by any centre separating collection of functionals. This fact plays an important role in the next chapter for our definition of the tensor product of von Neumann algebras (on which the product functionals are to be centre separating, 108 II .
Proposition Given a centre separating collection $\Omega$ of np-functionals on a von Neumann algebra $\mathscr{A}$, and an ultrastrongly dense subset $S$ of $\mathscr{A}$

1. $\Omega^{\prime}:=\left\{\omega\left(s^{*}(\cdot) s\right): \omega \in \Omega, s \in S\right\}$ is order separating, and
2. $\Omega^{\prime \prime}:=\left\{\sum_{n} \omega_{n}: \omega_{1}, \ldots, \omega_{N} \in \Omega^{\prime}\right\}$ is operator norm dense in $\left(\mathscr{A}_{*}\right)_{+}$.

III Proof We tackle 1 first. We already know from $30 \times$ that the collection $\Xi:=$ $\left\{\omega\left(a^{*}(\cdot) a\right): \omega \in \Omega, a \in \mathscr{A}\right\}$, which contains $\Omega^{\prime}$, is order separating; so to prove that $\Omega^{\prime}$ is itself order separating it suffices by 21 X to show that $\Omega^{\prime}$ is norm dense in $\Xi$. This is indeed the case since given $a \in \mathscr{A}$ and $\omega \in \Omega$, and a net $\left(s_{\alpha}\right)_{\alpha}$ in $S$ that converges ultrastrongly to $a$, the functionals $s_{\alpha} * \omega \equiv \omega\left(s_{\alpha}^{*}(\cdot) s_{\alpha}\right)$ converge in norm to $a * \omega$ as $\alpha \rightarrow \infty$ by 72 III .

IV (Concerning 2) Let $f: \mathscr{A} \rightarrow \mathbb{C}$ be an np-map; we must show that $f$ is in the norm closure $\overline{\Omega^{\prime \prime}}$ of $\Omega^{\prime \prime}$. Note that since $\Omega$ is centre separating, the map $\varrho_{\Omega}: \mathscr{A} \rightarrow \mathscr{B}\left(\mathscr{H}_{\Omega}\right)$ from 30 X is injective, and in fact restricts to a nmiuisomorphism from $\mathscr{A}$ onto $\varrho_{\Omega}(\mathscr{A})$ (cf. 48 VIIII ). So by $89 \mathrm{IX} f$ is of the form $f \equiv \sum_{n}\left\langle x_{n}, \varrho_{\Omega}(\cdot) x_{n}\right\rangle$ for some $x_{1}, x_{2}, \ldots \in \mathscr{H}_{\Omega}$ with $\sum_{n}\left\|x_{n}\right\|^{2}<\infty$, so that the partial sums $\sum_{n=1}^{N}\left\langle x_{n}, \varrho_{\Omega}(\cdot) x_{n}\right\rangle$ converge with respect to the operator norm to $f$ (by 38 VII ). Thus to show that $f$ is in $\overline{\Omega^{\prime \prime}}$ it suffices to show that each $\left\langle x_{n}, \varrho_{\Omega}(\cdot) x_{n}\right\rangle$ is in $\Omega^{\prime \prime}$ (since $\overline{\Omega^{\prime \prime}}$ is clearly closed under finite sums and norm limits). In effect we may assume without loss of generality that $f \equiv\left\langle x, \varrho_{\Omega}(\cdot) x\right\rangle$ for some $x \in \mathscr{H}_{\Omega}$. We reduce the problem some more. By definition of $\mathscr{H}_{\omega} \equiv \bigoplus_{\omega \in \Omega} \mathscr{H}_{\omega}$ and $\varrho_{\Omega}$, we have $f=\left\langle x, \varrho_{\Omega}(\cdot) x\right\rangle=\sum_{\omega \in \Omega}\left\langle x_{\omega}, \varrho_{\omega}(\cdot) x_{\omega}\right\rangle ;$ and so we may, by the same token, assume without loss of generality that $f=\left\langle x, \varrho_{\omega}(\cdot) x\right\rangle$ for some $\omega \in \Omega$ and $x \in \mathscr{H}_{\omega}$. Since such $x$ (by definition of $\mathscr{H}_{\omega}$, 30 VII ) is the norm limit of a sequence $\eta_{\omega}\left(a_{1}\right), \eta_{\omega}\left(a_{2}\right), \cdots$, where $a_{1}, a_{2}, \ldots \in \mathscr{A}$, the np-maps $a_{n} * \omega \equiv\left\langle\eta_{\omega}\left(a_{n}\right), \varrho_{\omega}(\cdot) \eta_{\omega}\left(a_{n}\right)\right\rangle$ converge to $\left\langle x, \varrho_{\omega}(\cdot) x\right\rangle=f$ in the operator norm as $n \rightarrow \infty$ by 38 VI and so we may assume without loss of generality that $f=a * \omega$ for some $a \in \mathscr{A}$ and $\omega \in \Omega$. Since $S$ is ultrastrongly dense in $\mathscr{A}$ we can find a net $\left(s_{\alpha}\right)_{\alpha}$ in $S$ that converges ultrastrongly to $a$. As the np-functionals $s_{\alpha} * \omega$ in $\Omega^{\prime} \subseteq \Omega^{\prime \prime}$ will then operator-norm converge to $f=a * \omega$ as $\alpha \rightarrow \infty$ by 72 III we conclude that $f \in \overline{\Omega^{\prime \prime}}$.

91 With this chapter ends perhaps the most hairy part of this thesis: we've developed the theory of von Neumann algebras starting from Kadison's characterization (see 42) to the point that we have a sufficiently firm hold on the normal functionals (see e.g. $86 \mathrm{IX}, 89 \mathrm{IX}$, the ultraweak and ultrastrong topologies (e.g. $74 \mathrm{IV}, 89 \mathrm{XI}, 90 \mathrm{II}$, the projections $561,59 \mathrm{I}, 65 \mathrm{IV}$, and the division structure ( $81 \mathrm{~V}, 821$ ) on a von Neumann algebra. In the next chapter we reap the benefits of our labour when we study an assortment of structures in the category $\mathbf{W}_{\text {CPsu }}^{*}$ of von Neumann algebras and ncpsu-maps.

## Chapter 4

## Assorted Structure in $\mathbf{W}_{\text {cpsu }}^{*}$

In the previous two chapters we have travelled through charted territory when developing the theory of $C^{*}$-algebras and von Neumann algebras adding some new landmarks and shortcuts of our own along the way. In this chapter we properly break new ground by revealing two entirely new features of the category $\mathbf{W}_{\text {CPSU }}^{*}$ of von Neumann algebras and the normal completely positive subunital linear maps between them, namely,

1. that the binary operation $*$ on the effects of a von Neumann algebra $\mathscr{A}$ given by $p * q=\sqrt{p} q \sqrt{p}$ (representing measurement of $p$ ) can be axiomatised, and
2. that the category $\mathbf{W}_{\text {CPSU }}^{*}$ has all the bits and pieces needed to be a model of Selinger and Valiron's quantum lambda calculus.

We'll deal with the first matter directly after this introduction in Section 4.1. The second matter is treated in Section 4.3, but only after we have given the tensor product of von Neumann algebras a complete overhaul in Section 4.2 Finally, as an offshoot of our model of the quantum lambda calculus we'll study all von Neumann algebras that admit a 'duplicator' in Section 4.4 - surprisingly, they're all of the form $\ell^{\infty}(X)$.

### 4.1 Measurement

93 The maps on a von Neumann algebra $\mathscr{A}$ of the form $a \mapsto \sqrt{p} a \sqrt{p}: \mathscr{A} \rightarrow \mathscr{A}$, where $p$ is an effect of $\mathscr{A}$, represent measurement of $p$, and are called assert maps in 26]. The importance of these maps to any logical description of quantum computation is not easily overstated. On the effects of $\mathscr{A}$ these maps are also studied in the guise of the binary operation $p * q=\sqrt{p} q \sqrt{p}$ called the sequential product (see e.g. [21]). We'll axiomatise this operation in this section in terms of the properties of the underlying assert maps.

Our first observation to this end is that any assert map factors as

$$
\mathscr{A} \xrightarrow{\pi: a \mapsto\lceil p\rceil a\lceil p\rceil}\lceil p\rceil \mathscr{A}\lceil p\rceil \xrightarrow{c: a \mapsto \sqrt{p} a \sqrt{p}} \mathscr{A},
$$

where both $\pi$ and $c$ obey a universal property: $c$ is a filter of $p$, see 961 , and $\pi$ is a corner of $\lceil p\rceil$, see 951 . Such maps that are the composition of a filter and a corner will be called pure, see 1001 . Since not only assert maps turn out to be pure, but also maps of the form $b^{*}(\cdot) b: \mathscr{A} \rightarrow \mathscr{A}$ for an arbitrary element $b$ of $\mathscr{A}$, we need another property of assert maps, namely that

$$
\sqrt{p} e_{1} \sqrt{p} \leqslant e_{2}^{\perp} \quad \Longleftrightarrow \quad \sqrt{p} e_{2} \sqrt{p} \leqslant e_{1}^{\perp}
$$

for all projections $e_{1}$ and $e_{2}$ of $\mathscr{A}$-which we'll describe by saying that

$$
\sqrt{p}(\cdot) \sqrt{p}: \mathscr{A} \rightarrow \mathscr{A}
$$

is $\diamond$-self-adjoint. Judging only by the name it may not surprise you that the map $b(\cdot) b: \mathscr{A} \rightarrow \mathscr{A}$ where $b \in \mathscr{A}$ is self-adjoint (but not necessarily positive) turns out to be $\diamond$-self-adjoint too, so that as a final touch we introduce the notion of $\diamond$-positive maps $f: \mathscr{A} \rightarrow \mathscr{A}$ that are simply maps of the form $f \equiv g g$ for some $\diamond$-self-adjoint $g$.

The main technical result, then, of this section is that any $\diamond$-positive map $f: \mathscr{A} \rightarrow \mathscr{A}$ is of the form $f=\sqrt{p}(\cdot) \sqrt{p}$ where $p=f(1)$; and, accordingly, our axioms (in 106I) that uniquely determine the sequential product $*$ on the effects of a von Neumann algebra $\mathscr{A}$ are: for every effect $p$ of $\mathscr{A}$,

1. $p * 1=p$,
2. $p * q=f(q)$ for all $q \in[0,1]_{\mathscr{A}}$ for some pure map $f: \mathscr{A} \rightarrow \mathscr{A}$,
3. $p=q * q$ for some $q$ from $[0,1]_{\mathscr{A}}$,
4. $p *(p * q)=(p * p) * q$ for all $q \in[0,1]_{\mathscr{A}}$,
5. $p * e_{1} \leqslant e_{2}^{\perp} \Longleftrightarrow p * e_{2} \leqslant e_{1}^{\perp}$ for all projections $e_{1}, e_{2}$ of $\mathscr{A}$.

While I would certainly not like to undersell the results mentioned above, I suspect that the notion of purity exposed along the way might turn out to be of far greater significance for the following reason. Our notion of purity can be described in wildly different terms: a map $f: \mathscr{A} \rightarrow \mathscr{B}$ is pure when given its Paschke dilation $\mathscr{A} \longrightarrow \varrho \rightarrow \mathscr{P} \longrightarrow c \rightarrow \mathscr{B}$ the map $\varrho$ is surjective (see 171 VII and [73]). Because of my faith in our notion of purity I've allowed myself to address some theoretical questions concerning it here that are not required for the main results of this thesis, but suppose a general interest in purity: I'll show that every pure map $f: \mathscr{A} \rightarrow \mathscr{B}$ is extreme among the ncp-maps $g: \mathscr{A} \rightarrow$ $\mathscr{B}$ with $f(1)=g(1)$, and, in fact, enjoys the possibly stronger property of being rigid (see 102 II and 102 IX .

### 4.1.1 Corner and Filter

Definition Given a projection $e$ of a von Neumann algebra $\mathscr{A}$, the corner of $e$ is the subset $e \mathscr{A} e$ of $\mathscr{A}$ (consisting of the elements of $\mathscr{A}$ of the form eae with $a \in \mathscr{A}$ ). In this context, the obvious map $e \mathscr{A} e \rightarrow \mathscr{A}$ is called the inclusion and the map $a \mapsto e a e, \mathscr{A} \rightarrow e \mathscr{A} e$ is called the projection.

Exercise Let $e$ be a projection from a von Neumann algebra $\mathscr{A}$.

1. Show that $a \in \mathscr{A}$ is an element of $e \mathscr{A} e$ iff $e a e=a$ iff $(a\rceil \cup\lceil a) \leqslant e$.
2. Show that the corner $e \mathscr{A} e$ is closed under addition, (scalar) multiplication, and involution.
3. Show that $e$ is a unit for $e \mathscr{A} e$, that is, $e a=a e=a$ for all $a \in e \mathscr{A} e$.
4. Show that $e \mathscr{A} e$ is norm and ultraweakly closed.
(Hint: use the fact that $e(\cdot) e: \mathscr{A} \rightarrow \mathscr{A}$ is normal and bounded.)
5. Show that $e \mathscr{A} e$ - endowed with the addition, (scalar) multiplication, involution and norm from $\mathscr{A}$, and with $e$ as its unit - is a $C^{*}$-algebra.
6. Show that the supremum of a bounded directed set $D$ of self-adjoint elements of $e \mathscr{A} e$ computed in $\mathscr{A}$ is itself in $e \mathscr{A} e$, and, in fact, the supremum of $D$ in $e \mathscr{A} e$.

93, 94..
7. Show that the inclusion $e \mathscr{A} e \rightarrow \mathscr{A}$ is an ncpsu-map.
8. Deduce from this that the restriction of an np-map $\omega: \mathscr{A} \rightarrow \mathbb{C}$ to a map $e \mathscr{A} e \rightarrow \mathbb{C}$ is an np-map.
Conclude that $e \mathscr{A} e$ is a von Neumann algebra.
9. Show that the projection $a \mapsto e a e, \mathscr{A} \rightarrow e \mathscr{A} e$ is an ncpu-map.
10. Show that every np-map $\omega:$ e $\mathscr{A} e \rightarrow \mathbb{C}$ is the restriction of the np-map $\omega(e(\cdot) e): \mathscr{A} \rightarrow \mathbb{C}$. Deduce from this that the ultraweak topology of $e \mathscr{A} e$ coincides (on $e \mathscr{A} e$ ) with the ultraweak topology on $\mathscr{A}$. Show that the ultrastrong topologies on $e \mathscr{A} e$ and $\mathscr{A}$ coincide in a similar fashion.

III Exercise Let $a$ be an element of a von Neumann algebra $\mathscr{A}$, and let $p$ and $q$ be projections of $\mathscr{A}$ with $a^{*} p a \leqslant q$.

1. Show that $a^{*} b a \in q \mathscr{A} q$ for every $b \in p \mathscr{A} p$.
2. Show that $a^{*}(\cdot) a$ gives an ncp-map $p \mathscr{A} p \rightarrow q \mathscr{A} q$.

95 Definition Let $p$ be an effect of a von Neumann algebra $\mathscr{A}$. A corner of $p$ is an ncp-map $\pi: \mathscr{A} \rightarrow \mathscr{C}$ to a von Neumann algebra $\mathscr{C}$ with $\pi\left(p^{\perp}\right)=0$, which is initial among such maps in the sense that every ncp-map $f: \mathscr{A} \rightarrow \mathscr{B}$ with $f\left(p^{\perp}\right)=0$ factors as $f=g \circ \pi$ for some unique ncp-map $g: \mathscr{C} \rightarrow \mathscr{B}$.

While most corners that we'll deal with are unital, there are also corners which are not unital (because there are non-unital ncp-isomorphisms). When we write "corner" we shall always mean a "unital corner" unless explicitly stated otherwise.

II Proposition Given an effect $p$ of a von Neumann algebra $\mathscr{A}$, and a partial isometry $u$ of $\mathscr{A}$ with $\lfloor p\rfloor=u u^{*}$, the map $\pi: \mathscr{A} \rightarrow u^{*} u \mathscr{A} u^{*} u$ given by $\pi(a)=$ $u^{*} a u$ is a corner of $p$.
III Proof By 94 III . $\pi$ is an ncp-map. To see that $\pi\left(p^{\perp}\right) \equiv u^{*} p^{\perp} u=0$, note that since $u^{*} u=u^{*} u u^{*} u$, we have $0=u^{*}\left(u u^{*}\right)^{\perp} u=u^{*}\lfloor p\rfloor^{\perp} u=u^{*}\left\lceil p^{\perp}\right\rceil u$, and so $0=\left\lceil u^{*}\left\lceil p^{\perp}\right\rceil u\right\rceil=\left\lceil u^{*} p^{\perp} u\right\rceil$ by 60 VII , giving $u^{*} p^{\perp} u=0$ by 59 III .

Let $\mathscr{B}$ be a von Neumann algebra, and let $f: \mathscr{A} \rightarrow \mathscr{B}$ be an ncp-map with $f\left(p^{\perp}\right)=0$. To show that $\pi$ is a corner, we must show that there is a unique ncp-map $g: u^{*} u \mathscr{A} u^{*} u \rightarrow \mathscr{B}$ with $f=g \circ \pi$. Uniqueness follows from surjectivity
of $\pi$. Concerning existence, define $g:=f \circ \zeta$, where $\zeta: u^{*} u \mathscr{A} u^{*} u \rightarrow \mathscr{A}$ is the ncp-map given by $\zeta(a)=u a u^{*}$ for $a \in \mathscr{A}$ (see 94 III$)$, so that it is immediately clear that $g$ is an ncp-map. It remains to be shown $f=g \circ \pi$, that is, $f(a)=$ $f\left(u u^{*} a u u^{*}\right)$ for all $a \in \mathscr{A}$. This follows from 63 IV because $f\left(\left(u u^{*}\right)^{\perp}\right)=0$, since $\left\lceil f\left(\left(u u^{*}\right)^{\perp}\right)\right\rceil=\left\lceil f\left(\lfloor p\rfloor^{\perp}\right)\right\rceil=\left\lceil f\left(\left\lceil p^{\perp}\right\rceil\right)\right\rceil=\left\lceil f\left(p^{\perp}\right)\right\rceil=\lceil 0\rceil=0$.

Definition A filter is an ncp-map $c: \mathscr{C} \rightarrow \mathscr{A}$ between von Neumann algebras
such that every ncp-map $f: \mathscr{B} \rightarrow \mathscr{A}$ with $f(1) \leqslant c(1)$ factors as $f=c \circ g$ for some unique ncp-map $g: \mathscr{B} \rightarrow \mathscr{C}$. We'll say that $c$ is a filter for $c(1)$.
To show that there is a filter for every positive element of a von Neumann algebra we need to the following result concerning ultraweak limits of ncp-maps.
Lemma Given von Neumann algebras $\mathscr{A}$ and $\mathscr{B}$ the pointwise ultraweak limit $f: \mathscr{A} \rightarrow \mathscr{B}$ of a net of positive linear maps $f_{\alpha}: \mathscr{A} \rightarrow \mathscr{B}$ is positive, and,

1. $f$ is completely positive provided that the $f_{\alpha}$ are completely positive, and
2. $f$ is normal provided that the $f_{\alpha}$ are normal and the ultraweak convergence of the $f_{\alpha}$ to $f$ is uniform on $[0,1]_{\mathscr{A}}$.

Proof Since given $a \in \mathscr{A}$ the element $f(a)$ is the ultraweak limit of the positive elements $f_{\alpha}(a)$, and therefore positive (by 44XI), we see that $f$ is positive.

Suppose that each $f_{\alpha}$ is completely positive. To show that $f$ is completely positive, we must prove, given $a_{1}, \ldots, a_{n} \in \mathscr{A}$ and $b_{1}, \ldots, b_{n} \in \mathscr{B}$, that the element $\sum_{i, j} b_{i}^{*} f\left(a_{i}^{*} a_{j}\right) b_{j}$ of $\mathscr{B}$ is positive. And indeed it is, being the ultraweak limit of the positive elements $\sum_{i, j} b_{i}^{*} f_{\alpha}\left(a_{i}^{*} a_{j}\right) b_{j}$, because $f_{\alpha}\left(a_{i}^{*} a_{j}\right)$ converges ultraweakly to $f\left(a_{i}^{*} a_{j}\right)$, and $b_{i}^{*}(\cdot) b_{j}: \mathscr{B} \rightarrow \mathscr{B}$ is ultraweakly continuous 45 IV for any $i$ and $j$.

If the $f_{\alpha}$ are normal, and converge uniformly on $[0,1]_{\mathscr{A}}$ ultraweakly to $f$, then $f$ is ultraweakly continuous on $[0,1]_{\mathscr{A}}$ (because the uniform limit of continuous functions is continuous), and thus normal (by 44 XV ).

Proposition Given an element $d$ of a von Neumann algebra $\mathscr{A}$, the map $c:(d\rceil \mathscr{A}(d\rceil \rightarrow \mathscr{A}$ given by $c(a)=d^{*} a d$ is a filter.
Proof Note that $c$ is an ncp-map by 94 III. Let $\mathscr{B}$ be a von Neumann algebra, and let $f: \mathscr{B} \rightarrow \mathscr{A}$ be an ncp-map with $f(1) \leqslant c(1)$. To show that $c$ is a filter, we must show that there is a unique ncp-map $g: \mathscr{B} \rightarrow(d\rceil \mathscr{A}(d\rceil$ with $f=c \circ g$. Uniqueness of $g$ follows from the observation that $c$ is injective by 60 VIII .

To establish the existence of such $g$, note that $f(b)$ is an element of $d^{*} \mathscr{A} d$, when $b$ is positive by 81 VI because $0 \leqslant f(b) \leqslant\|b\| f(1) \leqslant\|b\| c(1)=\|b\| d^{*} d$, and
thus for arbitrary $b \in \mathscr{B}$ too (being a linear combination of positive elements). We can thus define $g: \mathscr{B} \rightarrow(d\rceil \mathscr{A}(d\rceil$ by $g(b)=d^{*} \backslash f(b) / d$ for all $b \in \mathscr{B}$. It is clear that $g$ is linear and positive, and $c \circ g=f$.

To see that $g$ is normal, note that $d^{*} \backslash \cdot / d: d^{*}(\mathscr{A})_{1} d \rightarrow \mathscr{A}$ is ultrastrongly continuous by 81 IX , as is $f$ by 45 II (also) as map from $(\mathscr{B})_{1}$ to $d^{*}(\mathscr{A})_{1} d$, so that $g$ is ultrastrongly continuous on $(\mathscr{B})_{1}$, and therefore normal by 44 XV .

Finally, $g$ is completely positive by 111 , because it is by 81 VII the uniform ultrastrong limit of the by 94 III completely positive maps $\left(\sum_{n=1}^{N} t_{n}\right)^{*} f(\cdot)\left(\sum_{n=1}^{N} t_{n}\right)$, where $t_{1}, t_{2}, \ldots$ is an approximate pseudoinverse of $d$.

97 Before exploring their more technical aspects, we'll explain how corners and filters can be made to appear at opposite ends of a chain of adjunctions:


The category Eff has as objects pairs $(\mathscr{A}, p)$, where $\mathscr{A}$ is a von Neumann algebra, and $p \in[0,1]_{\mathscr{A}}$ is an effect from $\mathscr{A}$. A morphism $(\mathscr{A}, p) \longrightarrow(\mathscr{B}, q)$ in Eff is an ncpsu-map $f: \mathscr{B} \rightarrow \mathscr{A}$ with $p \leqslant f(q)+f(1)^{\perp}$ - that is,

$$
\omega(p) \leqslant \omega(f(q))+\omega(f(1))^{\perp} \quad \text { for every normal state } \omega: \mathscr{A} \rightarrow \mathbb{C} .
$$

The functor $\mathbf{E f f} \longrightarrow\left(\mathbf{W}_{\text {CPSU }}^{*}\right)^{\text {op }}$ in the middle of the diagram above maps a morphism $f:(\mathscr{A}, p) \rightarrow(\mathscr{B}, q)$ to the underlying map $f: \mathscr{B} \rightarrow \mathscr{A}$. The functors $\mathbf{0}$ and $\mathbf{1}$ on its sides map a von Neumann algebra $\mathscr{A}$ to $(\mathscr{A}, 0)$ and $(\mathscr{A}, 1)$, respectively, and send an ncpsu-map $f: \mathscr{A} \rightarrow \mathscr{B}$ to itself; this is possible since

$$
0 \leqslant f(0)+f(1)^{\perp} \quad \text { and } \quad 1 \leqslant f(1)+f(1)^{\perp}
$$

That $\mathbf{1}$ is right adjoint to the functor $\mathbf{E f f} \longrightarrow\left(\mathbf{W}_{\text {CPSU }}^{*}\right)^{\text {op }}$ follows from the observation that an ncpsu-map $f: \mathscr{B} \rightarrow \mathscr{A}$ is always a morphism $(\mathscr{A}, p) \rightarrow(\mathscr{B}, 1)$, whatever $p \in[0,1]_{\mathscr{A}}$ may be, because $p \leqslant f(1)+f(1)^{\perp}$. For a similar reason $\mathbf{0}$ is left adjoint to $\mathbf{E f f} \longrightarrow\left(\mathbf{W}_{\text {CPSU }}^{*}\right)^{\text {op }}$.

On the other hand, a morphism $(\mathscr{A}, 1) \rightarrow(\mathscr{B}, q)$ where $q \in[0,1]_{\mathscr{B}}$ is not just any ncpsu-map $f: \mathscr{B} \rightarrow \mathscr{A}$, but one with $1 \leqslant f(q)+f(1)^{\perp}$, that is, $f\left(q^{\perp}\right)=0$.

It's no surprise then that a corner $\pi: \mathscr{B} \rightarrow \mathscr{C}$ for $q \in[0,1]_{\mathscr{B}}$ considered as morphism $(\mathscr{C}, 1) \rightarrow(\mathscr{B}, q)$ is a universal arrow from 1 to $(\mathscr{B}, q)$.

On the other side there's a twist: a morphism $(\mathscr{A}, p) \rightarrow(\mathscr{B}, 0)$ where $p \in$ $[0,1]_{\mathscr{A}}$ is an ncpsu-map $f: \mathscr{A} \rightarrow \mathscr{B}$ with $p \leqslant f(0)+f(1)^{\perp}$, that is, $f(1) \leqslant p^{\perp}$. It follows that any filter $c: \mathscr{C} \rightarrow \mathscr{A}$ for $p^{\perp}$, when considered as morphism $(\mathscr{A}, p) \rightarrow(\mathscr{C}, 0)$, is a universal arrow from $(\mathscr{A}, p)$ to $\mathbf{0}$.

This chain of adjunctions not only exposes a hidden symmetry between filters and corners, but such chains appear in many other categories as well, see 6].

Definition Let $\mathscr{A}$ be a von Neumann algebra.

1. Given a positive element $p$ of $\mathscr{A}$ we denote by $c_{p}:\lceil p\rceil \mathscr{A}\lceil p\rceil \rightarrow \mathscr{A}$ the standard filter for $p$ given by $c_{p}(a)=\sqrt{p} a \sqrt{p}$ for all $a \in\lceil p\rceil \mathscr{A}\lceil p\rceil$.
2. Given an effect $p$ of $\mathscr{A}$ we denote by $\pi_{p}: \mathscr{A} \rightarrow\lfloor p\rfloor \mathscr{A}\lfloor p\rfloor$ the standard corner of $p$ given by $\pi_{p}(a)=\lfloor p\rfloor a\lfloor p\rfloor$.

Exercise Let $c: \mathscr{C} \rightarrow \mathscr{A}$ be a filter, where $\mathscr{C}$ and $\mathscr{A}$ are von Neumann algebras.

1. Show that, writing $p:=c(1)$, there is a unique ncp-map $\alpha: \mathscr{C} \rightarrow\lceil p\rceil \mathscr{A}\lceil p\rceil$ with $c=c_{p} \circ \alpha$; and that this $\alpha$ is a unital ncp-isomorphism.
2. Show that $c$ is injective (by proving first that $c_{p}$ is injective using 60 VIII ). Conclude that $c$ is faithful (so $\lceil f\rceil=1$ ), and that $c$ is mono in $\mathbf{W}_{\mathrm{CP}}^{*}$.
3. Show that $c$ is bipositive (by proving first that $c_{p}$ is bipositive using 81 VI ).

Exercise Show that the composition $d \circ c$ of filters $c: \mathscr{C} \rightarrow \mathscr{D}$ and $d: \mathscr{D} \rightarrow \mathscr{A}$ III between von Neumann algebras is again a filter.

Exercise Let $p$ be an effect of a von Neumann algebra $\mathscr{A}$, and let $\pi: \mathscr{A} \rightarrow \mathscr{C}$ IV be a corner of $p$.

1. Show that there is a unique ncp-map $\beta:\lfloor p\rfloor \mathscr{A}\lfloor p\rfloor \rightarrow \mathscr{C}$ with $\pi=\beta \circ \pi_{p}$; and that this $\beta$ is unital and an ncp-isomorphism.
2. Show that $\pi$ is surjective, and that $\pi$ is epi in $\mathbf{W}_{\text {CP }}^{*}$.
$\checkmark$ Exercise Show that an ncpu-map $\pi: \mathscr{A} \rightarrow \mathscr{B}$ between von Neumann algebras is a corner for an effect $p$ of $\mathscr{A}$ iff $\pi$ is a corner for $\lfloor p\rfloor$; in which case $\lceil\pi\rceil=\lfloor p\rfloor$. Thus a corner $\pi$ is a corner for $\lceil\pi\rceil$.
VI Exercise Show that the composition $\tau \circ \pi$ of corners $\pi: \mathscr{A} \rightarrow \mathscr{B}$ and $\tau: \mathscr{B} \rightarrow \mathscr{C}$ between von Neumann algebras is again a corner.
(Hint: prove and use the inequality $\lceil\tau\rceil \leqslant\left\lceil\pi\left(\lceil\tau \circ \pi\rceil^{\perp}\right)\right\rceil^{\perp}$.)
VII Theorem Given an ncp-map $f: \mathscr{A} \rightarrow \mathscr{B}$ between von Neumann algebras, a projection $e$ of $\mathscr{A}$ with $\lceil f\rceil \leqslant e$, and a positive element $p$ of $\mathscr{B}$ with $f(1) \leqslant p$, there is a unique ncp-map $g: e \mathscr{A} e \rightarrow\lceil p\rceil \mathscr{B}\lceil p\rceil$ such that

commutes, and it is given by $g(a)=\sqrt{p} \backslash f(a) / \sqrt{p}$ for all $a \in e \mathscr{A} e$.
VIII Proof Uniqueness of $g$ follows from the facts that $\pi_{e}$ is epi and $c_{p}$ is mono in $\mathbf{W}_{\mathrm{CP}}^{*}$, see IV and IT.

Concerning existence, since $\pi_{e}$ is a corner of $e, 951$, and $\lceil f\rceil \leqslant e$, or in other words, $f\left(e^{\perp}\right)=0$, there is a unique ncp-map $h: e \mathscr{A} e \rightarrow \mathscr{B}$ with $h \circ \pi_{e}=f$. Note that $h(a)=f(a)$ for all $a$ from $e \mathscr{A} e$.

As $h(1)=h\left(\pi_{e}(1)\right)=f(1) \leqslant p=c_{p}(1)$, and $c_{p}$ is a filter, 961 , there is a unique ncp-map $g: e \mathscr{A} e \rightarrow p \mathscr{B} p$ with $c_{p} \circ g=h$, which is (by the proof of 96 V given by $g(a)=\sqrt{p} \backslash h(a) / \sqrt{p} \equiv \sqrt{p} \backslash f(a) / \sqrt{p}$ for all $a$ from e $\mathscr{A} e$. Then $c_{p} \circ g \circ \pi_{e}=h \circ \pi_{e}=f$.
IX Corollary Given an ncp-map $f: \mathscr{A} \rightarrow \mathscr{B}$ between von Neumann algebras, there is a unique ncp-map $[f]:\lceil f\rceil \mathscr{A}\lceil f\rceil \rightarrow\lceil f(1)\rceil \mathscr{B}\lceil f(1)\rceil$ such that

commutes; and it is given by $[f](a)=\sqrt{f(1)} \backslash f(a) / \sqrt{f(1)}$ for all $a$ from $\lceil f\rceil \mathscr{A}\lceil f\rceil$.
Moreover, $[f]$ is unital and faithful.

Example For any faithful unital ncp-map $f: \mathscr{A} \rightarrow \mathscr{B}$ we have $[f]=f$. Such X map need not be an isomorphism; as one may take $f:(\lambda, \mu) \mapsto \frac{1}{2}(\lambda+\mu), \mathbb{C}^{2} \rightarrow \mathbb{C}$.
Example In the concrete case that $f \equiv a^{*}(\cdot) a: s \mathscr{A} s \rightarrow t \mathscr{A} t$, where $a$ is an element of a von Neumann algebra, and $s$ and $t$ are projections of $\mathscr{A}$ with $(a\rceil \leqslant s$ and $\lceil a) \leqslant t$, the map $[f]$ is closely related to the polar decomposition $a \equiv[a] \sqrt{a^{*} a}=\sqrt{a a^{*}}[a]$ of $a$, where $[a]=a / \sqrt{a^{*} a}$ (see 821).

Indeed, since $\lceil f\rceil=(a\rceil, f(1)=a^{*} a$, and $[f] \equiv \sqrt{a^{*} a} \backslash a^{*}(\cdot) a / \sqrt{a^{*} a} \equiv$ $[a](\cdot)[a]^{*}$, the picture becomes:


Note that in this example $[f]$ is an ncpu-isomorphism, because $[a]$ is a partial isometry with initial projection $\lceil a)$ and final projection ( $a\rceil$. Thus one can think of the diagram above as an isomorphism theorem of sorts, which applies only to certain ncp-maps that'll be called pure in a moment (see 100 III ).

### 4.1.2 Isomorphism

In case you were wondering, the ncpu-isomorphism we encountered in 98 XI is simply a nmiu-isomorphism (see $\triangle \mathbb{X}$, which follows from the following characterisation of multiplicativity.
Proposition For an ncpu-map $f: \mathscr{A} \rightarrow \mathscr{B}$ between von Neumann algebras the following are equivalent.

1. $f$ is multiplicative.
2. $f(a) f(b)=0$ for all $a, b \in \mathscr{A}$ with $a b=0$.
3. $\lceil f(p)\rceil\lceil f(q)\rceil=0$ for all projections $p$ and $q$ of $\mathscr{A}$ with $p q=0$.
4. $f$ maps projections to projections.
5. $\lceil f(a)\rceil=f(\lceil a\rceil)$ for all $a \in \mathscr{A}_{+}$.

III Proof (Based in part on the work of Gardner in 18]).

and $5 \Rightarrow 4$ are rather obvious.
$\lceil f(a)\rceil \stackrel{\underline{\text { ®VV }}}{ }\lceil f(\lceil a\rceil)\rceil=f(\lceil a\rceil)$ since $f(\lceil a\rceil)$ is a projection.
Let $p$ and $q$ be projections of $\mathscr{A}$ with $p q=0$. Then $p \leqslant q^{\perp}$, and so $f(p) \leqslant f\left(q^{\perp}\right)=f(q)^{\perp}$, which implies that $\lceil f(p)\rceil\lceil f(q)\rceil=f(p) f(q)=0$ since $f(p)$ and $f(q)$ are projections.
VII 3 2 Let $a, b \in \mathscr{A}$ with $a b=0$ be given. We must show that $f(a) f(b)=0$, and for this it suffices to show that $\lceil f(a))(f(b)\rceil=0$, because $f(a) f(b)=$ $f(a)\lceil f(a))(f(b)\rceil f(b)$. Since $a b=0$, we have $\lceil a)(b\rceil=0$ by 60 VIII , and so $\lceil f(\lceil a))\rceil\lceil f((a\rceil)\rceil=0$. Now, since $\lceil f(\lceil a))\rceil \leqslant\lceil f(a))$ and $\lceil f((a\rceil\rceil \leqslant(f(a)\rceil$ by 61 II , we get $\lceil f(a))(f(b)\rceil=\lceil f(a))\lceil f(\lceil a))\rceil\lceil f((a\rceil)\rceil(f(a)\rceil=0$.
VIII 2 1 We must show that $f(a) f(b)=f(a b)$ for all $a, b \in \mathscr{A}$. Since the linear span of projections is norm-dense in $\mathscr{A}$, it suffices to show that $f(a) f(e)=f(a e)$ for any $a \in \mathscr{A}$ and a projection $e$ of $\mathscr{A}$. Given such $a$ and $e$, we on the one hand have $a e^{\perp} e=0$, so that $f\left(a e^{\perp}\right) f(e)=0$, that is, $f(a) f(e)=f(a e) f(e)$; and on the other hand we have ae $e^{\perp}=0$, so that $f(a e) f\left(e^{\perp}\right)=0$, that is, $f(a e)=f(a e) f(e)$; so that we reach $f(a e)=f(a) f(e)$ as sum total, and the result that $f$ is multiplicative.

IX Theorem An ncpsu-isomorphism $f: \mathscr{A} \rightarrow \mathscr{B}$ between von Neumann algebras (so both $f$ and $f^{-1}$ are ncpsu-maps) is an nmiu-isomorphism.
$\times \quad$ Proof Since $f^{-1}(1) \leqslant 1$ and so $1=f\left(f^{-1}(1)\right) \leqslant f(1) \leqslant 1$, we see that $f(1)=1$, so both $f$ and $f^{-1}$ are unital. It remains to be shown that $f$ and $f^{-1}$ are multiplicative. Since by 55 X an effect $a$ of $\mathscr{A}$ is a projection iff 0 is the infimum of $a$ and $a^{\perp}$, and $f$ (as ncpu-isomorphism) preserves $(\cdot)^{\perp}$ and order, we see that $f$ maps projections to projections, and is thus multiplicative, by IT. It follows automatically that $f^{-1}$ is multiplicative too.

XI Exercise Show that any filter of a projection is multiplicative.
(Hint: the filter is a standard filter up to an ncpu-isomorphism, 98 II , which is a nmiu-isomorphism by $\triangle \mathbb{X}$ )
XII Exercise Show that for an ncp-map $f: \mathscr{A} \rightarrow \mathscr{B}$ between von Neumann algebras the following are equivalent.

1. $f$ is multiplicative.
2. $f$ sends projections to projections.
3. $\lceil f(a)\rceil=f(\lceil a\rceil)$ for all $a \in \mathscr{A}_{+}$.
(Hint: factor $f=\zeta \circ h$ where $\zeta$ is a filter for $f(1)$ and $h$ is an ncp-map.)

### 4.1.3 Purity

Definition Filters, corners, and their compositions we'll call pure.
Exercise Show that the following maps are pure.

1. An ncp-isomorphism between von Neumann algebras.
2. The identity map id: $\mathscr{A} \rightarrow \mathscr{A}$ on a von Neumann algebra $\mathscr{A}$.
3. The map $a^{*}(\cdot) a: \mathscr{A} \rightarrow \mathscr{A}$ for any element $a$ of a von Neumann algebra $\mathscr{A}$.

Proposition For an ncp-map $f: \mathscr{A} \rightarrow \mathscr{B}$ between von Neumann algebras the III following are equivalent.

1. $f$ is pure, i.e., $f$ is the composition of (perhaps many) filters and corners.
2. $f=c \circ \pi$ for a filter $c: \mathscr{C} \rightarrow \mathscr{B}$ and a corner $\pi: \mathscr{A} \rightarrow \mathscr{C}$.
3. $[f]$ from 98 IX is an ncpu-isomorphism.

Proof $3 \Longrightarrow 2$ and $2 \Longrightarrow 1$ are rather obvious.
1 2 Calling $f$ properly pure when $f \equiv c \circ \pi$ for some filter $c$ and corner $\pi$, we must show that every pure map is properly pure. For this it suffices to show that the composition of properly pure maps is again properly pure; which, since filters are closed under composition (by 98 III ), and corners are closed under composition (by 98 VI ), boils down to proving that the composition $\pi \circ c$ of a corner $\pi$ and a filter $c$ is properly pure. Since $\pi \equiv \alpha \circ \pi_{\lceil\pi\rceil}$ and $c \equiv c_{c(1)} \circ \beta$ for ncpu-isomorphisms $\alpha$ and $\beta$ (see 98 II and 98 IV ) it suffices to show that $f:=\pi_{s} c_{p}$ is properly pure for a positive element $p$ and a projection $s$ of a von Neumann algebra $\mathscr{A}$. Since such $f$ is of the form $f=s \sqrt{p}(\cdot) \sqrt{p} s:\lceil p\rceil \mathscr{A}\lceil p\rceil \rightarrow s \mathscr{A} s$, we know by 98 XI that $[f]$ is an ncpu-isomorphism, and thus that $f \equiv c_{f(1)} \circ[f] \circ \pi_{\lceil f\rceil}$ is properly pure.
2 2 Recall that $[f]$ is by definition the unique ncp-map with $f=c_{f(1)}[f] \pi_{[f\rceil}$, see 98TX. Note that since $f=c \circ \pi$, we have $\lceil f\rceil=\lceil\pi\rceil$ (because $\lceil c\rceil=1$ by 98 II ),
and $f(1)=c(1)$ (because $\pi(1)=1$ ). Since there are ncpu-isomorphisms $\alpha$ and $\beta$ with $\pi=\alpha \pi_{\lceil\pi\rceil}$ and $c=c_{c(1)} \beta$, we see that $f=c_{c(1)} \beta \alpha \pi_{\lceil\pi\rceil}$, and so $[f]=\beta \alpha$ by definition of $[f]$, so $[f]$ is an ncpu-isomorphism.
VII Exercise Use III to show that

1. a faithful pure map is a filter,
2. a unital pure map is a corner, and
3. a unital and faithful pure map is an ncpu-isomorphism.

### 4.1.4 Contraposition

101 Definition Given an ncp-map $f: \mathscr{A} \rightarrow \mathscr{B}$ between von Neumann algebras we define $f^{\diamond}: \operatorname{Proj}(\mathscr{A}) \rightarrow \operatorname{Proj}(\mathscr{B})$ by $f^{\diamond}(e)=\lceil f(e)\rceil$ for all $e \in \operatorname{Proj}(\mathscr{A})$.
II Proposition Given an ncp-map $f: \mathscr{A} \rightarrow \mathscr{B}$ between von Neumann algebras and a projection $e$ from $\mathscr{B}$ there is a least projection $f_{\diamond}(e)$ from $\mathscr{A}$ with $\left\lceil f\left(f_{\diamond}(e)^{\perp}\right)\right\rceil \leqslant$ $e^{\perp}$, namely $f_{\diamond}(e)=\lceil e f(\cdot) e\rceil$ (being the carrier of the ncp-map $e f(\cdot) e$ from 631); giving a map $f_{\diamond}: \operatorname{Proj}(\mathscr{B}) \rightarrow \operatorname{Proj}(\mathscr{A})$.
III Proof Since by definition $\lceil e f(\cdot) e\rceil$ is the greatest projection $s$ of $\mathscr{A}$ with $e f\left(s^{\perp}\right) e=0$ (see 631); and $e f\left(s^{\perp}\right) e=0$ iff $\left\lceil f\left(s^{\perp}\right)\right\rceil \leqslant\lceil e(\cdot) e\rceil^{\perp} \equiv e^{\perp}$; the projection $\left\lceil e f(\cdot) e \mid\right.$ satisfies the description of $f_{\diamond}(e)$.

IV Exercise Let $f: \mathscr{A} \rightarrow \mathscr{B}$ be an ncp-map between von Neumann algebras.

1. Show that $f^{\diamond}(s) \leqslant t^{\perp}$ iff $f_{\diamond}(t) \leqslant s^{\perp}$, for all $s \in \operatorname{Proj}(\mathscr{A})$ and $t \in \operatorname{Proj}(\mathscr{B})$.
2. Show that $f^{\diamond}(\bigcup E)=\bigcup_{e \in E} f^{\diamond}(e)$ for every set of projections $E$ from $\mathscr{A}$.

V Exercise Show that for ncp-maps $f, g: \mathscr{A} \rightarrow \mathscr{B}$ between von Neumann algebras $f^{\diamond}=g^{\diamond}$ iff $f_{\diamond}=g_{\diamond}$. In that case we say that $f$ and $g$ are equivalent.
VI Show that for ncp-maps $f: \mathscr{A} \rightarrow \mathscr{B}$ and $g: \mathscr{B} \rightarrow \mathscr{A}$ we have $f^{\diamond}=g_{\diamond}$ iff $f_{\diamond}=g^{\diamond}$ iff $\lceil f(s)\rceil \leqslant t^{\perp} \Longleftrightarrow\lceil g(t)\rceil \leqslant s^{\perp}$ for all projections $s$ from $\mathscr{A}$ and $t$ from $\mathscr{B}$.

In that case we say that $f$ and $g$ are contraposed.

## Examples

1. Given an element $a$ of a von Neumann algebra $\mathscr{A}$, the maps $a^{*}(\cdot) a$ and $a(\cdot) a^{*}$ on $\mathscr{A}$ are contraposed.
If $p$ and $q$ are projections of $\mathscr{A}$ with $a^{*} p a \leqslant q$ (as in 94 III), then the maps $a^{*}(\cdot) a: p \mathscr{A} p \rightarrow q \mathscr{A} q$ and $a(\cdot) a^{*}: q \mathscr{A} q \rightarrow p \mathscr{A} p$ are contraposed.
In particular, the standard corner $\pi_{s}: \mathscr{A} \rightarrow s \mathscr{A} s$ and the standard filter $c_{s}: s \mathscr{A} s \rightarrow \mathscr{A}$ for a projection $s$ from $\mathscr{A}$ are contraposed.
2. An ncp-isomorphism $f: \mathscr{A} \rightarrow \mathscr{B}$ between von Neumann algebras is contraposed to its inverse $f^{-1}: \mathscr{B} \rightarrow \mathscr{A}$.
3. There may be many maps equivalent to a given ncp-map $f: \mathscr{A} \rightarrow \mathscr{B}$ between von Neumann algebras: show that $(z f)^{\diamond}=f^{\diamond}$ for every positive central element $z$ of $\mathscr{B}$ with $\lceil z\rceil=1$.

Exercise Let $\mathscr{A} \longrightarrow^{f \rightarrow \mathscr{B}} \longrightarrow^{g \rightarrow \mathscr{C}}$ be ncp-maps between von Neumann algebras $\mathscr{A}, \mathscr{B}$ and $\mathscr{C}$.

1. Show that $(g \circ f)^{\diamond}=g^{\diamond} \circ f^{\diamond}\left(\right.$ using 60V), and $(g \circ f)_{\diamond}=f_{\diamond} \circ g_{\diamond}$.
2. Assuming that $f$ is equivalent to an ncp-map $f^{\prime}: \mathscr{A} \rightarrow \mathscr{B}$ and $g$ is equivalent to an ncp-map $g^{\prime}: \mathscr{B} \rightarrow \mathscr{C}$, show that $g \circ f$ is equivalent to $g^{\prime} \circ f^{\prime}$.
3. Assuming that $f$ is contraposed to an ncp-map $f^{\prime}: \mathscr{B} \rightarrow \mathscr{A}$ and $g$ is contraposed to an ncp-map $g^{\prime}: \mathscr{C} \rightarrow \mathscr{B}$, show that $g \circ f$ is contraposed to $f^{\prime} \circ g^{\prime}$.

Proposition Given ncp-maps $f, g: \mathscr{A} \rightarrow \mathscr{B}$ between von Neumann algebras

$$
(f+g)^{\diamond}(s)=f^{\diamond}(s) \cup g^{\diamond}(s) \quad \text { and } \quad(f+g)_{\diamond}(t)=f_{\diamond}(t) \cup g_{\diamond}(t)
$$

for all $s \in \operatorname{Proj}(\mathscr{A})$ and $t \in \operatorname{Proj}(\mathscr{B})$.
Proof Note that $(f+g)^{\diamond}(s)=\lceil(f+g)(s)\rceil=\lceil f(s)+g(s)\rceil=\lceil f(s)\rceil \cup\lceil g(s)\rceil=$ $f^{\diamond}(s) \cup g^{\diamond}(s)$ by 59III. Since $(f+g)_{\diamond}(t) \leqslant s^{\perp}$ iff $f^{\diamond}(s) \cup g^{\diamond}(s) \equiv(f+g)^{\diamond}(s) \leqslant$ $t^{\perp}$ iff both $f^{\diamond}(s) \leqslant t^{\perp}$ and $g^{\diamond}(s) \leqslant t^{\perp}$ iff both $f_{\diamond}(t) \leqslant s^{\perp}$ and $g_{\diamond}(t) \leqslant s^{\perp}$ iff $f_{\diamond}(t) \cup g_{\diamond}(t) \leqslant s^{\perp}$, we see that $(f+g)_{\diamond}(t)=f_{\diamond}(t) \cup g_{\diamond}(t)$.

Lemma Given contraposed maps $f: \mathscr{A} \rightarrow \mathscr{B}$ and $g: \mathscr{B} \rightarrow \mathscr{A}$ between von XI Neumann algebras, we have $\lceil f\rceil=\lceil g f\rceil$.
Proof $\lceil g f\rceil=(g f)_{\diamond}(1)=f_{\diamond}\left(g_{\diamond}(1)\right)=g^{\diamond}(\lceil g\rceil)=g^{\diamond}(1)=f_{\diamond}(1)=\lceil f\rceil$. $\square$

### 4.1.5 Rigidity

102 We now turn to a remarkable property shared by pure and nmiu-maps.
II Definition We say that an ncp-map $f: \mathscr{A} \rightarrow \mathscr{B}$ between von Neumann algebras is rigid when the only ncp-map $g: \mathscr{A} \rightarrow \mathscr{B}$ with $g(1)=f(1)$ and $\lceil f(p)\rceil=\lceil g(p)\rceil$ for all projections $p$ from $\mathscr{A}$ is $f$ itself.
III Proposition A rigid map $f: \mathscr{A} \rightarrow \mathscr{B}$ between von Neumann algebras is extreme among the ncp-maps $g: \mathscr{A} \rightarrow \mathscr{B}$ with $g(1)=f(1)$.
IV Proof Given $f \equiv \lambda g_{1}+\lambda^{\perp} g_{2}$ where $\lambda \in(0,1)$ and $g_{1}, g_{2}: \mathscr{A} \rightarrow \mathscr{B}$ are ncpmaps with $g_{i}(1)=f(1)$, we must show that $f=g_{1}=g_{2}$. Note that for every projection $s$ of $\mathscr{A}$ we have $f^{\diamond}(s)=\left(\lambda g_{1}+\lambda^{\perp} g_{2}\right)^{\diamond}(s)=g_{1}^{\diamond}(s) \cup g_{2}^{\diamond}(s)$ by 101 IX and 101 VII and in particular $g_{1}^{\diamond}(s) \leqslant f^{\diamond}(s)$. Then for $h:=\lambda g_{1}+\lambda^{\perp} f$ we have $h(1)=f(1)$ and $h^{\diamond}(s)=g_{1}^{\diamond}(s) \cup f^{\diamond}(s)=f^{\diamond}(s)$, so that $\lambda g_{1}+\lambda^{\perp} f \equiv h=f=$ $\lambda g_{1}+\lambda^{\perp} g_{2}$ by rigidity of $f$; and thus $f=g_{2}$. Similarly, $f=g_{1}$.
V Proposition A nmiu-map $\varrho: \mathscr{A} \rightarrow \mathscr{B}$ between von Neumann algebras is rigid.
VI Proof Let $g: \mathscr{A} \rightarrow \mathscr{B}$ be an ncpu-map with $\lceil\varrho(p)\rceil=\lceil g(p)\rceil$ for every projection $p$ of $\mathscr{A}$. To show that $\varrho$ is rigid, we must show that $g=\varrho$, and for this, it suffices to prove that $g(p)=\varrho(p)$ for every projection $p$ of $\mathscr{A}$. To this end, we'll show that $g$ is multiplicative, because then $g$ maps projections to projections, so that $g(p)=\lceil g(p)\rceil=\lceil\varrho(p)\rceil=\varrho(p)$ for every projection $p$ of $\mathscr{A}$. We'll show that $g$ is multiplicative using 9911 by proving that $\lceil g(p)\rceil\lceil g(q)\rceil=0$ for projections $p$ and $q$ of $\mathscr{A}$ with $p q=0$. Indeed, $\lceil g(p)\rceil\lceil g(q)\rceil=\lceil\varrho(p)\rceil\lceil\varrho(q)\rceil=\varrho(p) \varrho(q)=\varrho(p q)=\varrho(0)=0$.
VII Lemma Given an element $b$ of a von Neumann algebra $\mathscr{A}$ the ncp-map $a \mapsto$ $b^{*} a b, \quad(b\rceil \mathscr{A}(b\rceil \rightarrow \mathscr{A}$ is rigid.
VIII Proof Let $g:(b\rceil \mathscr{A}(b\rceil \rightarrow \mathscr{A}$ be an ncp-map with $g(1)=b^{*} b$ and $\left\lceil b^{*} p b\right\rceil=\lceil g(p)\rceil$ for every projection $p$ of $(b\rceil \mathscr{A}(b\rceil$. To prove that $c:=b^{*}(\cdot) b:(b\rceil \mathscr{A}(b\rceil \rightarrow \mathscr{A}$ is rigid, we must show that $g=c$. Since $c$ is a filter (by 96 V ) and $g(1)=b^{*} b$ there is a unique ncp-map $h:(b\rceil \mathscr{A}(b\rceil \rightarrow(b\rceil \mathscr{A}(b\rceil$ with $g=c \circ h$. Our task then is to show that $h=\mathrm{id}$, and for this it suffices to show that, for all $a \in(b\rceil \mathscr{A}(b\rceil$,

$$
\begin{equation*}
e_{n} h\left(e_{n} a e_{n}\right) e_{n}=e_{n} a e_{n} \tag{4.1}
\end{equation*}
$$

for some sequence of projections $e_{1}, e_{2}, \ldots$ of $(b\rceil \mathscr{A}(b\rceil$ that converges ultrastrongly to ( $b 7$, because by 45 VI the left-hand side of the equation above converges ultrastrongly to $g(a)$, while the right-hand side converges ultrastrongly
to $a$. We'll take $e_{N}:=\sum_{n=1}^{N}\left\lceil t_{n}\right)$, where $t_{1}, t_{2}, \ldots$ is an approximate pseudoinverse for $b$, because $(b\rceil=\sum_{n}\left\lceil t_{n}\right)$.

Since the identity on $e_{n} \mathscr{A} e_{n}$ is rigid by V , it suffices (for (4.1)) to show that $e_{n} h\left(e_{n}\right) e_{n}=e_{n}$ and $\left\lceil e_{n} h(p) e_{n}\right\rceil=p$ for every projection $p$ from $e_{n} \mathscr{A} e_{n}$. Writing $s_{N}:=\sum_{n=1}^{N} t_{n}$, we have $b s_{n}=e_{n}$, and so $\left\lceil e_{n} h(p) e_{n}\right\rceil=\left\lceil s_{n}^{*} b^{*} h(p) b s_{n}\right\rceil=$ $\left\lceil s_{n}^{*} g(p) s_{n}\right\rceil=\left\lceil s_{n}^{*}\lceil g(p)\rceil s_{n}\right\rceil=\left\lceil s_{n}^{*}\left\lceil b^{*} p b\right\rceil s_{n}\right\rceil=\left\lceil s_{n}^{*} b^{*} p b s_{n}\right\rceil=\left\lceil e_{n} p e_{n}\right\rceil$ for every projection $p$ from $(b\rceil \mathscr{A}(b\rceil$. In particular, $\left\lceil e_{n} h(p) e_{n}\right\rceil=p$ when $p$ is from $e_{n} \mathscr{A} e_{n}$; and we see $\left\lceil e_{n} h\left(e_{n}^{\perp}\right) e_{n}\right\rceil=\left\lceil e_{n} e_{n}^{\perp} e_{n}\right\rceil=0$ when we take $p=e_{n}^{\perp}$, so that $e_{n} h\left(e_{n}^{\perp}\right) e_{n}=0$, which yields $e_{n} h\left(e_{n}\right) e_{n}=e_{n}$.

Theorem Every pure map between von Neumann algebras is rigid.
Proof Let $f: \mathscr{A} \rightarrow \mathscr{B}$ be a pure map between von Neumann algebras, and let $g: \mathscr{A} \rightarrow \mathscr{B}$ be an ncp-map with $f(1)=g(1)$ and $f^{\diamond}=g^{\triangleright}$. To show that $f$ is rigid, we must prove that $f=g$. We know by 98 IX that $f$ can be written as $f \equiv c_{f(1)} \circ[f] \circ \pi_{\lceil f\rceil}$, and that $c_{f(1)}$ is rigid, by VII. which we'll use shortly. To this end, note that since $f^{\diamond}=g^{\diamond}$, we have $f_{\diamond}=g_{\diamond}$, and so $\lceil f\rceil=f_{\diamond}(1)=g_{\diamond}(1)=\lceil g\rceil$. As $\pi_{\lceil f\rceil}$ is a corner of $\lceil f\rceil=\lceil g\rceil$, there is a unique ncp-map $h:\lceil f\rceil \mathscr{A}\lceil f\rceil \rightarrow \mathscr{B}$ with $h \circ \pi_{\lceil f\rceil}=g$. Since then $h^{\diamond} \circ \pi_{\lceil f\rceil}^{\diamond}=g^{\diamond}=f^{\diamond}=c_{f(1)}^{\diamond} \circ[f]^{\diamond} \circ \pi_{\lceil f\rceil}^{\diamond}$, and $\pi_{\lceil f\rceil}^{\diamond}$ is clearly surjective, we get $h^{\diamond}=c_{f(1)}^{\diamond} \circ[f]^{\diamond}$, and thus $\left(h \circ[f]^{-1}\right)^{\diamond}=c_{f(1)}^{\diamond}$, using here that $[f]$ is invertible, because $f$ is pure. Now, using that $c_{f(1)}$ is rigid, and $h\left([f]^{-1}(1)\right)=h(1)=h\left(\pi_{\lceil f\rceil}(1)\right)=g(1)=f(1)=c_{f(1)}(1)$, we get $h \circ[f]^{-1}=$ $c_{f(1)}$, which yields $g=h \circ \pi_{\lceil f\rceil}=h \circ[f]^{-1} \circ[f] \circ \pi_{\lceil f\rceil}=c_{f(1)} \circ[f] \circ \pi_{\lceil f\rceil}=f$, and thus $f$ is rigid.

### 4.1.6 $\diamond$-Positivity

Definition We'll call an ncp-map $f: \mathscr{A} \rightarrow \mathscr{A}$ between von Neumann algebras

1. $\diamond$-self-adjoint if $f$ is pure and contraposed to itself $\left(f^{\diamond}=f_{\diamond}\right)$, and
2. $\diamond$-positive if $f \equiv g g$ for some $\diamond$-self-adjoint map $g: \mathscr{A} \rightarrow \mathscr{A}$.

We added " $\diamond$-" to "positive" not only to distinguish it from the pre-existing notion of positivity for maps between $C^{*}$-algebras, but also to contrast it with the notion of " $\dagger$-positivity" that appears in the following thesis (see 2141).

Examples Let $\mathscr{A}$ be a von Neumann algebra.

1. Given $a \in \mathscr{A} \mathbb{R}$ the map $a(\cdot) a: \mathscr{A} \rightarrow \mathscr{A}$ is $\diamond$-self-adjoint.
2. Given $a \in \mathscr{A}_{+}$the map $a(\cdot) a: \mathscr{A} \rightarrow \mathscr{A}$ is $\diamond$-positive.

III Exercise Let $f: \mathscr{A} \rightarrow \mathscr{A}$ be an ncp-map, where $\mathscr{A}$ is a von Neumann algebra.

1. Show that $\lceil f\rceil=\lceil f(1)\rceil$ when $f$ is $\diamond$-self-adjoint.
2. Assuming $f$ is $\diamond$-self-adjoint, show that $f f$ is $\diamond$-self-adjoint, and show that $\lceil f f\rceil=\lceil f\rceil$ (cf. 101 XI).
3. Show that $f$ is $\diamond$-self-adjoint when $f$ is $\diamond$-positive.

104 We now turn to the question roughly speaking to what extent a filter $c$ is determined by its action $c^{\diamond}: e \mapsto\lceil c(e)\rceil$ on projections; we will see (essentially in VII) that two filters $c_{1}$ and $c_{2}$ are equivalent, $c_{1}^{\diamond}=c_{2}^{\diamond}$, if and only if $c_{1}(1)$ and $c_{2}(1)$ are equal up to some central elements, that is, centrally similar.
II Definition We say that positive elements $p$ and $q$ of a von Neumann algebra $\mathscr{A}$ are centrally similar if $c p=d q$ for some positive central elements $c$ and $d$ of $\mathscr{A}$ with $\lceil p\rceil \leqslant\lceil c\rceil$ and $\lceil q\rceil \leqslant\lceil d\rceil$.

III Exercise Let $p$ and $q$ be positive elements of a von Neumann algebra $\mathscr{A}$.

1. Show that when $p$ and $q$ are centrally similar, every element $a$ of $\mathscr{A}$ that commutes with $p$ commutes with $q$ too; and in particular, $p q=q p$.
2. Show that when $p$ and $q$ are centrally similar, $\lceil p\rceil=\lceil q\rceil$.
3. Show that when $p$ and $q$ commute, and both $\frac{p \wedge q}{p}$ and $\frac{p \wedge q}{q}$ are central, $p$ and $q$ are centrally similar.
4. Show that when $p$ and $q$ are pseudoinvertible, then: $p$ and $q$ are centrally similar iff $p q^{\sim 1}$ is central iff $q p^{\sim 1}$ is central iff both $(p \wedge q) p^{\sim 1}$ and $(p \wedge q) q^{\sim 1}$ are central.
5. Assuming that $p$ and $q$ commute and $e_{1} \leqslant e_{2} \leqslant \cdots$ are projections commuting with $p$ and $q$ with $\bigcup_{n} e_{n}=\lceil p\rceil$ such that the $e_{n} p$ and $e_{n} q$ are pseudoinvertible, and centrally similar, show that $p$ and $q$ are centrally similar on the grounds that both $\frac{p \wedge q}{p}$ and $\frac{p \wedge q}{q}$ are central.
(Hint: $e_{n} \frac{p \wedge q}{p}=\frac{\left(e_{n} p\right) \wedge\left(e_{n} q\right)}{e_{n} p}$ are central, and converge ultraweakly to $\frac{p \wedge q}{p}$.)

Lemma Suppose that $\lceil q \vartheta(e) q\rceil \leqslant e$ and $\left\lceil q \vartheta\left(e^{\perp}\right) q\right\rceil \leqslant e^{\perp}$, where $e$ is a projection of a von Neumann algebra $\mathscr{A}, q$ is a positive element of $\mathscr{A}$, and $\vartheta: \mathscr{A} \rightarrow \mathscr{A}$ is a miu-map. Then $e q=q e$ and $\vartheta(e)=e$.
Proof We have $\vartheta(e) q e=\vartheta(e) q$, because $e \geqslant\lceil q \vartheta(e) q\rceil \equiv\lceil\vartheta(e) q)$ (see 59 VI$)$. Similarly, $\vartheta\left(e^{\perp}\right) q e^{\perp}=\vartheta\left(e^{\perp}\right) q$, because $e^{\perp} \geqslant\left\lceil q \vartheta\left(e^{\perp}\right) q\right\rceil \equiv\left\lceil\vartheta\left(e^{\perp}\right) q\right)$, and so $\vartheta\left(e^{\perp}\right) q e=0$, which implies $\vartheta(e) q e=q e$. Thus $q e=\vartheta(e) q e=\vartheta(e) q$, and so $q^{2} e=q \vartheta(e) q$ is self-adjoint, which gives us that $q^{2} e=\left(q^{2} e\right)^{*}=e q^{2}$. Since $q^{2}$ commutes with $e, q=\sqrt{q^{2}}$ commutes with $e$ too (see 23 VIII ). Finally, $\vartheta(e) q=q e=e q$ and $\lceil q\rceil=1$ imply that $\vartheta(e)=e$ by 60 VIII .

Corollary A positive element $q$ of a von Neumann algebra $\mathscr{A}$ with $\lceil q\rceil=1$ is central provided there is a miu-map $\vartheta: \mathscr{A} \rightarrow \mathscr{A}$ with $\lceil q \vartheta(e) q\rceil \leqslant e$ for every projection $e$ from $\mathscr{A}$; and in that case $\vartheta=\mathrm{id}$.

Proposition Positive elements $p$ and $q$ of a von Neumann algebra $\mathscr{A}$ with $\lceil p\rceil=$ $\lceil q\rceil=1$ are centrally similar when there is a miu-isomorphism $\vartheta: \mathscr{A} \rightarrow \mathscr{A}$ with $\lceil p e p\rceil=\lceil q \vartheta(e) q\rceil$ for all projections $e$ of $\mathscr{A}$; and in that case $\vartheta=\mathrm{id}$.
Proof Let $e$ be a projection from $\mathscr{A}$ with $e p=p e$. Since $1=\lceil p\rceil=\left\lceil p^{2}\right\rceil$ we have $e=\left\lceil e\left\lceil p^{2}\right\rceil e\right\rceil=\left\lceil e p^{2} e\right\rceil=\lceil p e p\rceil=\lceil q \vartheta(e) q\rceil$. Since $e^{\perp}$ commutes with $p$ too, we get $e^{\perp}=\left\lceil q \vartheta\left(e^{\perp}\right) q\right\rceil$ by the same token; and thus $e q=q e$ and $\vartheta(e)=e$ by IV. Since $p$ is the norm limit of linear combinations of such projections $e$, we get $p q=q p$ and $\vartheta(p)=p$.

Since $p$ and $q$ commute, we can find a sequence of projections $e_{1} \leqslant e_{2} \leqslant \cdots$ that commute with $p$ and $q$ with $\bigcup_{n} e_{n}=\lceil p\rceil$ and such that $p e_{n}$ and $q e_{n}$ are pseudoinvertible - one may, for example, take $e_{N}:=\sum_{n=1}^{N}\left\lceil t_{n}\right\rceil$ where $t_{1}, t_{2}, \ldots$ is an approximate pseudoinverse of $p \wedge q$ (see 80 IV ). Note that to prove that $p$ and $q$ are centrally similar, it suffices to show that $p e_{n}$ and $q e_{n}$ are centrally similar, by III. Further, to prove that $\vartheta(a)=a$ for some $a \in \mathscr{A}$, it suffices to show that $\vartheta\left(e_{n} a e_{n}\right)=e_{n} a e_{n}$, because $e_{n} a e_{n}$ converges ultrastrongly to $a$ by 45 VI , Note that $\vartheta\left(e_{n}\right)=e_{n}$, because $e_{n} p=p e_{n}$, and so $\vartheta$ maps $e_{n} \mathscr{A} e_{n}$ into $e_{n} \mathscr{A} e_{n}$.

Thus, by considering $e_{n} \mathscr{A} e_{n}$ instead of $\mathscr{A}$, and the restriction of $\vartheta$ to $e_{n} \mathscr{A} e_{n}$ instead of $\vartheta$, and $p e_{n}$ and $q e_{n}$ instead of $p$ and $q$, we reduce the problem to the case that $p$ and $q$ are invertible; and so we may assume without loss of generality that $p$ and $q$ are invertible to start with. Given a projection $e$ from $\mathscr{A}$ we have $\left\lceil p^{-1} q \vartheta(e) q p^{-1}\right\rceil=\left\lceil p^{-1}\lceil q \vartheta(e) q\rceil p^{-1}\right\rceil=\left\lceil p^{-1}\lceil p e p\rceil p^{-1}\right\rceil=e$; so by VI, we get that $\vartheta=\mathrm{id}$ and $p^{-1} q$ is central; and so $p$ and $q$ are centrally similar (by III).

Proposition A faithful $\diamond$-positive map $f: \mathscr{A} \rightarrow \mathscr{A}$ on a von Neumann algebra $\mathscr{A}$ IX is of the form $f=\sqrt{p}(\cdot) \sqrt{p}$ where $p:=f(1)$.
$\times$ Proof Note that $f$, being faithful and pure, is a filter (by 100 VII , and thus of the form $f \equiv \sqrt{p} \vartheta(\cdot) \sqrt{p}$ for some isomorphism $\vartheta: \mathscr{A} \rightarrow \mathscr{A}$. Our task then is to show that $\vartheta=\mathrm{id}$, and for this it suffices, by VII , to find some positive $q$ in $\mathscr{A}$ with $\lceil q\rceil=1$ and $f^{\diamond}(e) \equiv\lceil\sqrt{p} \vartheta(e) \sqrt{p}\rceil=\lceil q e q\rceil$ for all projections $e$ in $\mathscr{A}$.

Since $f$ is $\diamond$-positive, we have $f \equiv \xi \xi$ for some $\diamond$-self-adjoint map $\xi: \mathscr{A} \rightarrow$ $\mathscr{A}$. Since $1=\lceil f\rceil=f_{\diamond}(1)=\xi_{\diamond}\left(\xi_{\diamond}(1)\right) \leqslant \xi_{\diamond}(1)=\lceil\xi\rceil$ we have $\lceil\xi\rceil=$ 1 , and so, $\xi$, being pure and faithful, is a filter (by 100 VII$)$. Furthermore, as $\tilde{\xi}:=\sqrt{\xi(1)}(\cdot) \sqrt{\xi(1)}: \mathscr{A} \rightarrow \mathscr{A}$ is a filter of $\xi(1)$ too, there is an isomorphism $\alpha: \mathscr{A} \rightarrow \mathscr{A}$ with $\xi=\tilde{\xi} \alpha$. Now, $\tilde{\xi}_{\tilde{\beta}} \alpha^{\diamond}=\xi^{\diamond}=\xi_{\diamond}=\alpha_{\diamond} \tilde{\xi}_{\diamond}=\left(\alpha^{\diamond}\right)^{-1} \tilde{\xi}^{\diamond}$ implies $\tilde{\xi}^{\diamond}=\alpha^{\diamond} \tilde{\xi}^{\diamond} \alpha^{\diamond}$, and $f^{\diamond}=(\xi \xi)^{\diamond}=\tilde{\xi}^{\diamond} \alpha^{\diamond} \tilde{\xi}^{\diamond} \alpha^{\diamond}=\tilde{\xi}^{\diamond} \tilde{\xi}^{\diamond}=(\tilde{\xi} \tilde{\xi})^{\diamond}$. In other words, $\lceil\sqrt{p} \vartheta(e) \sqrt{p}\rceil=f^{\diamond}(e)=(\tilde{\xi} \tilde{\xi})^{\diamond}(e)=\lceil\xi(1) e \xi(1)\rceil$ for all projections $e$ of $\mathscr{A}$, which implies that $\vartheta=$ id by VII and hence that $f=\sqrt{p}(\cdot) \sqrt{p}$.

105 To strip from 104 IX the assumption that $f$ be faithful we employ this device:
II Definition Given an ncp-map $f: \mathscr{A} \rightarrow \mathscr{B}$ between von Neumann algebras we denote by $\langle f\rangle:\lceil f\rceil \mathscr{A}\lceil f\rceil \rightarrow\lceil f(1)\rceil \mathscr{B}\lceil f(1)\rceil$ the unique ncp-map such that

commutes. (Compare this with the definition of $[f]$ in 98 IX .)
III Exercise Let $f: \mathscr{A} \rightarrow \mathscr{B}$ be an ncp-map.

1. Show that $\langle f\rangle=\pi_{\lceil f(1)\rceil} \circ f \circ c_{\lceil f\rceil}$ (using, perhaps, that $\pi_{\lceil f\rceil} \circ c_{\lceil f\rceil}=\mathrm{id}$ ).
2. Show that $\langle f\rangle=\pi_{\lceil f(1)\rceil} \circ c_{f(1)} \circ[f]$.
(Thus $\langle f\rangle(a)=\sqrt{f(1)}[f\rceil(a) \sqrt{f(1)}$ for all $a$ from $\lceil f\rceil \mathscr{A}\lceil f\rceil$.)
3. Show that $\langle f\rangle$ is faithful, and $\langle f\rangle(1)=f(1)$.
4. Assuming that $f$ is pure, show that $\langle f\rangle$ is pure, and hence a filter (by 100 VIII ).

IV Exercise Let $f: \mathscr{A} \rightarrow \mathscr{A}$ be an ncp-map, where $\mathscr{A}$ is a von Neumann algebra.

1. Suppose that $f$ is $\diamond$-self-adjoint.

Recall that $\lceil f\rceil=\lceil f(1)\rceil$, and so $\langle f\rangle:\lceil f\rceil \mathscr{A}\lceil f\rceil \rightarrow\lceil f\rceil \mathscr{A}\lceil f\rceil$.
Prove that $\langle f\rangle$ is $\diamond$-self-adjoint.
2. Suppose again that $f$ is $\diamond$-self-adjoint, and recall from 103 III that $f^{2}$ is $\diamond$-self-adjoint, and $\left\lceil f^{2}\right\rceil=\lceil f\rceil$. Show that $\left\langle f^{2}\right\rangle=\langle f\rangle^{2}$.
3. Assuming that $f$ is $\diamond$-positive, show that $\langle f\rangle$ is $\diamond$-positive.

Theorem Given a positive element $p$ of a von Neumann algebra $\mathscr{A}$ there is a unique $\diamond$-positive map $f: \mathscr{A} \rightarrow \mathscr{A}$ with $f(1)=p$, namely $f=\sqrt{p}(\cdot) \sqrt{p}$.
Proof We've already seen in 1031 that $f=\sqrt{p}(\cdot) \sqrt{p}: \mathscr{A} \rightarrow \mathscr{A}$ is a $\diamond-$ positive map with $f(1)=p$. Concerning uniqueness, (given arbitrary $f$ ) the $\operatorname{map}\langle f\rangle:\lceil p\rceil \mathscr{A}\lceil p\rceil \rightarrow\lceil p\rceil \mathscr{A}\lceil p\rceil$ from $\Pi$ is $\diamond$-positive by $\mathbb{I V}$, and faithful by III, and so of the form $\langle f\rangle=\sqrt{p}(\cdot) \sqrt{p}$ by 104 IX (since $\langle f\rangle(1)=f(1)=p$ ); implying that $f=c_{\lceil p\rceil} \circ\langle f\rangle \circ \pi_{\lceil p\rceil}=\sqrt{p}\lceil p\rceil(\cdot)|p| \sqrt{p}=\sqrt{p}(\cdot) \sqrt{p}$.
Corollary ("Square Root Axiom") Given a positive element $p$ of a von Neumann algebra $\mathscr{A}$ there is a unique $\diamond$-positive map $g: \mathscr{A} \rightarrow \mathscr{A}$ with $g(g(1))=p$, namely $g=\sqrt[4]{p}(\cdot) \sqrt[4]{p}$.
Proof Any $\diamond$-positive map $g: \mathscr{A} \rightarrow \mathscr{A}$ with $g(g(1))=p$ will be of the form $g=\sqrt{g(1)}(\cdot) \sqrt{g(1)}$ by V so that $p=g(g(1))=g(1)^{2}$ implies that $g(1)=\sqrt{p}$ by 23 VII , thereby giving $g=\sqrt[4]{p}(\cdot) \sqrt[4]{p}$.

Theorem On the effects of every von Neumann algebra $\mathscr{A}$ there is a unique binary operation $*$ such that for all $p$ from $[0,1]_{\mathscr{A}}$,
A. $p * 1=p$,
B. $p * q=f(q)$ for all $q$ from $[0,1]_{\mathscr{A}}$ for some pure map $f: \mathscr{A} \rightarrow \mathscr{A}$,
C. $p *(p * q)=(p * p) * q$ for all $q$ from $[0,1]_{\mathscr{A}}$,
D. $p=q * q$ for some $q$ from $[0,1]_{\mathscr{A}}$,
E. $p * e_{1} \leqslant e_{2}^{\perp} \Longleftrightarrow p * e_{2} \leqslant e_{1}^{\perp}$ for all projections $e_{1}, e_{2}$ from $\mathscr{A}$;
namely, the sequential product, given by $p * q=\sqrt{p} q \sqrt{p}$ for all $p, q$ from $[0,1]_{\mathscr{A}}$. Proof Let $p$ from $[0,1]_{\mathscr{A}}$ be given. Pick $p^{\prime}$ from $[0,1]_{\mathscr{A}}$ with $p=p^{\prime} * p^{\prime}$ using D and find a pure map $f: \mathscr{A} \rightarrow \mathscr{A}$ with $f(q)=p^{\prime} * q$ for all $q$ from $[0,1]_{\mathscr{A}}$ using B. Then $f$ is $\diamond$-self-adjoint by E and so $f f$ is $\diamond$-positive. Since $f(f(1))=$ $p^{\prime} *\left(p^{\prime} * 1\right)=p^{\prime} * p^{\prime}=p$ by $f f=\sqrt{p}(\cdot) \sqrt{p}$ by 105 V so $p * q=\left(p^{\prime} * p^{\prime}\right) * q=$ $p^{\prime} *\left(p^{\prime} * q\right)=f(f(q))=\sqrt{p q} \sqrt{p}$ for all $q \in[0,1]_{\mathscr{A}}$ by C

III Exercise None of the axioms from I may be omitted (except perhaps D, see IV):

1. Show that $p * q:=\lceil p\rceil q\lceil p\rceil$ satisfies all axioms of $\prod$ except A .
2. Show that $p * q:=\lfloor p\rfloor q\lfloor p\rfloor+\sqrt{p-\lfloor p\rfloor} q \sqrt{p-\lfloor p\rfloor}$ satisfies all axioms except B
3. Show that if for every effect $p$ of $\mathscr{A}$ we pick a unitary $u_{p}$ from $\lceil p\rceil \mathscr{A}\lceil p\rceil$ then $*$ given by $p * q=\sqrt{p} u_{p}^{*} q u_{p} \sqrt{p}$ satisfies A and B
Show that this $*$ obeys C when $u_{p}^{2}=u_{p^{2}}$, and D when $p u_{p}=u_{p} p$, and E when $u_{p}^{*}=u_{p}$.
Conclude that when $u_{p}$ is defined by $u_{p}:=g(p)$, where $g:[0,1] \rightarrow\{-1,1\}$ is any Borel function with $g(2 / 3)=1$ and $g(4 / 9)=-1$ the operation $*$ (defined by $u_{p}$ as above) satisfies all conditions of $\square$ except $C$.
4. Show that there is a Borel function $g:[0,1] \rightarrow S^{1}$ with $g(1 / 2) \neq 1$ and $g\left(\lambda^{2}\right)=$ $g(\lambda)^{2}$ for all $\lambda \in[0,1]$, and that $*$ given by $p * q=\sqrt{p} g(p)^{*} q g(p) \sqrt{p}$ satisfies all conditions of Dexcept $E$

IV Problem Is there a binary operation $*$ on the effects $[0,1]_{\mathscr{A}}$ of a von Neumann algebra $\mathscr{A}$ that satisfies all axioms of $\square$ except $D$ ?
V Remark The axioms for the sequential product (on a single von Neumann algebra) presented here (in <br>) evolved from the following axioms for all sequential products on von Neumann algebras $\left(*_{\mathscr{A}}\right)_{\mathscr{A}}$ we previously published in 72 .

Ax. 1 For every effect $p$ of a von Neumann algebra $\mathscr{A}$ there is a filter $c: \mathscr{C} \rightarrow \mathscr{A}$ of $p$ and a corner $\pi: \mathscr{A} \rightarrow \mathscr{C}$ of $\lfloor p\rfloor$ with $p *_{\mathscr{A}} q=c(\pi(q))$ for all $q \in[0,1]_{\mathscr{A}}$.

Ax. $2 p *_{\mathscr{A}}\left(p *_{\mathscr{A}} q\right)=\left(p *_{\mathscr{A}} p\right) *_{\mathscr{A}} q$ for all effects $p$ and $q$ from a von Neumann algebra $\mathscr{A}$.

Ax. $3 f\left(p *_{\mathscr{A}} q\right)=f(p) *_{\mathscr{B}} f(q)$ for every nmisu-map $f: \mathscr{A} \rightarrow \mathscr{B}$ between von Neumann algebras and all effects $p$ and $q$ from $\mathscr{A}$.

Ax. $4 p *_{\mathscr{A}} e_{1} \leqslant e_{2}^{\perp} \Longleftrightarrow p *_{\mathscr{A}} e_{2} \leqslant e_{1}^{\perp}$ for every effect $p$ from a von Neumann algebra $\mathscr{A}$ and projections $e_{1}$ and $e_{2}$ from $\mathscr{A}$.

Note that Ax. 2 and Ax. 4 are mutatis mutandis the same as axioms C and E, respectively, and Ax. 1 is essentially the same as the combination of axioms A and B. In other words, we managed to get rid of Ax.3-and with it the need to
axiomatise all sequential products simultaneously - at the slight cost of adding axiom D, though that one might be superfluous as well (see IV).

We refer to $\S$ VI of $[72$ for comments on the relation of our axioms with those of Gudder and Latémolière [22] and for some more pointers to the literature.

### 4.2 Tensor product

The tensor product of von Neumann algebras $\mathscr{A}$ and $\mathscr{B}$ represented on Hilbert spaces $\mathscr{H}$ and $\mathscr{K}$, respectively, is usually defined as the von Neumann subalgebra of $\mathscr{B}(\mathscr{H} \otimes \mathscr{K})$ generated by the operators on $\mathscr{H} \otimes \mathscr{K}$ of the form $A \otimes B$ where $A \in \mathscr{A}$ and $B \in \mathscr{B}$. In line with the representation-avoiding treatment of von Neumann algebras from the previous chapter we'll take an entirely different approach by defining the tensor product of von Neumann algebras $\mathscr{A}$ and $\mathscr{B}$ abstractly as a miu-bilinear map $\otimes: \mathscr{A} \times \mathscr{B} \rightarrow \mathscr{A} \otimes \mathscr{B}$ whose range generates $\mathscr{A} \otimes \mathscr{B}$ and admits sufficiently many product functionals (see 108 II ); we'll only resort to the concrete representation of the tensor product mentioned above to show that such an abstract tensor product actually exists (see 111 VII ).

Moreover, we'll show that the tensor product has a universal property 112 XI yielding bifunctors on $\mathbf{W}_{\text {CPSU }}^{*}$ and $\mathbf{W}_{\text {MIU }}^{*}$ (see 115 IV ) turning them into a monoidal categories (see 119 V ). In the next chapter, we'll see that $\mathbf{W}_{\text {MIU }}^{*}$ is even monoidal closed (see 125 VIIII . This fact is one ingredient of our model for the quantum lambda calculus from [9] built of von Neumann algebras, but more of that later.

### 4.2.1 Definition

Definition A bilinear map $\beta: \mathscr{A} \times \mathscr{B} \rightarrow \mathscr{C}$ between von Neumann algebras is

1. unital when $\beta(1,1)=1$,
2. multiplicative if $\beta(a b, c d)=\beta(a, c) \beta(b, d)$ for all $a, b \in \mathscr{A}, c, d \in \mathscr{B}$,
3. involution preserving if $\beta(a, b)^{*}=\beta\left(a^{*}, b^{*}\right)$ for all $a \in \mathscr{A}, b \in \mathscr{B}$.
4. (This list is extended in 112 II)

We abbreviate these properties as in 10 II and say, for instance, that $\beta$ is miubilinear when it is unital, multiplicative and involution preserving.

II Definition A miu-bilinear map $\gamma: \mathscr{A} \times \mathscr{B} \rightarrow \mathscr{T}$ between von Neumann algebras is a tensor product of $\mathscr{A}$ and $\mathscr{B}$ when it obeys the following three conditions.

1. The range of $\gamma$ generates $\mathscr{T}$ (which means in this case that the linear span of the range of $\gamma$ is ultraweakly dense in $\mathscr{T}$.)
This implies that for all $f \in \mathscr{A}_{*}$ and $g \in \mathscr{B}_{*}$ there is at most one $h \in \mathscr{T}_{*}$ with, for all $a \in \mathscr{A}$ and $b \in \mathscr{B}$,

$$
h(\gamma(a, b))=f(a) g(b)
$$

which we'll call the product functional for $f$ and $g$, and denote by $\gamma(f, g)$ (when it exists).
2. For all np-functionals $\sigma: \mathscr{A} \rightarrow \mathbb{C}$ and $\tau: \mathscr{B} \rightarrow \mathbb{C}$ the product functional $\gamma(\sigma, \tau): \mathscr{T} \rightarrow \mathbb{C}$ exists and is positive.
3. The product functionals $\gamma(\sigma, \tau)$ of np-functionals $\sigma$ and $\tau$ form a faithful collection of np-functionals on $\mathscr{T}$.
(We'll see a slightly different characterisation of the tensor in which not all product functionals of np-functionals are required to exists upfront in 116 VII .)

III Remark This compact definition of the tensor product leaves four questions unanswered: whether such a tensor product of two von Neumann algebras always exists, whether it has some universal property, whether it is unique in some way, and whether it coincides with the usual definition. We'll shortly address all four questions.

### 4.2.2 Existence

109 We'll start with the existence of a tensor product of von Neumann algebras for which we'll first need the tensor product of Hilbert spaces.
II Definition We'll call a bilinear map $\gamma: \mathscr{H} \times \mathscr{K} \rightarrow \mathscr{T}$ between Hilbert spaces a tensor product when it obeys the following two conditions.

1. The linear span of the range of $\gamma$ is dense in $\mathscr{T}$.
2. $\left\langle\gamma(x, y), \gamma\left(x^{\prime}, y^{\prime}\right)\right\rangle=\left\langle x, x^{\prime}\right\rangle\left\langle y, y^{\prime}\right\rangle$ for all $x, x^{\prime} \in \mathscr{H}$ and $y, y^{\prime} \in \mathscr{K}$.

Exercise We're going to prove that every pair of Hilbert spaces $\mathscr{H}$ and $\mathscr{K}$ admits a tensor product.

1. Given sets $X$ and $Y$ show that $\gamma: \ell^{2}(X) \times \ell^{2}(Y) \rightarrow \ell^{2}(X \times Y)$ given by

$$
\gamma(f, g)=(f(x) g(y))_{x \in X, y \in Y}
$$

is a tensor product of $\ell^{2}(X)$ and $\ell^{2}(Y)$.
2. Show that a subset $\mathscr{E}$ of a Hilbert space $\mathscr{H}$ is an orthonormal basis (see 39 IV ) iff the map $T: \ell^{2}(\mathscr{E}) \rightarrow \mathscr{H}$ given by $T(x)=\sum_{e \in \mathscr{E}} x_{e} e$ is an isometric isomorphism.
3. Show that any pair $\mathscr{H}$ and $\mathscr{K}$ of Hilbert spaces has a tensor product (using the fact that every Hilbert space has an orthonormal basis).

Proposition Let $\gamma: \mathscr{H} \times \mathscr{K} \rightarrow \mathscr{T}$ be a tensor product of Hilbert spaces.

1. We have $\|\gamma(x, y)\|=\|x\|\|y\|$ for all $x \in \mathscr{H}$ and $y \in \mathscr{K}$.
2. Given orthonormal bases $\mathscr{E}$ and $\mathscr{F}$ of $\mathscr{H}$ and $\mathscr{K}$, respectively, the set

$$
\mathscr{G}:=\{\gamma(e, f): e \in \mathscr{E}, f \in \mathscr{F}\}
$$

is an orthonormal basis for $\mathscr{T}$.

Proof 11 We have $\|\gamma(x, y)\|^{2}=\langle\gamma(x, y), \gamma(x, y)\rangle=\langle x, x\rangle\langle y, y\rangle=\|x\|^{2}\|y\|^{2}$.
2 Since $\left\langle\gamma\left(e, e^{\prime}\right), \gamma\left(f, f^{\prime}\right)\right\rangle=\left\langle e, e^{\prime}\right\rangle\left\langle f, f^{\prime}\right\rangle$ where $e, e^{\prime} \in \mathscr{E}$ and $f, f^{\prime} \in \mathscr{F}$, the set $\mathscr{G}$ is clearly orthonormal. To see that $\mathscr{G}$ is maximal (and thus a basis) it suffices to show that the span of $\mathscr{G}$ is dense in $\mathscr{T}$, and for this it suffices to show that each $\gamma(x, y)$ where $x \in \mathscr{H}$ and $y \in \mathscr{K}$ is in the closure of the span of $\mathscr{G}$. Now, since $y=\sum_{f \in \mathscr{F}}\langle f, y\rangle f$, by 39 IV and $\langle x,(\cdot)\rangle$ is bounded by 1 we have $\gamma(x, y)=\sum_{f \in \mathscr{F}}\langle y, f\rangle \gamma(x, f)$. Since similarly $\gamma(x, f)=\sum_{e \in \mathscr{E}}\langle e, x\rangle \gamma(e, f)$ for all $f \in \mathscr{F}$, we see that $\gamma(x, y)$ is indeed in the closure of the span of $\mathscr{G}$.

Definition We'll say that a bilinear map $\eta: \mathscr{H} \times \mathscr{K} \rightarrow \mathscr{L}$ between Hilbert spaces is $\ell^{2}$-bounded by $B \in[0, \infty)$ when

$$
\left\|\sum_{i} \beta\left(x_{i}, y_{i}\right)\right\|^{2} \leqslant B^{2} \sum_{i, j}\left\langle x_{i}, x_{j}\right\rangle\left\langle y_{i}, y_{j}\right\rangle
$$

for all $x_{1}, \ldots, x_{n} \in \mathscr{H}$ and $y_{1}, \ldots, y_{n} \in \mathscr{K}$.

II Remark We added the prefix " $\ell^{2}$-" to clearly distinguish it from the boundedness of (sesquilinear) forms from 36 IV , which one might call " $\ell{ }^{\infty}$-boundedness."

This distinction is needed since for example given a Hilbert space $\mathscr{H}$ the bilinear map $(f, x) \mapsto f(x): \mathscr{H}^{*} \times \mathscr{H} \rightarrow \mathbb{C}$ is always $\ell^{\infty}$-bounded in the sense that $|f(x)| \leqslant\|f\|\|x\|$ for all $f \in \mathscr{H}^{*}$ and $x \in \mathscr{H}$, but it is not $\ell^{2}$-bounded when $\mathscr{H}$ is infinite dimensional
III Theorem A tensor product $\gamma: \mathscr{H} \times \mathscr{K} \rightarrow \mathscr{T}$ of Hilbert spaces is $\ell^{2}$-bounded, and initial as such in the sense that for any by $B \in[0, \infty) \ell^{2}$-bounded bilinear map $\beta: \mathscr{H} \times \mathscr{K} \rightarrow \mathscr{L}$ into a Hilbert space $\mathscr{L}$ there is a unique bounded linear $\operatorname{map} \beta_{\gamma}: \mathscr{T} \rightarrow \mathscr{L}$ with $\beta_{\gamma}(\gamma(x, y))=\beta(x, y)$ for all $x \in \mathscr{H}$ and $y \in \mathscr{K}$. Moreover, $\left\|\beta_{\gamma}\right\| \leqslant B$ for such $\beta$.
IV Proof Note that $\gamma$ is $\ell^{2}$-bounded, since for all $x_{1}, \ldots, x_{n} \in \mathscr{H}, y_{1}, \ldots, y_{n} \in \mathscr{K}$, we have $\left\|\sum_{i} \gamma\left(x_{i}, y_{i}\right)\right\|^{2}=\sum_{i, j}\left\langle\gamma\left(x_{i}, y_{i}\right), \gamma\left(x_{j}, y_{j}\right)\right\rangle=\sum_{i, j}\left\langle x_{i}, x_{j}\right\rangle\left\langle y_{i}, y_{j}\right\rangle$.

Let $\mathscr{E}$ and $\mathscr{F}$ be orthonormal bases for $\mathscr{H}$ and $\mathscr{K}$, respectively. Then since $\{\gamma(e, f): e \in \mathscr{E}, f \in \mathscr{F}\}$ is an orthonormal basis for $\mathscr{T}$ by 109 IV and $\beta_{\gamma}$ is fixed on it by $\beta_{\gamma}(\gamma(e, f))=\beta(e, f)$, uniqueness of $\beta_{\gamma}$ is clear.

Concerning existence of $\beta_{\gamma}$, note that since $t=\sum_{e \in \mathscr{E}, f \in \mathscr{F}}\langle\gamma(e, f), t\rangle \gamma(e, f)$ for all $t \in \mathscr{T}$ by 39 IV , we'd like to define $\beta_{\gamma}$ by

$$
\begin{equation*}
\beta_{\gamma}(t)=\sum_{e \in \mathscr{E}, f \in \mathscr{F}}\langle\gamma(e, f), t\rangle \gamma(e, f) ; \tag{4.2}
\end{equation*}
$$

but before we can do this we must first check that this series converges. To this end, note that since $\beta$ is $\ell^{2}$-bounded by $B$ we have, given $t \in \mathscr{T}$,

$$
\begin{aligned}
\left\|\sum_{e \in E, f \in F}\langle\gamma(e, f), t\rangle \beta(e, f)\right\|^{2} & =\left\|\sum_{e \in E, f \in F} \beta(e,\langle\gamma(e, f), t\rangle f)\right\|^{2} \\
& \leqslant B^{2} \sum_{e, e \in E, f^{\prime}, f \in F}\left\langle e^{\prime}, e\right\rangle\left\langle t, \gamma\left(e^{\prime}, f^{\prime}\right)\right\rangle\left\langle f^{\prime}, f\right\rangle\langle\gamma(e, f), t\rangle \\
& =B^{2} \sum_{e \in E, f \in F}|\langle\gamma(e, f), t\rangle|^{2}
\end{aligned}
$$

for all finite subsets $E \subseteq \mathscr{E}$ and $F \subseteq \mathscr{F}$. Since $\|t\|^{2}=\sum_{e \in \mathscr{E}, f \in \mathscr{F}}|\langle\gamma(e, f), t\rangle|^{2}$ by Parseval's identity (39IV), we see that the series from (4.2) converges defining $\beta_{\gamma}(t)$, and, moreover, that $\left\|\beta_{\gamma}(t)\right\|^{2} \leqslant B^{2}\|t\|^{2}$.

The resulting map $\beta_{\gamma}: \mathscr{T} \rightarrow \mathscr{L}$ is clearly linear, and bounded by $B$. Further, $\beta_{\gamma}(\gamma(e, f))=\beta(e, f)$ for all $e \in \mathscr{E}$ and $f \in \mathscr{F}$ implies that $\beta_{\gamma}(\gamma(x, y))=\beta(x, y)$ for all $x \in \mathscr{H}$ and $y \in \mathscr{K}$, and so we're done.

Exercise Show that the tensor product of Hilbert spaces $\mathscr{H}$ and $\mathscr{K}$ is unique in the sense that given tensor products $\gamma: \mathscr{H} \times \mathscr{K} \rightarrow \mathscr{T}$ and $\gamma^{\prime}: \mathscr{H} \times \mathscr{K} \rightarrow \mathscr{T}^{\prime}$ there is a unique isometric linear isomorphism $\varphi: \mathscr{T} \rightarrow \mathscr{T}^{\prime}$ with $\gamma^{\prime}(x, y)=$ $\varphi(\gamma(x, y))$ for all $x \in \mathscr{H}$ and $y \in \mathscr{K}$.
Notation Now that we've established that that the tensor product of Hilbert spaces $\mathscr{H}$ and $\mathscr{K}$ exists and is unique (up to unique isomorphism) we just pick one and denote it by $\otimes: \mathscr{H} \times \mathscr{K} \rightarrow \mathscr{H} \otimes \mathscr{K}$.

Essentially to turn $\otimes$ into a functor on the category of Hilbert spaces in $\nabla$ we'll need the following result (known as part of Schur's product theorem), which will be useful several times later on.
Lemma For any natural number $N$ the entrywise product ( $a_{n m} b_{n m}$ ) of positive $N \times N$-matrices $\left(a_{n m}\right)$ and $\left(b_{n m}\right)$ over $\mathbb{C}$ is positive.
Proof Let $z_{1}, \ldots, z_{N} \in \mathbb{C}$ be given. To show that $\left(a_{n m} b_{n m}\right)$ is positive, it suffices by 33 II to prove that $\sum_{n, m} \bar{z}_{n} a_{n m} b_{n m} z_{m} \geqslant 0$ for all $n, m$. Since $\left(a_{n m}\right)$ is a positive element of the $C^{*}$-algebra $M_{N}$ it's of the form $\left(a_{n m}\right)=C^{*} C$ for some $N \times N$-matrix $C \equiv\left(c_{n m}\right)$ over $\mathbb{C}$, so $a_{n m}=\sum_{k} \bar{c}_{k n} c_{k m}$ for all $n, m$. Similarly, there a $N \times N$-matrix $\left(d_{n m}\right)$ over $\mathbb{C}$ with $b_{n m}=\sum_{\ell} \bar{d}_{\ell n} d_{\ell m}$ for all $n, m$. Then

$$
\begin{aligned}
\sum_{n, m} \bar{z}_{n} a_{n m} b_{n m} z_{m} & =\sum_{n, m, k, \ell} \bar{z}_{n} \bar{c}_{k n} c_{k m} \bar{d}_{\ell n} d_{\ell m} z_{m} \\
& =\sum_{k, \ell}\left(\sum_{n} \bar{z}_{n} \bar{c}_{k n} \bar{d}_{\ell n}\right)\left(\sum_{m} z_{m} c_{k m} d_{\ell m}\right) \\
& =\sum_{k, \ell}\left|\sum_{n} z_{n} c_{k n} d_{\ell n}\right|^{2} \geqslant 0
\end{aligned}
$$

and so $\left(a_{n m} b_{n m}\right)$ is positive.
Exercise Given square matrices $\left(a_{n m}\right) \leqslant\left(\tilde{a}_{n m}\right)$ and $\left(b_{n m}\right) \leqslant\left(\tilde{b}_{n m}\right)$ over $\mathbb{C}$ of IV the same dimensions, show that $\left(a_{n m} b_{n m}\right) \leqslant\left(\tilde{a}_{n m} \tilde{b}_{n m}\right)$.
Proposition Given bounded linear maps $A: \mathscr{H} \rightarrow \mathscr{H}^{\prime}$ and $B: \mathscr{K} \rightarrow \mathscr{K}^{\prime}$ between Hilbert spaces there is a unique bounded linear map

$$
A \otimes B: \mathscr{H} \otimes \mathscr{K} \rightarrow \mathscr{H}^{\prime} \otimes \mathscr{K}^{\prime}
$$

with $(A \otimes B)(x \otimes y)=(A x) \otimes(B y)$ for all $x \in \mathscr{H}$ and $y \in \mathscr{K}$.
Proof In view of 110 III the only thing we need to prove is that the bilinear map VI $\otimes \circ(A \times B): \mathscr{H} \times \mathscr{K} \rightarrow \mathscr{H} \otimes \mathscr{K}$ is $\ell^{2}$-bounded (for then $\left.A \otimes B=(\otimes \circ(A \times B))_{\otimes .}.\right)$

So let $x_{1}, \ldots, x_{n} \in \mathscr{H}$ and $y_{1}, \ldots, y_{n} \in \mathscr{K}$ be given, and note that

$$
\begin{aligned}
\left\|\sum_{i}(\otimes \circ(A \times B))\left(x_{i}, y_{i}\right)\right\|^{2} & =\left\|\sum_{i}\left(A x_{i}\right) \otimes\left(B y_{i}\right)\right\|^{2} \\
& =\sum_{i, j}\left\langle A x_{i}, A x_{j}\right\rangle\left\langle B y_{i}, B y_{j}\right\rangle \\
& \leqslant\|A\|^{2}\|B\|^{2} \sum_{i, j}\left\langle x_{i}, x_{j}\right\rangle\left\langle y_{i}, y_{j}\right\rangle
\end{aligned}
$$

so $\otimes \circ(A \times B)$ is bounded by $\|A\|\|B\|$. The last step in the display above is justified by IV, and the inequalities $\left(\left\langle A x_{i}, A x_{j}\right\rangle\right) \leqslant\left(\|A\|^{2}\left\langle x_{i}, x_{j}\right\rangle\right)$ and $\left(\left\langle B y_{i}, B y_{j}\right\rangle\right) \leqslant\left(\|B\|^{2}\left\langle y_{i}, y_{j}\right\rangle\right)$.
VII Theorem Let $\mathscr{A}$ and $\mathscr{B}$ be von Neumann algebras of bounded operators on Hilbert spaces $\mathscr{H}$ and $\mathscr{K}$, respectively. Sending operators $A \in \mathscr{A}$ and $B \in \mathscr{B}$ to $A \otimes B: \mathscr{H} \otimes \mathscr{K} \rightarrow \mathscr{H} \otimes \mathscr{K}$ from $\bar{V}$ gives a miu-bilinear map

$$
\otimes: \mathscr{A} \times \mathscr{B} \longrightarrow \mathscr{B}(\mathscr{H} \otimes \mathscr{K})
$$

Letting $\mathscr{T}$ be the von Neumann subalgebra of $\mathscr{B}(\mathscr{H} \otimes \mathscr{K})$ generated by the range of $\otimes$, the restriction $\gamma: \mathscr{A} \times \mathscr{B} \rightarrow \mathscr{T}$ of $\otimes$ is a tensor product of $\mathscr{A}$ and $\mathscr{B}$.
VIII Proof We'll check that the three conditions of 108 II hold; we leave it to the reader to verify that $\otimes$ is miu-bilinear.
IX (Condition 1 The range of $\gamma$ being the same as the range of $\otimes$ generates $\mathscr{T}$ simply by the way $\mathscr{T}$ was defined.
$\times$ (Condition 2 Let $\sigma: \mathscr{A} \rightarrow \mathbb{C}$ and $\tau: \mathscr{B} \rightarrow \mathbb{C}$ be np-maps. We must find an npfunctional $\omega$ on $\mathscr{T}$ with $\omega(A \otimes B)=\sigma(A) \tau(B)$ for all $A \in \mathscr{A}, B \in \mathscr{B}$. Note that by $89 \mathrm{IX} \sigma$ and $\tau$ are of the form $\sigma \equiv \sum_{n}\left\langle x_{n},(\cdot) x_{n}\right\rangle$ and $\tau \equiv \sum_{n}\left\langle y_{n},(\cdot) y_{n}\right\rangle$ for some $x_{1}, x_{2}, \ldots \in \mathscr{H}$ and $y_{1}, y_{2}, \ldots \in \mathscr{K}$ with $\sum_{n}\left\|x_{n}\right\|^{2}<\infty$ and $\sum_{m}\left\|y_{m}\right\|^{2}<$ $\infty$. So as $\sum_{n, m}\left\|x_{n} \otimes y_{m}\right\|^{2} \equiv \sum_{n}\left\|x_{n}\right\|^{2} \sum_{m}\left\|y_{m}\right\|^{2}<\infty$, we can define an npfunctional $\omega$ on $\mathscr{T}$ by $\omega(T):=\sum_{n, m}\left\langle x_{n} \otimes y_{m}, T x_{n} \otimes y_{m}\right\rangle$; which does the job: $\omega(A \otimes B)=\sum_{n, m}\left\langle x_{n}, A x_{n}\right\rangle\left\langle y_{m}, B y_{m}\right\rangle=\sigma(A) \tau(B)$ for all $A \in \mathscr{A}$ and $B \in \mathscr{B}$.
XI (Condition 3 It remains to be shown that the product functionals on $\mathscr{T}$ form a faithful collection. These functionals are - as we've just seen - all of the form $\sum_{m, n}\left\langle x_{n} \otimes y_{n},(\cdot) x_{n} \otimes y_{m}\right\rangle$ for some $x_{1}, x_{2}, \ldots \in \mathscr{H}$ and $y_{1}, y_{2}, \ldots \in \mathscr{K}$ (and, conversely, it's easily seen that a functional of that form is a product functional). It suffices, then, to show that the subset of product functionals of the form $\langle x \otimes y,(\cdot) x \otimes y\rangle$ where $x \in \mathscr{H}$ and $y \in \mathscr{K}$ is faithful. To this end, let $T \in \mathscr{T}_{+}$ with $\langle x \otimes y, T x \otimes y\rangle=0$ for all $x \in \mathscr{H}$ and $y \in \mathscr{K}$ be given in order to
show that $T=0$. Note that since $\|\sqrt{T} x \otimes y\|^{2}=\langle x \otimes y, T x \otimes y\rangle=0$, and so $\sqrt{T} x \otimes y=0$ for all $x \in \mathscr{H}, y \in \mathscr{K}$, we have $\sqrt{T}=0$ (since the linear span of the $x \otimes y$ is dense in $\mathscr{H} \otimes \mathscr{K})$, and thus $T=0$.

Exercise Given von Neumann algebras $\mathscr{A}$ and $\mathscr{B}$ (which are not a priori represented on Hilbert spaces) construct a tensor product $\gamma: \mathscr{A} \times \mathscr{B} \rightarrow \mathscr{T}$ of $\mathscr{A}$ and $\mathscr{B}$ using 48 VIII and VII .

### 4.2.3 Universal Property

Before we bring our categorical faculties to bear upon the tensor product for von Neumann algebras we quickly review the (algebraic) tensor product of plain vector spaces $V$ and $W$ first - it is a vector space $V \odot W$ equipped with a bilinear mapping $\odot: V \times W \rightarrow V \odot W$ which is universal in the sense that for every bilinear mapping $\beta: V \times W \rightarrow Z$ into some vector space $Z$ there is a unique linear map $\beta_{\odot}: V \odot W \rightarrow Z$ with $\beta_{\odot}(v \odot w)=\beta(v, w)$ for all $v \in V$ and $w \in W$. This property uniquely determines the algebraic tensor product in the sense that for any bilinear map $\tilde{\odot}: V \times W \rightarrow V \tilde{\odot} W$ into a vector space $V \tilde{\odot} W$ which shares this property there is a unique linear isomorphism $\varphi: V \odot W \rightarrow V \tilde{\odot} W$ with $\varphi(v \odot w)=v \tilde{\odot} w$ for all $v \in V$ and $w \in W$.

In fact, one may take this property as a neat abstract definition of the algebraic tensor product. However, to see that the darn thing actually exists, one still needs a concrete description such as this one: take given a basis $B$ of $V$ and a basis $C$ of $W$ the bilinear map $\odot$ on $V \times W$ to the vector space $(B \times C) \cdot \mathbb{C}$ with basis $B \times C$ determined by $b \odot c=(b, c)$ for $b \in B$ and $c \in C$. This shows us not only that the algebraic tensor product exists, but also that $\odot$ is injective (among other things).

This is all, of course, well known, and we already saw in 110 III that the tensor product for Hilbert spaces has a similar universal property; the interesting thing here is that with some work one can see that a tensor product $\gamma: \mathscr{A} \times \mathscr{B} \rightarrow \mathscr{T}$ of von Neumann algebras $\mathscr{A}$ and $\mathscr{B}$ has a similar universal property too! We'll see that any bilinear map $\beta: \mathscr{A} \times \mathscr{B} \rightarrow \mathscr{C}$ into a von Neumann algebra $\mathscr{C}$ which is sufficiently regular extends uniquely along $\gamma$ to a ultraweakly continuous map $\beta_{\gamma}: \mathscr{T} \rightarrow \mathbb{C}$, where regular will mean that the extension $\beta_{\odot}: \mathscr{A} \odot \mathscr{B} \rightarrow \mathscr{C}$ from the algebraic tensor product is ultraweakly continuous and bounded with respect to the norm and ultraweak topology induced on $\mathscr{A} \odot \mathscr{B}$ by $\mathscr{T}$ via $\gamma$.

To prevent a circular description here, we'll first describe the norm and ultraweak topology that the tensor product induces on $\mathscr{A} \odot \mathscr{B}$ directly, which
turns out to be independent (as it should) from the choice of $\gamma$. This description is essentially based on the fact that the product functionals on $\mathscr{T}$ are centre separating; and that this determines both norm and ultraweak topology is just a general observation concerning centre separating sets, as we saw in 90 II

Definitions Let $\mathscr{A}$ and $\mathscr{B}$ be von Neumann algebras.

1. A basic functional is a map $\omega: \mathscr{A} \odot \mathscr{B} \rightarrow \mathbb{C}$ with $\omega \equiv(\sigma \odot \tau)\left(t^{*}(\cdot) t\right)$ for some np-maps $\sigma: \mathscr{A} \rightarrow \mathbb{C}, \tau: \mathscr{B} \rightarrow \mathbb{C}$, and $t \in \mathscr{A} \odot \mathscr{B}$.
A simple functional is a finite sum of basic functionals.
2. Each basic functional $\omega: \mathscr{A} \odot \mathscr{B} \rightarrow \mathbb{C}$ gives us an operation $[\cdot, \cdot]_{\omega}$, that will turn out to be an inner product in V by $[s, t]_{\omega}:=\omega\left(s^{*} t\right)$ (cf. 30 II ), and an associated semi-norm denoted by $\|t\|_{\omega}:=[t, t]_{\omega}^{1 / 2}=\omega\left(t^{*} t\right)^{1 / 2}$. The tensor product norm on $\mathscr{A} \odot \mathscr{B}$ is the norm (seeVIII) given by

$$
\|t\|=\sup _{\omega}\|t\|_{\omega},
$$

where $\omega$ ranges over all basic functionals on $\mathscr{A} \odot \mathscr{B}$ with $\omega(1) \leqslant 1$.
3. Note that having endowed $\mathscr{A} \odot \mathscr{B}$ with the tensor product norm we can speak of bounded functionals on $\mathscr{A} \odot \mathscr{B}$, and the operator norm on them; and note that the basic and simple functionals are bounded.

The ultraweak tensor product topology is the least topology on $\mathscr{A} \odot \mathscr{B}$ that makes all operator norm limits of simple functionals continuous.
4. A bilinear map $\beta: \mathscr{A} \times \mathscr{B} \rightarrow \mathscr{C}$ to a von Neumann algebra $\mathscr{C}$ is called
(a) (continues the list from 1081)
(b) bounded when the unique extension $\beta_{\odot}: \mathscr{A} \odot \mathscr{B} \rightarrow \mathscr{C}$ is bounded,
(c) normal when $\beta_{\odot}$ is continuous with respect to the ultraweak tensor product topology on $\mathscr{A} \odot \mathscr{B}$ and the ultraweak topology on $\mathscr{C}$,
(d) completely positive when $\sum_{i, j, k} c_{k}^{*} \beta\left(a_{i}^{*} a_{j}, b_{i}^{*} b_{j}\right) c_{k} \geqslant 0$ for all tuples $a_{1}, \ldots, a_{N} \in \mathscr{A}, b_{1}, \ldots, b_{N} \in \mathscr{B}$, and $c_{1}, \ldots, c_{N} \in \mathscr{C}$.

III Lemma Given $C^{*}$-algebras $\mathscr{A}$ and $\mathscr{B}$ we have $(\sigma \odot \tau)\left(t^{*} t\right) \geqslant 0$ for all $t \in \mathscr{A} \odot \mathscr{B}$ and p-maps $\sigma: \mathscr{A} \rightarrow \mathbb{C}$ and $\tau: \mathscr{B} \rightarrow \mathbb{C}$.

Proof Note that writing $t \equiv \sum_{n} a_{n} \odot b_{n}$, where $a_{1}, \ldots, a_{N} \in \mathscr{A}, b_{1}, \ldots, b_{N} \in \mathscr{B}$, we have $(\sigma \odot \tau)\left(t^{*} t\right)=\sum_{n, m} \sigma\left(a_{n}^{*} a_{m}\right) \tau\left(b_{n}^{*} b_{m}\right)$. Since $\left(a_{n}^{*} a_{m}\right)$ is a positive matrix over $\mathscr{A}$, and $\sigma: \mathscr{A} \rightarrow \mathbb{C}$ is completely positive (by 34IX), the matrix $\left(\sigma\left(a_{n}^{*} a_{m}\right)\right)$ is positive. Since $\left(\tau\left(b_{n}^{*} b_{m}\right)\right)$ is positive by the same token, the entrywise product $\left(\sigma\left(a_{n}^{*} a_{m}\right) \tau\left(b_{n}^{*} a_{m}\right)\right)$ is positive too (by 111II). Whence $(\sigma \odot \tau)\left(t^{*} t\right)=\sum_{n, m} \sigma\left(a_{n}^{*} a_{m}\right) \tau\left(b_{n}^{*} b_{m}\right) \geqslant 0$.
Exercise Use $\Pi$ to show that $[\cdot, \cdot]_{\omega}$ from $\Pi$ is an inner product.
Lemma Product functionals on $\mathscr{A} \odot \mathscr{B}$ formed from separating collections $\Omega$ and $\Xi$ of linear functionals on $C^{*}$-algebras $\mathscr{A}$ and $\mathscr{B}$, respectively, are separating in the sense that given $t \in \mathscr{A} \odot \mathscr{B}$ the condition that $(\sigma \odot \tau)(t)=0$ for all $\sigma \in \Omega$ and $\tau \in \Xi$ entails that $t=0$.
Proof Write $t \equiv \sum_{n} a_{n} \odot b_{n}$ for some $a_{1}, \ldots, a_{N} \in \mathscr{A}$ and $b_{1}, \ldots, b_{N} \in \mathscr{B}$. Note that (by replacing them if necessary) we may assume that the $a_{1}, \ldots, a_{N}$ are linearly independent. Let $\tau \in \Xi$ be given. Since $0=(\sigma \odot \tau)(t)=$ $\sum_{n} \sigma\left(a_{n}\right) \tau\left(b_{n}\right)=\sigma\left(\sum_{n} a_{n} \tau\left(b_{n}\right)\right)$ for all $\sigma$ from the separating collection $\Omega$, we have $0=\sum_{n} a_{n} \tau\left(b_{n}\right)$, and so- $a_{1}, \ldots, a_{N}$ being linearly independent-we get $0=\tau\left(b_{1}\right)=\cdots=\tau\left(b_{N}\right)$. Since this holds for any $\tau$ in the separating collection $\Xi$ we get $0=b_{1}=\cdots=b_{N}$, and thus $t=\sum_{n} a_{n} \odot b_{n}=0$.
Exercise Show that the tensor product norm from $\Pi$ is, indeed, a norm.
Exercise Note that given np-functionals $\sigma: \mathscr{A} \rightarrow \mathbb{C}$ and $\tau: \mathscr{B} \rightarrow \mathbb{C}$ on von IX Neumann algebras, the functional $\sigma \odot \tau: \mathscr{A} \odot \mathscr{B} \rightarrow \mathbb{C}$ is ultraweakly continuous and bounded, almost by definition.

Show that $f \odot g$ is bounded and ultraweakly continuous too for all $f \in \mathscr{A}_{*}$ and $g \in \mathscr{B}_{*}$ (perhaps using 72 XI).
Exercise We're going to show that the ultraweak tensor product topology and tensor product norm from $\triangle$ actually describe the norm and ultraweak topology on $\mathscr{A} \odot \mathscr{B}$ induced by a tensor product $\mathscr{A} \times \mathscr{B} \rightarrow \mathscr{T}$ (via $\gamma_{\odot}$ ) by establishing the two closely related facts that $\gamma_{\odot}: \mathscr{A} \odot \mathscr{B} \rightarrow \mathscr{T}$ is an isometry and an ultraweak embedding, and that certain functionals $\omega: \mathscr{A} \odot \mathscr{B} \rightarrow \mathbb{C}$ can be extended uniquely to $\mathscr{T}$ along $\gamma_{\odot}$.

1. Show using 9011 that the collection $\Omega$ of np-functionals on $\mathscr{T}$ of the form $\gamma(\sigma, \tau)\left(\gamma_{\odot}(s)^{*}(\cdot) \gamma_{\odot}(s)\right)$, where $\sigma: \mathscr{A} \rightarrow \mathbb{C}, \tau: \mathscr{B} \rightarrow \mathbb{C}$ are np-functionals and $s \in \mathscr{A} \odot \mathscr{B}$, is order separating, and that every np-functional on $\mathscr{T}$ is the operator norm limit of finite sums of functionals from $\Omega$.

Show that $\omega \circ \gamma_{\odot}$ is a basic functional (see II) for every $\omega \in \Omega$, and that
every basic functional is of this form for some unique $\omega \in \Omega$.
2. Show that the subset $\Omega_{1}$ of $\Omega$ of unital maps is order separating, and so determines the norm on $\mathscr{T}$ via $\|a\|^{2}=\left\|a^{*} a\right\|=\sup _{\omega \in \Omega_{1}} \omega\left(a^{*} a\right)$ for all $a \in \mathscr{T}$ (see 21 VII ).
Prove that $\left\|\gamma_{\odot}(s)\right\|=\sup _{\omega \in \Omega_{1}} \omega\left(s^{*} s\right)^{1 / 2}=\sup _{\omega \in \Omega_{1}}\|s\|_{\omega \circ \tau_{\odot}}=\|s\|$ for all $s \in \mathscr{A} \odot \mathscr{B}$, and conclude that $\gamma_{\odot}$ is an isometry.
3. Show that $\left\|f \circ \gamma_{\odot}\right\| \leqslant\|f\|$ for every $f \in \mathscr{T}_{*}$, and deduce from this that when $\omega: \mathscr{T} \rightarrow \mathbb{C}$ is an np-functional its restriction $\omega \circ \gamma_{\odot}$ is the operator norm limit of simple functionals on $\mathscr{A} \odot \mathscr{B}$ implying that $\omega \circ \gamma_{\odot}$-and thus $\gamma_{\odot}$ itself-is ultraweakly continuous.
4. In order to show that $\gamma_{\odot}$ is an ultraweak embedding, we'll need the equality $\left\|f \circ \gamma_{\odot}\right\|=\|f\|$ for all $f \in \mathscr{T}_{*}$.
In order to show this in turn, recall (from 86 IX ) that there is a partial isometry $u$ in $\mathscr{T}$ with $f(u)=\|f\|$ (see 86XIV).
Show that given $\varepsilon>0$ there is a net $\left(s_{\alpha}\right)_{\alpha}$ in $\mathscr{A} \odot \mathscr{B}$ with $\left\|s_{\alpha}\right\| \leqslant 1+\varepsilon$ for all $\alpha$ such that $\gamma_{\odot}\left(s_{\alpha}\right)$ converges ultrastrongly to $t$ as $\alpha \rightarrow \infty$ (cf. 74 VI ).
Deduce that $\|f\|=f(u)=|f(u)|=\lim _{\alpha}\left|f\left(\gamma_{\odot}\left(s_{\alpha}\right)\right)\right| \leqslant\left\|f \circ \gamma_{\odot}\right\|(1+\varepsilon)$, and conclude that $\|f\|=\left\|f \circ \gamma_{\odot}\right\|$.
5. Show that any functional $\omega^{\prime}: \mathscr{A} \odot \mathscr{B} \rightarrow \mathbb{C}$ that is the operator norm limit of simple functionals on $\mathscr{A} \odot \mathscr{B}$ can be extended uniquely along $\gamma_{\odot}$ to an np-functional on $\mathscr{T}$ (using the fact that the operator norm limit of np-functionals is an np-functional again, see 87 III .
Deduce from this that $\gamma_{\odot}$ is a ultraweak topological embedding.
(Note that by 77 V any bounded ultraweakly continuous functional on $\mathscr{A} \odot$ $\mathscr{B}$ can be extended uniquely to a normal functional on $\mathscr{T}$.)

XI Theorem A tensor product $\gamma: \mathscr{A} \times \mathscr{B} \rightarrow \mathscr{T}$ of von Neumann algebras $\mathscr{A}$ and $\mathscr{B}$ has this universal property: for every normal bounded bilinear map $\beta: \mathscr{A} \times \mathscr{B} \rightarrow \mathscr{C}$ to a von Neumann algebra $\mathscr{C}$ there is a unique ultraweakly continuous map $\beta_{\gamma}: \mathscr{T} \rightarrow \mathscr{C}$ with $\beta_{\gamma} \circ \gamma=\beta$. Moreover, $\left\|\beta_{\gamma}\right\|=\left\|\beta_{\odot}\right\|$.
XII Proof Since $\beta_{\odot}: \mathscr{A} \odot \mathscr{B} \rightarrow \mathscr{C}$ is ultraweakly continuous and bounded, and $\mathscr{A} \odot \mathscr{B}$ can by Xbe considered an ultraweakly dense $*$-subalgebra of $\mathscr{T}$ via $\gamma_{\odot}$, the theorem follows from 77 V except for some trivial details.

We'll need some observations concerning completely positive bilinear maps.
Exercise Show that a mi-bilinear map $\beta: \mathscr{A} \times \mathscr{B} \rightarrow \mathscr{C}$ between von Neumann algebras is completely positive.

Notation Given a bilinear map $\beta: \mathscr{A} \times \mathscr{B} \rightarrow \mathscr{C}$ between von Neumann algebras, we define $M_{N} \beta: M_{N} \mathscr{A} \times M_{N} \mathscr{B} \rightarrow M_{N} \mathscr{C}$ by $\left(M_{N} \beta\right)(A, B)=\left(\beta\left(A_{i j}, B_{i j}\right)\right)_{i j}$ for each $N$.

Exercise Show that for a bilinear map $\beta: \mathscr{A} \times \mathscr{B} \rightarrow \mathscr{C}$ between von Neumann algebras the following are equivalent.

1. $\beta$ is completely positive.
2. $M_{N} \beta$ is completely positive for each $N$.
3. $\left(M_{N} \beta\right)(A, B) \geqslant 0$ for all $A \in M_{N}(\mathscr{A})_{+}, B \in M_{N}(\mathscr{B})_{+}$and $N$.

Deduce as a corollary that $h \circ \beta \circ(f \times g)$ is completely positive when $f: \mathscr{A}^{\prime} \rightarrow \mathscr{A}$, $g: \mathscr{B}^{\prime} \rightarrow \mathscr{B}$ and $h: \mathscr{C} \rightarrow \mathscr{C}^{\prime}$ are cp-maps between von Neumann algebras.

Exercise Let $\gamma: \mathscr{A} \times \mathscr{B} \rightarrow \mathscr{T}$ be a tensor product of von Neumann algebras, $\beta: \mathscr{A} \times \mathscr{B} \rightarrow \mathscr{C}$ a normal bounded bilinear map, and $\beta_{\gamma}: \mathscr{T} \rightarrow \mathscr{C}$ its extension along $\gamma_{\odot}$ from 112 XI . Show that

1. $\beta_{\gamma}$ is multiplicative iff $\beta$ is multiplicative (see 112 II);
2. $\beta_{\gamma}$ is involution preserving iff $\beta$ is involution preserving;
3. $\beta_{\gamma}$ is unital iff $\beta$ is unital;
4. $\beta_{\gamma}$ is positive iff $\sum_{i, j} \beta\left(a_{i}^{*} a_{j}, b_{i}^{*} b_{j}\right) \geqslant 0$ for all tuples $a_{1}, \ldots, a_{N}$ from $\mathscr{A}$ and $b_{1}, \ldots, b_{N}$ from $\mathscr{B}$;
5. $\beta_{\gamma}$ is completely positive iff $\beta$ is completely positive.

Exercise Show that the tensor product of von Neumann algebras $\mathscr{A}$ and $\mathscr{B}$ is unique in the sense that when $\gamma: \mathscr{A} \times \mathscr{B} \rightarrow \mathscr{T}$ and $\gamma^{\prime}: \mathscr{A} \times \mathscr{B} \rightarrow \mathscr{T}^{\prime}$ are tensor products of $\mathscr{A}$ and $\mathscr{B}$, then there is a unique nmiu-isomorphism $\varphi: \mathscr{T} \rightarrow \mathscr{T}^{\prime}$ with $\varphi(\gamma(a, b))=\gamma^{\prime}(a, b)$ for all $a \in \mathscr{A}$ and $b \in \mathscr{B}$.

### 4.2.4 Functoriality

115 Notation Now that we've established that that the tensor product of von Neumann algebras $\mathscr{A}$ and $\mathscr{B}$ exists and is unique (up to unique nmiu-isomorphism) we just pick one and denote it by $\otimes: \mathscr{A} \times \mathscr{B} \rightarrow \mathscr{A} \otimes \mathscr{B}$.

II Proposition Given ncp-maps $f: \mathscr{A} \rightarrow \mathscr{C}$ and $g: \mathscr{B} \rightarrow \mathscr{D}$ between von Neumann algebras there is a unique ncp-map $f \otimes g: \mathscr{A} \otimes \mathscr{B} \rightarrow \mathscr{C} \otimes \mathscr{D}$ with

$$
(f \otimes g)(a \otimes b)=f(a) \otimes f(b)
$$

for all $a \in \mathscr{A}$ and $b \in \mathscr{B}$. Moreover,

1. $f \otimes g$ is multiplicative when $f$ and $g$ are multiplicative;
2. $f \otimes g$ is involution preserving when $f$ and $g$ are involution preserving; and
3. $f \otimes g$ is (sub)unital when $f$ and $g$ are (sub)unital.

III Proof As uniqueness of $f \otimes g$ is rather obvious, we leave it at that. To establish existence of $f \otimes g$, it suffices to show that the bilinear map $\beta: \mathscr{A} \times \mathscr{B} \rightarrow$ $\mathscr{C} \otimes \mathscr{D}$ given by $\beta(a, b)=f(a) \otimes g(b)$, which is completely positive by 113 IV , is bounded and normal; because then we may take $f \otimes g:=\beta_{\otimes}$ as in 112 XI and all the properties claimed for $f \otimes g$ will then follow with the very least of effort from 1141 .

To see that $\beta$ is bounded, we'll prove that $\left\|\beta_{\odot}(s)\right\| \leqslant\|f\|\|g\|\|s\|$ given an element $s$ of $\mathscr{A} \otimes \mathscr{B}$, and for this it suffices (by the definition of the tensor product norm, 112 II to show that $\omega\left(\beta_{\odot}(s)^{*} \beta_{\odot}(s)\right) \leqslant\|f\|^{2}\|g\|^{2}\|s\|^{2}$ given a basic functional $\omega$ on $\mathscr{A} \odot \mathscr{B}$ with $\omega(1) \leqslant 1$. We'll prove in a moment that $\left\|\omega \circ \beta_{\odot}\right\| \leqslant\|f\|\|g\|$ and $\beta_{\odot}(s)^{*} \beta_{\odot}(s) \leqslant\|f\|\|g\| \beta_{\odot}\left(s^{*} s\right)$, because with these two claims we get $\omega\left(\beta_{\odot}(s)^{*} \beta_{\odot}(s)\right) \leqslant\|f\|\|g\| \omega\left(\beta_{\odot}\left(s^{*} s\right)\right) \leqslant\|f\|\|g\|\left\|\omega \circ \beta_{\odot}\right\|\|s\|^{2} \leqslant$ $\|f\|^{2}\|g\|^{2}\|s\|^{2}$ - which is the result desired.

Concerning the first promise, that $\left\|\omega \circ \beta_{\odot}\right\| \leqslant\|f\|\|g\|$, note that writing $\omega \equiv(\sigma \odot \tau)\left(t^{*}(\cdot) t\right)$, where $\sigma$ and $\tau$ are np-maps on $\mathscr{C}$ and $\mathscr{D}$, respectively, and $t \equiv \sum_{i j} c_{i} \odot d_{i}$ is from $\mathscr{C} \odot \mathscr{D}$, we have

$$
\omega \circ \beta_{\odot}=\sum_{i j} \sigma\left(c_{i}^{*} f(\cdot) c_{j}\right) \odot \tau\left(d_{i}^{*} g(\cdot) d_{j}\right)
$$

and so $\omega \circ \beta_{\odot}$ is ultraweakly continuous and bounded by 112IX, because the $\sigma\left(c_{i}^{*} f(\cdot) c_{j}\right)$ and $\tau\left(d_{i}^{*} g(\cdot) d_{j}\right)$ are bounded ultraweakly continuous functionals.

Although the bound for $\omega \circ \beta_{\odot}$ thus obtained is in all probability nowhere near $\|f\|\|g\|$, it does allow us by 112 XI to extend $\omega \circ \beta_{\odot}$ to an ultraweakly continuous functional $\omega^{\prime}:=(\omega \circ \beta)_{\otimes}$ on $\mathscr{C} \otimes \mathscr{D}$ with the same norm, $\left\|\omega^{\prime}\right\|=$ $\left\|\omega \circ \beta_{\odot}\right\|$. Since this extension $\omega^{\prime}$ is completely positive (because $\beta$ and thus $\omega \circ \beta$ are completely positive, see 113 IV its norm is by 34 XVI given by $\left\|\omega^{\prime}\right\|=\omega^{\prime}(1) \equiv$ $\omega(f(1) \otimes g(1)) \leqslant\|f\|\|g\|$, where we used that $\omega(1) \leqslant 1$. Thus $\left\|\omega \circ \beta_{\odot}\right\|=\left\|\omega^{\prime}\right\| \leqslant$ $\|f\|\|g\|$, as was claimed.

Incidentally, since each $\omega \circ \beta_{\odot}$ is ultraweakly continuous, so is $\beta_{\odot}$, and thus $\beta$ is normal. The only thing that remains is to make good on our last promise, that $\beta_{\odot}(s)^{*} \beta_{\odot}(s) \leqslant\|f\|\|g\| \beta_{\odot}\left(s^{*} s\right)$. To this end, write $s \equiv \sum_{i} a_{i} \odot b_{i}$, and consider the matrices $A$ and $B$ given by

$$
A:=\left(\begin{array}{cccc}
a_{1} & a_{2} & \cdots & a_{n} \\
0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0
\end{array}\right) \quad B:=\left(\begin{array}{cccc}
b_{1} & b_{2} & \cdots & b_{n} \\
0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0
\end{array}\right)
$$

and the cp-map $h: M_{n}(\mathscr{C} \otimes \mathscr{D}) \rightarrow \mathscr{C} \otimes \mathscr{D}$ given by $h(C)=\langle(1, \ldots, 1), C(1, \ldots, 1)\rangle=$ $\sum_{i j} C_{i j}$. We make these arrangements so that we may apply the inequality $\left(M_{n} f\right)(A)^{*}\left(M_{n} f\right)(A) \leqslant\left\|\left(M_{n} f\right)(1)\right\|\left(M_{n} f\right)\left(A^{*} A\right)$ easily derived from 34XIV, Indeed, noting also $\left\|\left(M_{n} f\right)(1)\right\|=\|f(1)\|=\|f\|$, we have

$$
\begin{aligned}
\beta_{\odot}(s)^{*} \beta_{\odot}(s) & =\sum_{i j} f\left(a_{i}\right)^{*} f\left(a_{j}\right) \otimes g\left(b_{i}\right)^{*} g\left(b_{j}\right) \\
& =h\left(\left(M_{n} f\right)(A)^{*}\left(M_{n} f\right)(A) \quad\left(M_{n} \otimes\right) \quad\left(M_{n} g\right)(B)^{*}\left(M_{n} g\right)(B)\right) \\
& \leqslant\|f\|\|g\| h\left(\left(M_{n} f\right)\left(A^{*} A\right) \quad\left(M_{n} \otimes\right) \quad\left(M_{n} g\right)\left(B^{*} B\right)\right) \\
& =\|f\|\|g\| \sum_{i j} f\left(a_{i}^{*} a_{j}\right) \otimes g\left(b_{i}^{*} b_{j}\right) \\
& =\|f\|\|g\| \beta_{\odot}\left(s^{*} s\right),
\end{aligned}
$$

which concludes this proof.
Exercise Show that the assignments $(\mathscr{A}, \mathscr{B}) \mapsto \mathscr{A} \otimes \mathscr{B}$, and $(f, g) \mapsto f \otimes g$ give IV a bifunctor $\otimes: \mathbf{C} \times \mathbf{C} \rightarrow \mathbf{C}$ where $\mathbf{C}$ can be $\mathbf{W}_{\mathrm{MIU}}^{*}, \mathbf{W}_{\mathrm{CP}}^{*}, \mathbf{W}_{\mathrm{CPU}}^{*}$ or $\mathbf{W}_{\mathrm{CPSU}}^{*}$.
Proposition Given injective nmiu-maps $f: \mathscr{A} \rightarrow \mathscr{C}$ and $g: \mathscr{B} \rightarrow \mathscr{D}$, the nmiu- V map $f \otimes g: \mathscr{A} \otimes \mathscr{B} \rightarrow \mathscr{C} \otimes \mathscr{D}$ is injective.
Proof The trick is to consider the von Neumann subalgebra $\mathscr{T}$ generated by the elements of $\mathscr{C} \otimes \mathscr{D}$ of the form $f(a) \otimes g(b)$ where $a \in \mathscr{A}$ and $b \in \mathscr{B}$, and to show that the miu-bilinear map $\gamma: \mathscr{A} \times \mathscr{B} \rightarrow \mathscr{T}$ given by $\gamma(a, b)=f(a) \otimes g(b)$ is a tensor product of $\mathscr{A}$ and $\mathscr{B}$. Indeed, if this is achieved, then there is, by 114 II
a unique nmiu-map $\varphi: \mathscr{A} \otimes \mathscr{B} \rightarrow \mathscr{T}$ with $\varphi(a \otimes b)=\gamma(a, b)=f(a) \otimes g(b)$, so that the following diagram commutes.


The map on the bottom side of this rectangle above is none other than $f \otimes g$, and is thus, being the composition of the isomorphism $\varphi$ with the inclusion $\mathscr{T} \subseteq \mathscr{C} \otimes \mathscr{D}$, injective.

It remains to be shown that $\gamma$ is a tensor product, that is, obeys the conditions from 108 II . Condition 1 holds simply by definition of $\mathscr{T}$. To see that $\gamma$ obeys condition 2, let np-functionals $\tilde{\sigma}: \mathscr{A} \rightarrow \mathbb{C}$ and $\tilde{\tau}: \mathscr{B} \rightarrow \mathbb{C}$ be given; we must find an np-functional $\gamma(\tilde{\sigma}, \tilde{\tau})$ on $\mathscr{T}$ with $\gamma(\tilde{\sigma}, \tilde{\tau})(a \otimes b)=\gamma(a, b)$.

By ultraweak permanence $\tilde{\sigma}$ and $\tilde{\tau}$ can be extended along $f$ and $g$, respectively, see 89XII, giving us np-functionals $\sigma: \mathscr{C} \rightarrow \mathbb{C}$ and $\tau: \mathscr{D} \rightarrow \mathbb{C}$ with $\tilde{\sigma}=\sigma \circ f$ and $\tilde{\tau}=\tau \circ g$. Now simply take $\gamma(\tilde{\sigma}, \tilde{\tau})$ to be the restriction of $\sigma \otimes \tau$ to $\mathscr{T}$, which does the job.

Finally, concerning condition 3, let $z$ be a central projection of $\mathscr{T}$ with $\gamma(\tilde{\sigma}, \tilde{\tau})(z)=0$ for all $\tilde{\sigma}$ and $\tilde{\tau}$ of aforementioned type. We must show that $z=0$, and for this it suffices to show that $(\sigma \otimes \tau)(z)=0$ for all np-functionals $\sigma$ and $\tau$ on $\mathscr{C}$ and $\mathscr{D}$, respectively. Since for such $\sigma$ and $\tau$ we have $\gamma(\tilde{\sigma}, \tilde{\tau})(\gamma(a, b))=$ $\sigma(f(a)) \tau(g(b))=(\sigma \otimes \tau)(\gamma(a, b))$ for all $a \in \mathscr{A}$ and $b \in \mathscr{B}$, we have $\gamma(\tilde{\sigma}, \tilde{\tau})(t)=$ $(\sigma \otimes \tau)(t)$ for all $t \in \mathscr{T}$, and, in particular, $0=\gamma(\tilde{\sigma}, \tilde{\tau})(z)=(\sigma \otimes \tau)(z)$. Hence $z=0$.

### 4.2.5 Miscellaneous Properties

116 Lemma Given von Neumann algebras $\mathscr{A}$ and $\mathscr{B}$, we have $\|f \otimes g\|=\|f\|\|g\|$ for all $f \in \mathscr{A}_{*}$ and $g \in \mathscr{B}_{*}$.
II Proof The trick is to use the polar decomposition for normal functionals, 86 IX . On its account we can find partial isometries $u \in \mathscr{A}$ and $v \in \mathscr{B}$ such that $f(u(\cdot))$ and $g(v(\cdot))$ are positive, and $f \equiv f\left(u u^{*}(\cdot)\right), g \equiv g\left(v v^{*}(\cdot)\right)$. Then $u \otimes v$ is a partial isometry such that $(f \otimes g)((u \otimes v)(\cdot))$ is positive, and $f \otimes g=$ $(f \otimes g)\left((u \otimes v)(u \otimes v)^{*}(\cdot)\right)$ so that $\|f \otimes g\|=(f \otimes g)(u \otimes v)=f(u) g(v)=\|f\|\|g\|$ by 86 XIV.

Exercise There are some easily obtained facts concerning the tensor product $\mathscr{A} \otimes \mathscr{B}$ of von Neumann algebras that nevertheless deserve explicit mention.

1. Show that $a \otimes b \geqslant 0$ for all $a \in \mathscr{A}_{+}$and $b \in \mathscr{B}_{+}$; and conclude that $a_{1} \otimes b_{1} \leqslant a_{2} \otimes b_{2}$ for all $a_{1} \leqslant a_{2}$ from $\mathscr{A}$ and $b_{1} \leqslant b_{2}$ from $\mathscr{B}$.
2. Show that $\|a \otimes b\|=\|a\|\|b\|$ for all $a \in \mathscr{A}$ and $b \in \mathscr{B}$.

Conclude that $\otimes: \mathscr{A} \times \mathscr{B} \rightarrow \mathscr{A} \otimes \mathscr{B}$ is norm continuous.
(Warning: as $\otimes$ is not linear this is not entirely trivial.)
3. Show that $\otimes: \mathscr{A}_{*} \times \mathscr{B}_{*} \rightarrow(\mathscr{A} \otimes \mathscr{B})_{*}$ is norm continuous (using 冋).
4. Show that $\otimes: \mathscr{A} \times \mathscr{B} \rightarrow \mathscr{A} \otimes \mathscr{B}$ is ultraweakly continuous.
(Hint: since we already know that $\otimes_{\odot}: \mathscr{A} \odot \mathscr{B} \rightarrow \mathscr{A} \otimes \mathscr{B}$ is ultraweakly continuous, by 112 X , an equivalent question is whether $\odot: \mathscr{A} \times \mathscr{B} \rightarrow$ $\mathscr{A} \odot \mathscr{B}$ is ultraweakly continuous, which may be boiled down to the fact that $(a, b) \mapsto \sum_{i j} \sigma\left(a_{i}^{*} a a_{j}\right) \tau\left(b_{i}^{*} b b_{j}\right): \mathscr{A} \times \mathscr{B} \rightarrow \mathbb{C}$ is ultraweakly continuous, where $\sigma$ and $\tau$ are np-functionals on $\mathscr{A}$ and $\mathscr{B}$, respectively, and $a_{1}, \ldots, a_{n} \in \mathscr{A}$, and $b_{1}, \ldots, b_{n} \in \mathscr{B}$.)
5. Show that $a \otimes(\cdot): \mathscr{B} \rightarrow \mathscr{A} \otimes \mathscr{B}$ is a ncp-map for every $a \in \mathscr{A}$, and that $1 \otimes(\cdot): \mathscr{B} \rightarrow \mathscr{A} \otimes \mathscr{B}$ is a nmiu-map.

Proposition Let $\mathscr{A}$ and $\mathscr{B}$ be von Neumann algebras.

1. If $S$ and $T$ are ultraweakly dense subsets of $\mathscr{A}$ and $\mathscr{B}$, respectively, then $\{s \otimes t: s \in S, t \in T\}$ is ultraweakly dense in $\mathscr{A} \otimes \mathscr{B}$.
2. If $\Omega$ and $\Theta$ are centre separating collections of np-functionals on $\mathscr{A}$ and $\mathscr{B}$, respectively, then $\{\omega \otimes \vartheta: \omega \in \Omega, \vartheta \in \Theta\}$ is centre separating for $\mathscr{A} \otimes \mathscr{B}$.

Proof Concerning 1, since the elements of $\mathscr{A} \otimes \mathscr{B}$ of the form $a \otimes b$ lie ultraweakly dense in $\mathscr{A} \otimes \mathscr{B}$ where $a \in \mathscr{A}$ and $b \in \mathscr{B}$, it suffices to show that such element $a \otimes$ $b$ is the ultraweak limit of elements of the form $s \otimes t$ where $s \in S$ and $t \in T$. This is indeed the case as there are nets $\left(s_{\alpha}\right)_{\alpha}$ and $\left(t_{\beta}\right)_{\beta}$ in $S$ and $T$ that converge to $a$ and $b$, respectively, and so, because $\otimes$ is ultraweakly continuous by III. we see that $s_{\alpha} \otimes t_{\beta}$ converges ultraweakly to $a \otimes b$ as $\alpha, \beta \rightarrow \infty$.

Concerning 2 , let $t$ be a positive element of $\mathscr{A} \otimes \mathscr{B}$ with $(\omega \otimes \vartheta)\left(s^{*} t s\right)=0$ for all $\omega \in \Omega, \vartheta \in \Theta$, and $s \in \mathscr{A} \otimes \mathscr{B}$; we must show that $t=0$. For this it suffices
to show that $(\sigma \otimes \tau)(t)=0$ for all np-functionals $\sigma: \mathscr{A} \rightarrow \mathbb{C}$ and $\tau: \mathscr{B} \rightarrow \mathbb{C}$ (since the product functionals $\sigma \otimes \tau$ form a faithful collection.) Now, since $\Omega$ is centre separating such $\sigma$ may by 9011 be obtained as operator norm limit of finite sums of functionals of the form $\omega\left(a^{*}(\cdot) a\right)$ where $\omega \in \Omega$ and $a \in \mathscr{A}$. Since a np-functional $\tau: \mathscr{B} \rightarrow \mathbb{C}$ can be obtained in a similar fashion from $\Theta$, and $\otimes: \mathscr{A}_{*} \otimes \mathscr{B}_{*} \rightarrow(\mathscr{A} \otimes \mathscr{B})_{*}$ is operator norm continuous (by III), we see that a product functional $\sigma \otimes \tau$ can be obtained as the operator norm limit of finite sums of functionals of the form $\omega\left(a^{*}(\cdot) a\right) \otimes \vartheta\left(b^{*}(\cdot) b\right) \equiv(\omega \otimes \vartheta)((a \otimes$ $\left.b)^{*}(\cdot)(a \otimes b)\right)$; and since those functionals map $t$ to 0 , by assumption, we conclude that $(\sigma \otimes \tau)(t)=0$ too.

VI To obtain certain examples the following characterisation of the tensor product of von Neumann algebras proves useful.
VII Theorem Given centre separating collections $\Sigma$ and $\Gamma$ of np-functionals on von Neumann algebras $\mathscr{A}$ and $\mathscr{B}$, respectively, a miu-bilinear map $\gamma: \mathscr{A} \times \mathscr{B} \rightarrow \mathscr{T}$ is a tensor product iff all of the following conditions hold.

1. The range of $\gamma$ generates $\mathscr{T}$.
2. For all $\sigma \in \Sigma$ and $\tau \in \Gamma$ the product functional $\gamma(\sigma, \tau): \mathscr{T} \rightarrow \mathbb{C}$ exists (see 108 II ) and is positive.
3. The set $\{\gamma(\sigma, \tau): \sigma \in \Sigma, \tau \in \Gamma\}$ is centre separating for $\mathscr{T}$.

VIII Proof A tensor product $\gamma$ obeys these conditions by definition and by IV, so we only need to show that a $\gamma$ that obeys these conditions is a tensor product, and for this it suffices to show that $\gamma$ can be extended to a nmiu-isomorphism $\gamma_{\otimes}: \mathscr{A} \otimes \mathscr{B} \rightarrow \mathscr{T}$. To extend $\gamma$ to just a miu-map $\gamma_{\otimes}$ (to begin with) it suffices by 112 XI and 114 I to show that $\gamma_{\odot}: \mathscr{A} \odot \mathscr{B} \rightarrow \mathscr{T}$ is bounded with respect to the tensor product norm on $\mathscr{A} \odot \mathscr{B}$ and continuous with respect to the tensor product topology on $\mathscr{A} \odot \mathscr{B}$ and the ultraweak topology on $\mathscr{T}$.

To see that $\gamma_{\odot}$ is bounded, let $t \in \mathscr{A} \odot \mathscr{B}$ be given; we'll show that $\left\|\gamma_{\odot}(t)\right\|^{2} \equiv$ $\left\|\gamma_{\odot}\left(t^{*} t\right)\right\| \leqslant\|t\|^{2}$ where $\|t\|$ is the tensor product norm of $t$. Since by 90 II the np-functionals on $\mathscr{T}$ of the form

$$
\begin{equation*}
\gamma(\sigma, \tau)\left(\gamma_{\odot}(s)^{*}(\cdot) \gamma_{\odot}(s)\right) \tag{4.3}
\end{equation*}
$$

where $\sigma \in \Sigma, \tau \in \Gamma$ and $s \in \mathscr{A} \odot \mathscr{B}$, are order separating, also with the restriction that $1=\gamma(\sigma, \tau)\left(\gamma_{\odot}\left(s^{*} s\right)\right) \equiv(\sigma \odot \tau)\left(s^{*} s\right)$, and therefore determine the norm of $t^{*} t$ as in 21 VII , it suffices to show that $\gamma(\sigma, \tau)\left(\gamma_{\odot}(s)^{*} \gamma_{\odot}\left(t^{*} t\right) \gamma_{\odot}(s)\right) \leqslant\|t\|^{2}$
given such $\sigma, \tau$, and $s\left(\right.$ with $\left.(\sigma \odot \tau)\left(s^{*} s\right)=1\right)$. But since $\gamma(\sigma, \tau)\left(\gamma_{\odot}(s)^{*} \gamma_{\odot}\left(t^{*} t\right) \gamma_{\odot}(s)\right)=$ $(\sigma \odot \tau)\left(s^{*} t^{*} t s\right)=\|t\|_{(\sigma \odot \tau)\left(s^{*}(\cdot) s\right)}^{2} \leqslant\|t\|^{2}$ by the definition of the tensor product norm (see 112 II ), this is indeed the case.

To see that $\gamma_{\odot}: \mathscr{A} \odot \mathscr{B} \rightarrow \mathscr{T}$ is ultraweakly continuous it suffices to show that $\omega \circ \gamma_{\odot}$ is the operator norm limit of finite sums of basic functionals on $\mathscr{A} \odot \mathscr{B}$ (see 112 II) given any np-functional $\omega: \mathscr{T} \rightarrow \mathbb{C}$. Since by 90 II such $\omega$ is the norm limit of finite sums of functionals on $\mathscr{T}$ of the form displayed in (4.3), and $\gamma_{\odot}$ is bounded, we may assume without loss of generality that $\omega$ itself is as shown in 4.3). Since $\omega \circ \gamma_{\odot} \equiv(\sigma \odot \tau)\left(s^{*}(\cdot) s\right)$ is then a basic functional $\gamma_{\odot}$ is ultraweakly continuous.

Having established boundedness and continuity of $\gamma_{\odot}$ we obtain our nmiumap $\gamma_{\otimes}: \mathscr{A} \otimes \mathscr{B} \rightarrow \mathscr{T}$ with $\gamma_{\otimes}(a \otimes b)=\gamma(a, b)$ for all $a \in \mathscr{A}$ and $b \in \mathscr{B}$. To show that $\gamma$ is a tensor product, it suffices to show that $\gamma_{\otimes}$ is a nmiuisomorphism, and for this, it suffices to show that $\gamma_{\otimes}$ is a bijection. In fact, we only need to show that $\gamma_{\otimes}$ is injective, because since the elements of $\mathscr{T}$ of the form $\gamma(a, b) \equiv \gamma_{\otimes}(a \otimes b)$ generate $\mathscr{T}$ (by assumption), and are in the range of $\gamma_{\otimes}$ (which is a von Neumann subalgebra of $\mathscr{T}$ by 48 VI ), $\gamma_{\otimes}$ will be surjective.

To show that $\gamma_{\otimes}$ is injective, it suffices to show that $\left\lceil\gamma_{\otimes}\right\rceil \equiv \llbracket \gamma_{\otimes} \rrbracket=1$ (see 69 IV ). Since the product functionals on $\mathscr{A} \otimes \mathscr{B}$ of the form $\sigma \otimes \tau$ where $\sigma \in \Sigma$ and $\tau \in \Gamma$ are centre separating (by $\mathbb{I V}$ ), and $\llbracket \gamma_{\otimes} \mathbb{\|}$ is central, it suffices to show that $(\sigma \otimes \tau)\left(\llbracket \gamma_{\otimes} \rrbracket^{\perp}\right)=0$ given $\sigma \in \Sigma$ and $\tau \in \Gamma$. But this is easy $(\sigma \otimes \tau)\left(\llbracket \gamma_{\otimes} \rrbracket^{\perp}\right)=\gamma(\sigma, \tau)\left(\gamma_{\otimes}\left(\llbracket \gamma_{\otimes} \rrbracket^{\perp}\right)\right)=0$. Whence $\gamma$ is a tensor product.

Using the characterization from 116 VII it is pretty easy to see that the tensor product distributes over (infinite) direct sums (see III) after some unsurprising observations regarding direct sums (in TI).
Exercise Let $\left(\mathscr{A}_{i}\right)_{i \in I}$ be a collection of von Neumann algebras.

1. Show that given a generating subset $A_{i}$ for each von Neumann algebra $\mathscr{A}_{i}$ the set $\bigcup_{i \in I} \kappa_{i}\left(A_{i}\right)$ generates $\bigoplus_{i \in I} \mathscr{A}_{i}$, where $\kappa_{i}: \mathscr{A}_{i} \rightarrow \bigoplus_{i \in I} \mathscr{A}_{i}$ denotes the np-map given by $\left(\kappa_{i}(a)\right)_{i}=a$ and $\left(\kappa_{i}(a)\right)_{j}=0$ when $j \neq i$.
2. Show that given a centre separating collection $\Omega_{i}$ of np-functionals on $\mathscr{A}_{i}$ for each $i \in I$ the collection $\left\{\omega \circ \pi_{i}: \omega \in \Omega_{i}, i \in I\right\}$ is centre separating for $\bigoplus_{i \in I} \mathscr{A}_{i}$.

Proposition Given von Neumann algebras $\mathscr{A}$ and $\left(\mathscr{B}_{i}\right)_{i \in I}$ the bilinear map

$$
\gamma: \mathscr{A} \times \bigoplus_{i} \mathscr{B}_{i} \longrightarrow \bigoplus_{i} \mathscr{A} \otimes \mathscr{B}_{i},(a, b) \mapsto\left(a_{i} \otimes b\right)_{i}
$$

is a tensor product. (Whence $\mathscr{A} \otimes \bigoplus_{i} \mathscr{B}_{i} \cong \bigoplus_{i} \mathscr{A} \otimes \mathscr{B}_{i}$.)
IV Proof We use 116 VII to show that $\gamma$ is a tensor product. Note that $\gamma$ is clearly miu-bilinear, and that the elements of the form $\gamma\left(a, \kappa_{i}(b)\right)=\kappa(a \otimes b)$ from the range of $\gamma$ where $a \in \mathscr{A}, i \in I$, and $b \in \mathscr{B}_{i}$ generate $\bigoplus_{i} \mathscr{A} \otimes \mathscr{B}_{i}$ by 冋. Further, since given $i \in I$ and np-functionals $\sigma: \mathscr{A} \rightarrow \mathbb{C}$ and $\tau: \mathscr{B}_{i} \rightarrow \mathbb{C}$ the product functional $\gamma\left(\sigma, \tau \circ \pi_{i}\right)$ exists being simply $(\sigma \otimes \tau) \circ \pi_{i}: \bigoplus_{i} \mathscr{A} \otimes \mathscr{B}_{i} \rightarrow \mathbb{C}$, and such product functionals form a centre separating collection by We see that $\gamma$ is indeed a tensor product.

118 The tensor interacts with projections as expected.
II Lemma Let $\mathscr{A}$ and $\mathscr{B}$ be von Neumann algebras.

1. We have $\lceil a \otimes b\rceil=\lceil a\rceil \otimes\lceil b\rceil$ for all $a \in \mathscr{A}_{+}$and $b \in \mathscr{B}_{+}$.
2. We have $\llbracket a \otimes b \rrbracket=\llbracket a \rrbracket \otimes \llbracket b \rrbracket$ for all $a \in \mathscr{A}$ and $b \in \mathscr{B}$.

III Proof Let $a_{\text {f }} \mathscr{A}_{+}$and $b \in \mathscr{B}_{+}$be given. Since the map (•) $\otimes b: \mathscr{A} \rightarrow \mathscr{A} \otimes \mathscr{B}$ is $\mathrm{np},\lceil a \otimes b\rceil \xlongequal{\underline{\boxed{60 V}}}\lceil\lceil a\rceil \otimes b\rceil$. Since similarly $\lceil\lceil a\rceil \otimes b\rceil=\lceil\lceil a\rceil \otimes\lceil b\rceil\rceil \equiv\lceil a\rceil \otimes\lceil b\rceil$ using here that $\lceil a\rceil \otimes\lceil b\rceil$ is already a projection, we get $\lceil a\rceil \otimes\lceil b\rceil=\lceil a \otimes b\rceil$.

Let $a \in \mathscr{A}$ and $b \in \mathscr{B}$ be given in order to prove that $\llbracket a \otimes b \rrbracket=\llbracket a \rrbracket \otimes \llbracket b \rrbracket$. Since $\llbracket a \rrbracket \otimes 1$ commutes with all elements of $\mathscr{A} \otimes \mathscr{B}$ of the form $a^{\prime} \otimes b^{\prime}$, and thus with all elements of $\mathscr{A} \otimes \mathscr{B}$, we see that $\llbracket a \rrbracket \otimes 1$ is central. Since similarly $1 \otimes \| b \rrbracket$ is central, we see that $\llbracket a \rrbracket \otimes \llbracket b \rrbracket=(\llbracket a \rrbracket \otimes 1) \otimes(1 \otimes \llbracket b \rrbracket)$ is central too. Since in addition $\llbracket a \rrbracket \otimes \llbracket b \rrbracket$ is a projection, and $(\llbracket a \rrbracket \otimes \llbracket b \rrbracket)(a \otimes b)=(\llbracket a \rrbracket a) \otimes(\llbracket b \rrbracket b)=$ $a \otimes b$ we see that $\llbracket a \otimes b \rrbracket \leqslant \llbracket a \rrbracket \otimes \llbracket b \rrbracket$ (by definition, see 68 II ).

So all that remains is to show that $\llbracket a \rrbracket \otimes \llbracket b \rrbracket \leqslant \llbracket a \otimes b \rrbracket$. Recall that $\llbracket a \rrbracket=$ $\bigcup_{\tilde{a} \in \mathscr{A}}\left\lceil\tilde{a}^{*} a^{*} a \tilde{a}\right\rceil$ by 681 . Using this, a similar expression for $\llbracket b \rrbracket$, and 60 IX , we see that $\llbracket a \rrbracket \otimes \llbracket b \rrbracket=\bigcup_{\tilde{a} \in \mathscr{A}} \bigcup_{\tilde{b} \in \mathscr{B}}\left\lceil\left(\tilde{a}^{*} a^{*} a \tilde{a}\right) \otimes\left(\tilde{b}^{*} b^{*} b \tilde{b}\right)\right\rceil$, and so it suffices to show that $\left.\left\lceil\left(\tilde{a}^{*} a^{*} a \tilde{a}\right) \otimes\left(\tilde{b}^{*} b^{*} b \tilde{b}\right)\right\rceil \leqslant \llbracket a \otimes b\right\rceil$ given $\tilde{a} \in \mathscr{A}$ and $\tilde{b} \in \mathscr{B}$. This is indeed the case since $\left.\left\lceil\left(\tilde{a}^{*} a^{*} a \tilde{a}\right) \otimes\left(\tilde{b}^{*} b^{*} b \tilde{b}\right)\right\rceil=\left\lceil(\tilde{a} \otimes \tilde{b})^{*}(a \otimes b)^{*}(a \otimes b)(\tilde{a} \otimes \tilde{b})\right\rceil \leqslant \llbracket a \otimes b\right\rceil$ (by 681, again.)

IV Exercise Let $f: \mathscr{A} \rightarrow \mathscr{B}$ and $g: \mathscr{C} \rightarrow \mathscr{D}$ be np-maps between von Neumann algebras. We're going to prove that $\lceil f \otimes g\rceil=\lceil f\rceil \otimes\lceil g\rceil$.

1. Show that $(f \otimes g)(\lceil f\rceil \otimes\lceil g\rceil)=1 \otimes 1$, and conclude that $\lceil f \otimes g\rceil \leqslant\lceil f\rceil \otimes\lceil g\rceil$.
2. Assume for the moment that $\mathscr{A}$ and $\mathscr{C}$ are von Neumann algebras of bounded operators on Hilbert spaces $\mathscr{H}$ and $\mathscr{K}$, respectively, and that $f$
and $g$ are vector functionals, that is, $\mathscr{B}=\mathscr{D}=\mathbb{C}$, and $f=\langle x,(\cdot) x\rangle$ for some $x \in \mathscr{H}$, and $g=\langle y,(\cdot) y\rangle$ for some $y \in \mathscr{K}$.
Show that $\lceil f\rceil=\bigcup_{a \in \mathscr{A} \square}\left\lceil a^{*}|x\rangle\langle x| a\right\rceil$ using 88 IV and 88 VI .
3. With the same assumptions as in the previous point, suppose, furthermore, without loss of generality that $\mathscr{A} \otimes \mathscr{B}$ is given as the von Neumann subalgebra of $\mathscr{B}(\mathscr{H} \otimes \mathscr{K})$ generated by the operators $A \otimes B$ where $A \in \mathscr{A}$ and $B \in \mathscr{B}$ (cf. 111 VIII ).
Show that $f \otimes g=\langle x \otimes y,(\cdot) x \otimes y\rangle$.
Given $a \in \mathscr{A}^{\square}$ and $b \in \mathscr{B}^{\square}$ show that $a \otimes b \in(\mathscr{A} \otimes \mathscr{B})^{\square}$, and thus

$$
\left\lceil a^{*}|x\rangle\langle x| a\right\rceil \otimes\left\lceil b^{*}|y\rangle\langle y| b\right\rceil \leqslant\lceil f \otimes g\rceil .
$$

Deduce from this that $\lceil f\rceil \otimes\lceil g\rceil \leqslant\lceil f \otimes g\rceil$, so $\lceil f\rceil \otimes\lceil g\rceil=\lceil f \otimes g\rceil$.
4. Let $f$ and $g$ be arbitrary again, and assume now that $f$ and $g$ are functionals, that is, $\mathscr{B}=\mathscr{D}=\mathbb{C}$. Show that $\lceil f \otimes g\rceil=\lceil f\rceil \otimes\lceil g\rceil$.
5. Let $f$ and $g$ be arbitrary again, and recall from 66 IV that $1=\bigcup_{\sigma}\lceil\sigma\rceil$ when $\sigma$ ranges over the np-functionals $\sigma$ on $\mathscr{B}$.
Show that $1 \otimes 1=\bigcup_{\sigma, \tau}\lceil\sigma \otimes \tau\rceil$ where $\sigma$ and $\tau$ range over the np-functionals on $\mathscr{B}$ and $\mathscr{D}$, respectively.
Show using 101 IV and 101 VIII that $\lceil f \otimes g\rceil \equiv(f \otimes g)_{\diamond}(1 \otimes 1)=\lceil f\rceil \otimes\lceil g\rceil$.
6. Show that $(f \otimes g)_{\diamond}(s \otimes t)=f_{\diamond}(s) \otimes g_{\diamond}(t)$ for projections $s \in \mathscr{B}$ and $t \in \mathscr{D}$.

### 4.2.6 Monoidal Structure

Up to this point we have only written about the tensor product $\mathscr{A} \otimes \mathscr{B}$ of two von Neumann algebras (to save ink), but all of it, as you will no doubt have observed already, can be easily adapted to deal with a tensor product $\otimes: \mathscr{A}_{1} \times \ldots \times \mathscr{A}_{n} \rightarrow \mathscr{A}_{1} \otimes \cdots \otimes \mathscr{A}_{n}$ of a tuple $\mathscr{A}_{1}, \ldots, \mathscr{A}_{n}$ of von Neumann algebras, which will then, of course, be a multilinear map instead of a bilinear map, etc.

What is less obvious is that there should be any relation between $(\mathscr{A} \otimes \mathscr{B}) \otimes$ $\mathscr{C}$, and $\mathscr{A} \otimes(\mathscr{B} \otimes \mathscr{C})$ and $\mathscr{A} \otimes \mathscr{B} \otimes \mathscr{C}$; but there is.

II Proposition Given von Neumann algebras $\mathscr{A}, \mathscr{B}$ and $\mathscr{C}$, the trilinear map $\gamma:(a, b, c) \mapsto(a \otimes b) \otimes c, \mathscr{A} \times \mathscr{B} \times \mathscr{C} \rightarrow(\mathscr{A} \otimes \mathscr{B}) \otimes \mathscr{C}$ is a tensor product.
III Proof We need to verify the three conditions from 108 II (adapted to trilinear maps). The first condition, that the elements of the form $(a \otimes b) \otimes c$ generate $(\mathscr{A} \otimes \mathscr{B}) \otimes \mathscr{C}$ follows by 116 IV since the elements of the form $a \otimes b$ generate $\mathscr{A} \otimes \mathscr{B}$ (and $\mathscr{C}$ generates $\mathscr{C})$. The second condition is met by defining $\gamma(\sigma, \tau, v):=$ $(\sigma \otimes \tau) \otimes v$ for all np-functionals $\sigma: \mathscr{A} \rightarrow \mathbb{C}, \tau: \mathscr{B} \rightarrow \mathbb{C}$ and $v: \mathscr{C} \rightarrow \mathbb{C}$. Finally, these product functionals $\gamma(\sigma, \tau, v)$ are center separating by 116 IV because the functionals on $\mathscr{A} \otimes \mathscr{B}$ of the form $\sigma \otimes \tau$ are center separating (and so is the set of all np-functionals on $\mathscr{C}$ ), which was the third condition.

IV Corollary There is a unique nmiu-isomorphism

$$
\alpha: \mathscr{A} \otimes(\mathscr{B} \otimes \mathscr{C}) \longrightarrow(\mathscr{A} \otimes \mathscr{B}) \otimes \mathscr{C},
$$

with $\alpha(a \otimes(b \otimes c))=(a \otimes b) \otimes c$ for all $a \in \mathscr{A}, b \in \mathscr{B}, c \in \mathscr{C}$, for any von Neumann algebras $\mathscr{A}, \mathscr{B}, \mathscr{C}$.
$\vee$ Exercise Show that $\mathbf{W}_{\mathrm{MIU}}^{*}, \mathbf{W}_{\mathrm{CP}}^{*}, \mathbf{W}_{\mathrm{CPU}}^{*}$ and $\mathbf{W}_{\mathrm{CPSU}}^{*}$ endowed with the tensor product are symmetric monoidal categories with $\mathbb{C}$ as unit.

### 4.3 Quantum Lambda Calculus

120 In this section we provide the parts needed to built a model of the quantum lambda calculus using von Neumann algebras. We will not venture to describe the quantum lambda calculus in all its details here, nor will we describe how to built the model from these parts (as we did in [9]); we'll just touch upon the two key ingredients: the interpretation of "!" and "—" - with them the expert can easily produce the model.

Let us, nevertheless, try to give some impression to those who are not familiar with the quantum lambda calculus. The quantum lambda calculus is a type theory proposed by Selinger and Valiron in 64, 65 to describe programs for quantum computers especially designed to include not only function types ( - ) and classical data types (such as bit), but also quantum data types (such as qbit), so that there can be a term such as new: bit $\multimap$ qbit that represents the program that initialises a qubit in the given state. There are of course also terms such as 0 : bit and 1 : bit, so that new 0 : qbit represents a qubit in state $|0\rangle$. The addition of quantum data to a type theory is a very delicate matter for if one were to allow for example in this system a variable to be used twice (a thing
usually beyond dispute) it would not take much more to construct a program that duplicates the contents of a qubit, which is nonphysical.

Still, classical data such as a bit can be duplicated freely, so to accommodate this the type !bit is used. More precisely, the type ! $A$ represents that part of the type of $A$ that is duplicable, so that !bit is the proper type for a bit, and !qbit is empty. For example, the term that represents the measurement of a qubit is meas: qbit $\multimap$ !bit, where the ! indicates that the bit resulting from the measurement may be duplicated freely.

The model we alluded to assigns to each type $A$ a von Neumann algebra $\llbracket A \rrbracket$, e.g. $\llbracket q \mathrm{bit} \rrbracket=M_{2}$ and $\llbracket \mathrm{bit} \rrbracket=\mathbb{C}^{2}$. A (closed) term $t: A$ is interpreted as an npsufunctional $\llbracket t: A \rrbracket: \llbracket A \rrbracket \rightarrow \mathbb{C}$, so for example $\llbracket 0:$ bit $\rrbracket:(x, y) \mapsto x: \mathbb{C}^{2} \rightarrow \mathbb{C}$. When $t: A$ has free variables $x_{1}: B_{1}, \ldots, x_{N}: B_{N}$ the interpretation becomes an ncpsu-map $\llbracket t \rrbracket: \llbracket A \rrbracket \rightarrow \llbracket B_{1} \rrbracket \otimes \cdots \otimes \llbracket B_{N} \rrbracket$, so for example,

$$
\llbracket x: \text { qbit } \vdash \text { meas } x \rrbracket:(x, y) \mapsto\left(\begin{array}{ll}
x & 0 \\
0 & y
\end{array}\right): \mathbb{C}^{2} \rightarrow M_{2} .
$$

In short, there are no surprises here. As said, the difficulty lies in the definition of $\llbracket!A \rrbracket$ and $\llbracket A \multimap B \rrbracket$, for which we will provide the following three ingredients.

- The observation (by to Kornell, 44 ) that the category $\left(\mathbf{W}_{\text {MIU }}^{*}\right)^{\text {op }}$ is monoidal closed, that is, that for every von Neumann algebra $\mathscr{B}$, the functor $\mathscr{B} \otimes$ $(\cdot): \mathbf{W}_{\text {MIU }}^{*} \rightarrow \mathbf{W}_{\text {MIU }}^{*}$ has a left adjoint $(\cdot)^{* \mathscr{B}}$.
- The following two adjunctions.


The interpretation of $\llbracket!A \rrbracket$ and $\llbracket A \multimap B \rrbracket$ will then be

$$
\llbracket!A \rrbracket=\ell^{\infty}(\operatorname{nsp}(\llbracket A \rrbracket)) \quad \text { and } \quad \llbracket A \multimap B \rrbracket=\mathcal{F}(\llbracket B \rrbracket)^{* \llbracket A \rrbracket} .
$$

Note that $\llbracket!A \rrbracket$ will always be a discrete von Neumann algebra no matter how complicated $\llbracket A \rrbracket$ may be, so that although this does the job perhaps a more interesting interpretation of ! may be chosen as well. This is not the case: in the next section we'll show that any von Neumann algebra that carries a $\otimes$-monoid structure (such as $\llbracket!A \rrbracket)$ is commutative and discrete, and that $\ell^{\infty}(\operatorname{nsp}(\mathscr{A}))$ is moreover the free $\otimes$-monoid on $\mathscr{A}$.

In this section, we'll need the following result from the literature on von Neumann algebras.

II Proposition Given Hilbert spaces $\mathscr{H}$ and $\mathscr{K}$, and von Neumann subalgebras $\mathscr{A}_{1}$ and $\mathscr{A}_{2}$ of $\mathscr{B}(\mathscr{H})$ and von Neumann subalgebras $\mathscr{B}_{1}$ and $\mathscr{B}_{2}$ of $\mathscr{B}(\mathscr{K})$, we have

$$
\left(\mathscr{A}_{1} \otimes \mathscr{B}_{1}\right) \cap\left(\mathscr{A}_{2} \otimes \mathscr{B}_{2}\right)=\left(\mathscr{A}_{1} \cap \mathscr{A}_{2}\right) \otimes\left(\mathscr{B}_{1} \cap \mathscr{B}_{2}\right) .
$$

Here $\mathscr{A}_{1} \otimes \mathscr{B}_{1}$ denotes not just any tensor product of $\mathscr{A}_{1}$ and $\mathscr{B}_{1}$, but instead the "concrete" tensor product of $\mathscr{A}_{1}$ and $\mathscr{B}_{1}$ : the least von Neumann subalgebra of $\mathscr{B}(\mathscr{H} \otimes \mathscr{K})$ that contains all operators of the form $A \otimes B$ where $A \in \mathscr{A}_{1}$ and $B \in \mathscr{B}_{1}$.
III Proof See Corollary IV.5.10 of 67.

### 4.3.1 First Adjunction

122 Definition We write nsp $:=\mathbf{W}_{\text {MIU }}^{*}(\cdot, \mathbb{C})$ for the functor $\left(\mathbf{W}_{\text {MIU }}^{*}\right)^{\text {op }} \rightarrow$ Set which maps a von Neumann algebra $\mathscr{A}$ to its set of nmiu-functionals, $\operatorname{nsp}(\mathscr{A})$, and sends a nmiu-map $f: \mathscr{A} \rightarrow \mathscr{B}$ to the map $\operatorname{nsp}(f): \operatorname{nsp}(\mathscr{B}) \rightarrow \operatorname{nsp}(\mathscr{A})$ given by $\operatorname{nsp}(f)(\varphi)=\varphi \circ f$ for $\varphi \in \operatorname{nsp}(\mathscr{B})$.
II Proposition Given a set $X$ the map

$$
\eta: X \rightarrow \operatorname{nsp}\left(\ell^{\infty}(X)\right) \quad \text { given by } \quad \eta(x)(h)=h(x)
$$

is universal in the sense that for every map $f: X \rightarrow \operatorname{nsp}(\mathscr{A})$, where $\mathscr{A}$ is a von Neumann algebra, there is a unique nmiu-map $g: \mathscr{A} \rightarrow \ell^{\infty}(X)$ such that

commutes. Moreover, and as a result, the assignment $X \mapsto \ell^{\infty}(X)$ extends to a functor $\ell^{\infty}$ : Set $\rightarrow\left(\mathbf{W}_{\text {MIU }}^{*}\right)^{\text {op }}$ that is left adjoint to nsp, and is given by $\ell^{\infty}(f)(h)=h \circ f$ for any map $f: X \rightarrow Y$ and $h \in \ell^{\infty}(Y)$.
III Proof Note that if we identify $\ell^{\infty}(X)$ with the $X$-fold product of $\mathbb{C}$, we see that $\eta(x): \ell^{\infty}(X) \equiv \bigoplus_{x \in X} \mathbb{C} \rightarrow \mathbb{C}$ is simply the $x$-th projection, and thus a nmiu-map (see 47 IV ). Hence we do indeed get a map $\eta: X \rightarrow \operatorname{nsp}\left(\ell^{\infty}(X)\right)$.

To see that $\eta$ has the desired universal property, let $f: X \rightarrow \operatorname{nsp}(\mathscr{A})$ be given, and define $g: \mathscr{A} \rightarrow \ell^{\infty}(X)$ by $g(a)(x)=f(x)(a)$. One can now either prove directly that $g$ is nmiu, or reduce this in a slightly roundabout way from
the known fact that $\ell^{\infty}(X)$ is the $X$-fold product of $\mathbb{C}$ with the $\eta(x)$ as projections; indeed $g$ is simply the unique nmiu-map with $\eta(x) \circ g=f(x)$ for all $x \in X$, that is, $g=\langle f(x)\rangle_{x \in X}$. In any case, we see that $\operatorname{nsp}(g)(\eta(x)) \equiv \eta(x) \circ g=f(x)$ for all $x \in X$, and so $\operatorname{nsp}(g) \circ \eta=f$. Concerning uniqueness of such $g$, note that given a nmiu-map $g^{\prime}: \mathscr{A} \rightarrow \ell^{\infty}(X)$ with nsp $\left(g^{\prime}\right) \circ \eta=f$ we have $\eta(x) \circ g^{\prime}=$ $\operatorname{nsp}\left(g^{\prime}\right)(\eta(x))=f(x)$ for all $x \in X$, and so $g^{\prime}=\langle f(x)\rangle_{x \in X}=g$.

Hence $\eta$ is a universal arrow from $X$ to nsp. That as a result the assignment $X \mapsto \ell^{\infty}(X)$ extends to a functor Set $\rightarrow\left(\mathbf{W}_{\text {MIU }}^{*}\right)^{\text {op }}$ by sending $f: X \rightarrow Y$ to the unique nmiu-map $\ell^{\infty}(f): \ell^{\infty}(Y) \rightarrow \ell^{\infty}(X)$ with $\operatorname{nsp}\left(\ell^{\infty}(f)\right) \circ \eta_{X}=\eta_{Y} \circ f$ is a known and easily checked fact (where $\eta_{X}:=\eta$ and $\eta_{Y}: Y \rightarrow \operatorname{nsp}\left(\ell^{\infty}(Y)\right.$ ) is what you'd expect). Finally, applying $x \in X$ and $h \in \ell^{\infty}(Y)$ we get $\ell^{\infty}(f)(h)(x)=$ $\left.\eta_{X}(x)\left(\ell^{\infty}(f)(h)\right)\right)=\operatorname{nsp}\left(\ell^{\infty}(x)\right)\left(\eta_{X}(x)\right)(h)=\eta_{Y}(f(x))(h)=h(f(x))$.
Lemma A nmiu-functional $\varphi$ on a direct sum $\bigoplus_{i} \mathscr{A}_{i}$ of von Neumann algebras is of the form $\varphi \equiv \varphi^{\prime} \circ \pi_{i}$ for some $i$ and nmiu-functional $\varphi^{\prime}$ on $\mathscr{A}_{i}$.
Proof Let $e_{j}$ denote the element of $\bigoplus_{i} \mathscr{A}_{i}$ given by $e_{j}(j)=1$ and $e_{j}(i)=0$ for all $i \neq j$. Note that given $i$ and $j$ with $i \neq j$ we have $e_{i} e_{j}=0$ and so $0=$ $\varphi\left(e_{i} e_{j}\right)=\varphi\left(e_{i}\right) \varphi\left(e_{j}\right)$; from this we see that there is at most one $i$ with $\varphi\left(e_{i}\right) \neq 0$. Since for this $i$ we have $e_{i}^{\perp}=\sum_{j \neq i} e_{j}$ and so $\varphi\left(e_{i}^{\perp}\right)=\sum_{j \neq i} \varphi\left(e_{j}\right)=0$, we see that $\varphi(a)=\varphi\left(e_{i} a\right)$ for all $a \in \bigoplus_{i} \mathscr{A}_{i}$. Letting $\kappa_{i}: \mathscr{A}_{i} \rightarrow \bigoplus_{j} \mathscr{A}_{j}$ be the nmisumap given by $\kappa_{i}(a)(i)=a$ and $\kappa_{i}(a)(j)=0$ for $j \neq i$ we have $\varphi=\varphi \circ \kappa_{i} \circ \pi_{i}$. Hence taking $\varphi^{\prime}:=\varphi \circ \kappa_{i}$ does the job.
Exercise Deduce from $\boxed{\boxed{V}}$ that the functor nsp: $\left(\mathbf{W}_{\text {MIU }}^{*}\right)^{\text {op }} \rightarrow$ Set preserves coproducts, and that the map $\eta: X \rightarrow \mathrm{nsp}\left(\ell^{\infty}(X)\right)$ from $\Pi$ is a bijection.

Show that $\ell^{\infty}:$ Set $\rightarrow\left(\mathbf{W}_{\text {MIU }}^{*}\right)^{\text {op }}$ is full and faithful. Whence Set is (isomorphic to) a coreflective subcategory of $\left(\mathbf{W}_{\text {MIU }}^{*}\right)^{\text {op }}$ via $\ell^{\infty}:$ Set $\rightarrow\left(\mathbf{W}_{\text {MIU }}^{*}\right)^{\text {op }}$.

Exercise We're going to prove that $\ell^{\infty}(X \times Y) \cong \ell^{\infty}(X) \otimes \ell^{\infty}(Y)$.

1. Given an element $x$ of a set $X$ let $\hat{x}$ denote the element of $\ell^{\infty}(X)$ that equals 1 on $x$ and is zero elsewhere.
Show that $\{\hat{x}: x \in X\}$ generates $\ell^{\infty}(X)$.
2. Show that the projections $\pi_{x}: \ell^{\infty}(X) \equiv \bigoplus_{y \in X} \mathbb{C} \rightarrow \mathbb{C}$ form an order separating collection of nmiu-functionals on $\ell^{\infty}(X)$.
3. Using this, and 116 VII , prove that given sets $X$ and $Y$ the map
$\otimes: \ell^{\infty}(X) \times \ell^{\infty}(Y) \rightarrow \ell^{\infty}(X \times Y)$
given by $(f \otimes g)(x, y)=f(x) g(y)$ is a tensor product.
Conclude that $\ell^{\infty}(X \times Y) \cong \ell^{\infty}(X) \otimes \ell^{\infty}(Y)$.
(In fact, it follows that $\ell^{\infty}$ is strong monoidal.)

II Exercise Let $\mathscr{A}$ and $\mathscr{B}$ be von Neumann algebras. We're going to show that $\operatorname{nsp}(\mathscr{A} \otimes \mathscr{B}) \cong \operatorname{nsp}(\mathscr{A}) \times \operatorname{nsp}(\mathscr{B})$.

1. Given a nmiu-functional $\varphi: \mathscr{A} \otimes \mathscr{B} \rightarrow \mathbb{C}$ show that $\sigma:=\varphi((\cdot) \otimes 1)$ and $\tau:=\varphi(1 \otimes(\cdot))$ are nmiu-functionals on $\mathscr{A}$ and $\mathscr{B}$, respectively; and show that $\varphi=\sigma \otimes \tau$ (by proving that $\varphi(a \otimes b)=\sigma(a) \tau(b)$.)
2. Show that $\sigma, \tau \mapsto \sigma \otimes \tau$ gives a bijection $\operatorname{nsp}(\mathscr{A}) \times \operatorname{nsp}(\mathscr{B}) \rightarrow \operatorname{nsp}(\mathscr{A} \otimes \mathscr{B})$. (This makes nsp strong monoidal.)

### 4.3.2 Second Adjunction

124 Lemma If a von Neumann algebra $\mathscr{A}$ is generated by $S \subseteq \mathscr{A}$, then

$$
\# \mathscr{A} \leqslant 2^{2^{\# C+\# S}}
$$

where $\# S$ denotes the cardinality of $S$, and so on.
$\|^{\prime} \quad$ Proof Note that the $*$-subalgebra $S^{\prime}$ of $\mathscr{A}$ generated by $S$ is ultraweakly dense in $\mathscr{A}$. Since every element of $S^{\prime}$ can be formed from the infinite set $S \cup \mathbb{C}$ using the finitary operations of addition, multiplication, and involution, $\# S^{\prime} \leqslant$ $\# \mathbb{C}+\# S$. Since every element of $\mathscr{A}$ is the ultraweak limit of a filter (see [76, §12]) on $S^{\prime}$ of which there no more than $2^{2^{\# S^{\prime}}}$, we conclude $\# \mathscr{A} \leqslant 2^{2^{\# \mathrm{C}+\# S}}$.
III Theorem The inclusion $\mathbf{W}_{\text {MIU }}^{*} \rightarrow \mathbf{W}_{\text {CPSU }}^{*}$ has a left adjoint $\mathcal{F}: \mathbf{W}_{\text {CPSU }}^{*} \rightarrow \mathbf{W}_{\text {MIU }}^{*}$.
IV Proof Note that since the category $\mathbf{W}_{\text {MIU }}^{*}$ has all products 47 IV , and equalisers (47V), W W ${ }_{\mathrm{MIU}}^{*}$ has all limits (by Theorem V2.1 and Exercise V4.2 of [46]). Moreover, the inclusion $U: \mathbf{W}_{\text {MIU }}^{*} \rightarrow \mathbf{W}_{\text {CPSU }}^{*}$ preserves these limits (see 47 IV and 47 V . So by Freyd's adjoint functor theorem (Theorem V6.1 of 46]) it suffices to check the solution set condition, that is, that
for every von Neumann algebra $\mathscr{A}$ there be a set $I$, and for each $i \in I$ an ncpsu-map $f_{i}: \mathscr{A} \rightarrow \mathscr{A}_{i}$ into a von Neumann algebra $\mathscr{A}_{i}$ such that every ncpsu-map $f: \mathscr{A} \rightarrow \mathscr{B}$ into some von Neumann algebra $\mathscr{B}$ is of the form $f \equiv h \circ f_{i}$ for some $i \in I$ and nmiu-map $h: \mathscr{A}_{i} \rightarrow \mathscr{B}$.

To this end, given a von Neumann algebra $\mathscr{A}$, let $\kappa:=2^{2^{\# \mathbb{C}+\# \mathscr{A}}}$, define

$$
\begin{gathered}
I=\{(\mathscr{C}, \gamma): \mathscr{C} \text { is a von Neumann algebra on a subset of } \kappa, \\
\text { and } \gamma: \mathscr{A} \rightarrow \mathscr{C} \text { is an ncpsu-map }\},
\end{gathered}
$$

and set $f_{i}:=\gamma$ for every $i \equiv(\mathscr{C}, \gamma) \in I$.
Let $f: \mathscr{A} \rightarrow \mathscr{B}$ be an ncpsu-map into a von Neumann algebra $\mathscr{B}$. The von Neumann algebra $\mathscr{B}^{\prime}$ generated by $f(\mathscr{A})$ has cardinality below $\kappa$ by П, and so by relabelling the elements of $\mathscr{B}^{\prime}$ we may find a von Neumann algebra $\mathscr{C}$ on a subset of $\kappa$ isomorphic to $\mathscr{B}^{\prime}$ via some nmiu-isomorphism $\Phi: \mathscr{B}^{\prime} \rightarrow \mathscr{C}$. Then the map $\gamma: \mathscr{A} \rightarrow \mathscr{C}$ given by $\gamma(a)=\Phi(f(a))$ for all $a \in \mathscr{A}$ is ncpsu, so that $i:=(\mathscr{C}, \gamma) \in I$, and, moreover, the assignment $c \mapsto \Phi^{-1}(c)$ gives a nmiu-map $h: \mathscr{C} \rightarrow \mathscr{B}$ with $h \circ f_{i} \equiv h \circ \gamma=f$. Hence $U: \mathbf{W}_{\text {MIU }}^{*} \rightarrow \mathbf{W}_{\text {CPSU }}^{*}$ obeys the solution set condition, and therefore has a left adjoint.
Remark A bit more can be said about the adjunction between the inclusion $U: \mathbf{W}_{\text {MIU }}^{*} \rightarrow \mathbf{W}_{\text {NCPSU }}^{*}$ and $\mathcal{F}$ : since $\mathbf{W}_{\text {MIU }}^{*}$ has the same objects as $\mathbf{W}_{\text {CPSU }}^{*}$, the category $\left(\mathbf{W}_{\text {CPSU }}^{*}\right)^{\text {op }}$ is, for very general reasons, equivalent to the Kleisli category of the (by the adjunction induced) monad $\mathcal{F} U$ on $\left(\mathbf{W}_{\text {MIU }}^{*}\right)^{\text {op }}$ in a certain natural way (see e.g. Theorem 9 of 71 ).

### 4.3.3 Free Exponential

We'll prove Kornell's result (from 44]) that the functor $\mathscr{B} \otimes(\cdot): \mathbf{W}_{\text {MIU }}^{*} \rightarrow \mathbf{W}_{\text {MIU }}^{*}$ has a left adjoint $(\cdot)^{* \mathscr{B}}$ for every von Neumann algebra $\mathscr{B}$. Kornell original proof is rather complex, and so is ours, unfortunately, but we've managed to peel off one layer of complexity from the original proof by way of Freyd's Adjoint Functor Theorem, reducing the problem to the facts that $\mathscr{B} \otimes(\cdot): \mathbf{W}_{\text {MIU }}^{*} \rightarrow$ $\mathbf{W}_{\text {MIU }}^{*}$ preserves products, equalisers, and satisfies the solution set condition.
Lemma A von Neumann algebra $\mathscr{A}$ can be faithfully represented on a Hilbert space which contains no more than $2^{\# \mathscr{A}}$ vectors.
Proof If $\mathscr{A}=\{0\}$, then the result is obvious, so let us assume that $\mathscr{A} \neq\{0\}$. Then $\mathscr{A}$ is infinite, and so $\aleph_{0} \cdot \# \mathscr{A}=\# \mathscr{A}$.

Let $\Omega$ be the set of np-functionals on $\mathscr{A}$. Recall that by the GNS-construction (see $48 \mathrm{VIIII} \mathscr{A}$ can be faithfully represented on the Hilbert space $\mathscr{H}_{\Omega} \equiv \bigoplus_{\omega \in \Omega} \mathscr{H}_{\omega}$. Since every element of $\mathscr{H}_{\omega}$ is the limit of a sequence of elements from $\mathscr{A}$, we have $\# \mathscr{H}_{\omega} \leqslant \aleph_{0}^{\# \mathscr{A}} \leqslant\left(2^{\aleph_{0}}\right)^{\# \mathscr{A}}=2^{\# \mathscr{A}}$, because $\aleph_{0} \cdot \# \mathscr{A}=\# \mathscr{A}$. Since every normal state is a map $\omega: \mathscr{A} \rightarrow \mathbb{C}$, we have $\# \Omega \leqslant \# \mathbb{C}^{\# \mathscr{A}}=\left(2^{\aleph_{0}}\right)^{\# \mathscr{A}}=2^{\# \mathscr{A}}$, because $\aleph_{0} \cdot \# \mathscr{A}=\# \mathscr{A}$. Hence $\# \mathscr{H}=\sum_{\omega \in \Omega} \# \mathscr{H}_{\omega} \leqslant 2^{\# \mathscr{A}} \cdot 2^{\# \mathscr{A}}=2^{\# \mathscr{A}}$.

IV Lemma Every nmiu-map $h: \mathscr{D} \rightarrow \mathscr{A} \otimes \mathscr{C}$, where $\mathscr{A}, \mathscr{C}$ and $\mathscr{D}$ are von Neumann algebras, factors as $\mathscr{D}-\tilde{h} \rightarrow \tilde{\mathscr{A}} \otimes \mathscr{C} \rightarrow \iota \mathrm{id} \rightarrow \mathscr{A} \otimes \mathscr{C}$, where $\tilde{\mathscr{A}}$ is a von Neumann algebra, and $\iota$ and $\tilde{h}$ are nmiu-maps, such that for all nmiu-maps $f, g: \mathscr{A} \rightarrow \mathscr{B}$ into some von Neumann algebra $\mathscr{B}$ with $(f \otimes \mathrm{id}) \circ h=(g \otimes \mathrm{id}) \circ h$ we have $f \circ \iota=g \circ \iota$.

Moreover, $\mathscr{A}$ can be generated by less than $\# \mathscr{D} \cdot 2^{\# \mathscr{C}}$ elements.
$\checkmark$ Proof Assume (without loss of generality) that $\mathscr{C}$ is a von Neumann algebra of operators on a Hilbert space $\mathscr{H}$ with no more than $2^{\# \mathscr{C}}$ vectors, see $I$.

For every vector $\xi \in \mathscr{H}$ let $r_{\xi}: \mathscr{A} \otimes \mathscr{C} \rightarrow \mathscr{A}$ be the unique np-map given by $r_{\xi}(a \otimes c)=\langle\xi, c \xi\rangle a$ for all $a \in \mathscr{A}$ and $c \in \mathscr{C}$ (see 112 XI and 1141 , and let $\tilde{\mathscr{A}}$ be the least Neumann subalgebra of $\mathscr{A}$ that contains $S:=\bigcup_{\xi \in \mathscr{H}} r_{\xi}(h(\mathscr{D}))$, and let $\iota: \tilde{\mathscr{A}} \rightarrow \mathscr{A}$ be the inclusion (so $\iota$ is nmiu). Note that $S$ (which generates $\tilde{\mathscr{A}}$ ) has no more than $\# \mathscr{D} \cdot \# \mathscr{H} \leqslant \# \mathscr{D} \cdot 2^{\# \mathscr{C}}$ elements.

Let $f, g: \mathscr{A} \rightarrow \mathscr{B}$ be nmiu-maps into a von Neumann algebra $\mathscr{B}$ such that $(f \otimes \mathrm{id}) \circ h=(g \otimes \mathrm{id}) \circ h$. We must show that $f \circ \iota=g \circ \iota$. By definition of $\tilde{\mathscr{A}}$ (and the fact that $f$ and $g$ are nmiu), it suffices to show that $f \circ r_{\xi} \circ h=$ $g \circ r_{\xi} \circ h$ for all $\xi \in \mathscr{H}$. Note that given such $\xi$, we have $f \circ r_{\xi}=r_{\xi}^{\prime} \circ(f \otimes \mathrm{id})$, where $r_{\xi}^{\prime}: \mathscr{B} \otimes \mathscr{C} \rightarrow \mathscr{B}$ is the np-map given by $r_{\xi}^{\prime}(b \otimes c)=\langle\xi, c \xi\rangle b$. Since similarly, $g \circ r_{\xi}=r_{\xi}^{\prime} \circ(g \otimes \mathrm{id})$, we get $f \circ r_{\xi} \circ h=r_{\xi}^{\prime} \circ(f \otimes \mathrm{id}) \circ h=r_{\xi}^{\prime} \circ(g \otimes \mathrm{id}) \circ h=$ $g \circ r_{\xi} \circ h$.

It remains only to be shown that $h(\mathscr{D}) \subseteq \tilde{\mathscr{A}} \otimes \mathscr{C}$, because we may then simply let $\tilde{h}$ be the restriction of $h$ to $\mathscr{A} \otimes \mathscr{C}$. It is enough to prove that $h(\mathscr{D}) \subseteq \tilde{\mathscr{A}} \otimes \mathscr{B}(\mathscr{H})$, because $\tilde{\mathscr{A}} \otimes \mathscr{C}=(\tilde{\mathscr{A}} \otimes \mathscr{B}(\mathscr{H})) \cap(\mathscr{A} \otimes \mathscr{C})($ see 121II) and we already know that $h(\mathscr{D}) \subseteq \mathscr{A} \otimes \mathscr{C}$. Let $\left(e_{k}\right)_{k}$ be orthonormal basis of $\mathscr{H}$. Since $1=\sum_{k}\left|e_{k}\right\rangle\left\langle e_{k}\right|$ in $\mathscr{B}(\mathscr{H})$, we have, for all $d \in \mathscr{D}$,

$$
\begin{aligned}
h(d) & =\left(\sum_{k} 1 \otimes\left|e_{k}\right\rangle\left\langle e_{k}\right|\right) h(d)\left(\sum_{\ell} 1 \otimes\left|e_{\ell}\right\rangle\left\langle e_{\ell}\right|\right) \\
& =\sum_{k} \sum_{\ell}\left(1 \otimes\left|e_{k}\right\rangle\left\langle e_{k}\right|\right) h(d)\left(1 \otimes\left|e_{\ell}\right\rangle\left\langle e_{\ell}\right|\right) .
\end{aligned}
$$

We are done if we can prove that, for all $\xi, \zeta \in \mathscr{H}$,

$$
\begin{equation*}
(1 \otimes|\xi\rangle\langle\xi|) h(d)(1 \otimes|\zeta\rangle\langle\zeta|) \in \tilde{\mathscr{A}} \otimes \mathscr{B}(\mathscr{H}) . \tag{4.4}
\end{equation*}
$$

By an easy computation, we see that, for all $e \in \mathscr{A} \otimes \mathscr{C}$ of the form $e \equiv a \otimes c$,

$$
(1 \otimes|\xi\rangle\langle\xi|) e(1 \otimes|\zeta\rangle\langle\zeta|)=\frac{1}{4} \sum_{k=0}^{3} i^{k} r_{i^{k} \xi+\zeta}(e) \otimes|\xi\rangle\langle\zeta| .
$$

It follows that the equation above holds for all $e \in \mathscr{A} \otimes \mathscr{C}$. Choosing $e=h(d)$ we see that 4.4 holds, because $r_{i^{k} \xi+\zeta}(h(d)) \in \tilde{\mathscr{A}}$.
Proposition Let $e: \mathscr{E} \rightarrow \mathscr{A}$ be an equaliser of nmiu-maps $f, g: \mathscr{A} \rightarrow \mathscr{B}$ between von Neumann algebras. Then $e \otimes \mathrm{id}: \mathscr{E} \otimes \mathscr{C} \rightarrow \mathscr{A} \otimes \mathscr{C}$ is an equaliser of $f \otimes \mathrm{id}$ and $g \otimes$ id for every von Neumann algebra $\mathscr{C}$.
Proof Let $h: \mathscr{D} \rightarrow \mathscr{A} \otimes \mathscr{C}$ be a nmiu-map with $(f \otimes i d) \circ h=(g \otimes \mathrm{id}) \circ h$. We must show that there is a unique nmiu-map $k: \mathscr{D} \rightarrow \mathscr{E} \otimes \mathscr{C}$ such that $h=(e \otimes \mathrm{id}) \circ k$. Note that since the equaliser map $e$ is injective, $e \otimes \mathrm{id}: \mathscr{E} \otimes \mathscr{C} \rightarrow \mathscr{A} \otimes \mathscr{C}$ is injective (by 115 V ) and thus uniqueness of $k$ is clear. Concerning existence, by $\mathbb{\boxed { V }}, h$ factors as $\mathscr{D} — \tilde{h} \rightarrow \tilde{A} \otimes \mathscr{C}-\iota \otimes \mathrm{id} \rightarrow \mathscr{A} \otimes \mathscr{C}$ where $\tilde{h}$ and $\iota$ are nmiu-maps, and moreover, we have $f \circ \iota=g \circ \iota$. Since $e$ is an equaliser of $f$ and $g$, there is a unique nmiu-map $\tilde{\iota}: \tilde{\mathscr{A}} \rightarrow \mathscr{E}$ with $e \circ \tilde{\iota}=\iota$. Now, define $k:=(\tilde{\iota} \otimes \mathrm{id}) \circ \tilde{h}: \mathscr{D} \rightarrow \mathscr{E} \otimes \mathscr{C}$. Then $(e \otimes \mathrm{id}) \circ k=((e \circ \tilde{\iota}) \otimes \mathrm{id}) \circ \tilde{h}=(\iota \otimes \mathrm{id}) \circ \tilde{h}=h$.

Theorem (Kornell) The functor $(-) \otimes \mathscr{A}: \mathbf{W}_{\text {MIU }}^{*} \rightarrow \mathbf{W}_{\text {MIU }}^{*}$ has a left adjoint for every von Neumann algebra $\mathscr{A}$.
Proof The category $\mathbf{W}_{\mathrm{MIU}}^{*}$ is (small-)complete, and $(-) \otimes \mathscr{A}: \mathbf{W}_{\text {MIU }}^{*} \rightarrow \mathbf{W}_{\text {MIU }}^{*}$ IX preserves (small-)products and equalisers. Thus, by Freyd's (General) Adjoint Functor Theorem [47, Thm. V.6.2], it suffices to check the following Solution Set Condition (where we've used that $\mathbf{W}_{\text {MIU }}^{*}$ is locally small).

- For each $\mathscr{B} \in \mathbf{W}_{\text {MIU }}^{*}$, there is a small subset $\mathcal{S}$ of objects in $\mathbf{W}_{\text {MIU }}^{*}$ such that every arrow $h: \mathscr{B} \rightarrow \mathscr{C} \otimes \mathscr{A}$ can be written as a composite $h=\left(t \otimes \mathrm{id}_{\mathscr{A}}\right) \circ f$ for some $\mathscr{D} \in \mathcal{S}, f: \mathscr{B} \rightarrow \mathscr{D} \otimes \mathscr{A}$, and $t: \mathscr{D} \rightarrow \mathscr{C}$.

Let $\mathscr{B}$ be an arbitrary von Neumann algebra. We claim that the following set $\mathcal{S}$ satisfies the required condition:

$$
\mathcal{S}=\{\mathscr{D} \mid \mathscr{D} \text { is a von Neumann algebra on } \kappa\}, \text { where } \kappa=2^{2^{\# C \cdot \# \mathscr{B} \cdot 2^{\# \cdot \mathscr{A}}} .}
$$

Indeed, suppose that $h: \mathscr{B} \rightarrow \mathscr{C} \otimes \mathscr{A}$ is given. By $\mathbb{I V} h$ factors as

$$
\mathscr{B} \longrightarrow \tilde{C} \otimes \mathscr{A} \longrightarrow \iota \otimes \mathrm{id} \longrightarrow \mathscr{C} \otimes \mathscr{A},
$$

where $\tilde{\mathscr{C}}$ is a von Neumann algebra generated by no more than $\# \mathscr{B} \cdot 2^{\# \mathscr{A}}$ elements. It follows that $\tilde{\mathscr{C}}$ has no more than $\kappa$ elements (by 1241 ). Thus we may assume without loss of generality that $\tilde{\mathscr{C}}$ is a subset of $\kappa$, that is, $\tilde{\mathscr{C}} \in \mathcal{S}$.

X Remark It should be noted that analogues of the first and second adjunctions can be found in the setting of $C^{*}$-algebras, which raises the question as to whether a variation on the free exponential exist for $C^{*}$-algebras, that is, is there a tensor $\otimes$ on $\mathbf{C}_{\text {MIU }}^{*}$ such that $(-) \otimes \mathscr{A}: \mathbf{C}_{\text {MIU }}^{*} \rightarrow \mathbf{C}_{\text {MIU }}^{*}$ has a left adjoint?

Such a tensor does not exist if we require that on commutative $C^{*}$-algebras it is given by the product of the spectra (as is the case for the projective and injective tensors of $C^{*}$-algebras) in the sense that there is a natural isomorphism $\Phi_{X, Y}: C(X) \otimes C(Y) \longrightarrow \simeq C(X \times Y)$ between the obvious functors of type $\mathbf{C H} \times \mathbf{C H} \rightarrow\left(\mathbf{C}_{\mathrm{MIU}}^{*}\right)^{\text {op }}$. Indeed, if $(-) \otimes \mathscr{A}: \mathbf{C}_{\mathrm{MIU}}^{*} \rightarrow \mathbf{C}_{\mathrm{MIU}}^{*}$ had a left adjoint and so would preserve all limits for all $C^{*}$-algebras $\mathscr{A}$, then the functor $(-) \times X: \mathbf{C H} \rightarrow \mathbf{C H}$ would preserve all colimits for every compact Hausdorff space $X$, which it does not, because if it did the square $\beta \mathbb{N} \times \beta \mathbb{N}$ of the StoneCech compactification $\beta \mathbb{N}$ of the natural numbers (being the $\mathbb{N}$-fold coproduct of the one-point space) would be homeomorphic to the Stone-Čech compactification $\beta(\mathbb{N} \times \mathbb{N})$ of $\mathbb{N} \times \mathbb{N}$, which it is not (by Theorem 1 of 19$]$ ).

Whence $\mathbf{C}_{\text {CPSU }}^{*}$ does not form a model of the quantum lambda calculus in the same way $\mathbf{W}_{\text {CPSU }}^{*}$ that does.

### 4.4 Duplicators and Monoids

When asked for an interpretation of the type $!A$ as a von Neumann algebra

$$
\begin{equation*}
\llbracket!A \rrbracket=\bigoplus_{n} \llbracket A \rrbracket^{\otimes n} \tag{4.5}
\end{equation*}
$$

definitely seems like a suitable answer given the cue that $!A$ should represent as many instances of $A$ as needed, which makes the interpretation we actually use in our model of the quantum lambda calculus (namely $\left.\llbracket!A \rrbracket=\ell^{\infty}(\operatorname{nsp}(\llbracket A \rrbracket))\right)$ rather suspect. To address such concerns we'll show that any von Neumann algebra that carries a $\otimes$-monoid structure (in $\mathbf{W}_{\text {MIU }}^{*}$ as $\llbracket!A \rrbracket$ should) must be nmiu-isomorphic to $\ell^{\infty}(X)$ for some set $X$ (see 127 III$)$ ruling out the interpretation (4.5) for all but the most trivial cases. We'll show in fact that $\ell^{\infty}(\operatorname{nsp}(\mathscr{A}))$ is the free $\otimes$-monoid over $\mathscr{A}$ in $\mathbf{W}_{\text {MIU }}^{*}$ (see 132 IV ) exonerating it in our minds from all doubts.

### 4.4.1 Duplicators

Definition A von Neumann algebra $\mathscr{A}$ is duplicable if there is a duplicator
on $\mathscr{A}$, that is, an npsu-map $\delta: \mathscr{A} \otimes \mathscr{A} \rightarrow \mathscr{A}$ with a unit $u \in[0,1]_{\mathscr{A}}$ satisfying

$$
\delta(a \otimes u)=a=\delta(u \otimes a) \quad \text { for all } a \in \mathscr{A} .
$$

(Note that we require of $\delta$ neither associativity nor commutativity.)
Remark The unit $u$ can be identified with a positive subunital map $\tilde{u}: \mathbb{C} \rightarrow \mathscr{A}$ via $\tilde{u}(\lambda)=\lambda u$. The definition is motivated by the fact that the interpretation of $!A$ must carry a commutative monoid structure in $\mathbf{W}_{\text {MIU }}^{*}$. The condition is weaker, requiring the maps to be only positive subunital, and dropping associativity and commutativity. Nevertheless this is sufficient to prove the following.
Theorem A von Neumann algebra $\mathscr{A}$ is duplicable if and only if $\mathscr{A}$ is nmiuisomorphic to $\ell^{\infty}(X)$ for some set $X$. In that case, the duplicator $(\delta, u)$ is unique, given by $\delta(a \otimes b)=a \cdot b$ and $u=1$.
Thus, to interpret duplicable types, we can really only use von Neumann algebras of the form $\ell^{\infty}(X)$. It also follows that a von Neumann algebra is duplicable precisely when it is a (commutative) monoid in $\mathbf{W}_{\text {MIU }}^{*}$, or in the symmetric monoidal category $\mathbf{W}_{\text {CPSU }}^{*}$ of von Neumann algebras and normal completely positive subunital (CPsU) maps.
To prove 111 we proceed as follows. First we prove in 128 VIII every duplicable von Neumann algebra $\mathscr{A}$ is commutative (and that the duplicator is given by multiplication). This reduces the problem to a measure theoretic one, because $\mathscr{A} \cong \bigoplus_{i} L^{\infty}\left(X_{i}\right)$ for some finite complete measure spaces $X_{i}$ (by 70 III$)$. Since each of the $L^{\infty}\left(X_{i}\right)$ s will be duplicable (see 128XIII) we may assume without loss of generality that $\mathscr{A} \cong L^{\infty}(X)$ for some finite complete measure space $X$. Since $X$ splits into a discrete and a continuous part (see 129 VI ), and the result is obviously true for discrete spaces, we only need to show that $L^{\infty}(C)=\{0\}$ for any continuous complete finite measure space $C$ for which $L^{\infty}(C)$ is duplicable. In fact, we'll show that $\mu(C)=0$ for such $C$ (see 129 VIII).
Lemma Let $\delta$ be a duplicator with unit $u$ on a von Neumann algebra $\mathscr{A}$. Then $u=1$ and $\delta(1 \otimes 1)=1$.
Proof Since $1=\delta(u \otimes 1) \leqslant \delta(1 \otimes 1) \leqslant 1$, we have $\delta(1 \otimes 1)=1$, and so $\delta\left(u^{\perp} \otimes 1\right)=0$. But, because $u^{\perp}=\delta\left(u^{\perp} \otimes u\right) \leqslant \delta\left(u^{\perp} \otimes 1\right)=0$, we have $u^{\perp}=0$, and thus $u=1$. Hence $1=\delta(1 \otimes u)=\delta(1 \otimes 1)$.

128 To prove that a duplicable von Neumann algebra is commutative we'll need the following two results from the theory on $C^{*}$-algebras.

II Theorem (Tomiyama) Given a $C^{*}$-subalgebra $\mathscr{B}$ of a $C^{*}$-algebra $\mathscr{A}$, any linear map $f: \mathscr{A} \rightarrow \mathscr{B}$ with $f(f(a))=f(a)$ and $\|f(a)\| \leqslant\|a\|$ for all $a \in \mathscr{A}$ must be positive and obey $b f(a) b^{\prime}=f\left(b a b^{\prime}\right)$ for all $a \in \mathscr{A}$ and $b, b^{\prime} \in \mathscr{B}$.
III Proof See 68 or 10.5.86 of 43 .
IV Theorem (Russo-Dye) We have $\|f\| \leqslant 1$ for any pu-map $f: \mathscr{A} \rightarrow \mathscr{B}$ between $C^{*}$-algebras $\mathscr{A}$ and $\mathscr{B}$.
$\checkmark$ Proof See Corollary 1 of 61 .
VI Lemma Let $\mathscr{A}$ be a $C^{*}$-algebra, and let $f: \mathscr{A} \oplus \mathscr{A} \rightarrow \mathscr{A}$ be a pu-map with $f(a, a)=a$ for all $a \in \mathscr{A}$. Then $p:=f(1,0)$ is central, and

$$
f(a, b)=a p+b p^{\perp}
$$

for all $a, b \in \mathscr{A}$.
VII Proof (Based on Lemma 8.3 of 26.)
Note that $(c, d) \mapsto(f(c, d), f(c, d))$ gives a pu-map $f^{\prime}$ from $\mathscr{A} \oplus \mathscr{A}$ onto its $C^{*}$-subalgebra $\{(a, a): a \in \mathscr{A}\}$ with $f^{\prime}\left(f^{\prime}(c, d)\right)=f^{\prime}(c, d)$ for all $c, d \in \mathscr{A}$. Since $\left\|f^{\prime}\right\| \leqslant 1$ by Russo-Dye's theorem (IV), Tomiyama's theorem (II) implies that for all $a, b, c, d \in \mathscr{A}$,

$$
(a, a) f^{\prime}(c, d)(b, b)=f^{\prime}(a c b, a d b), \quad \text { and so } \quad a f(c, d) b=f(a c b, a d b)
$$

In particular, $a p \equiv a f(1,0)=f(a, 0)=f(1,0) a \equiv p a$ for all $a \in \mathscr{A}$, and so $p$ is central. Similarly, $f(0, b)=b p^{\perp}$ for all $b \in \mathscr{A}$. Then $f(a, b)=f(a, 0)+f(0, b)=$ $a p+b p^{\perp}$ for all $a, b \in \mathscr{A}$.
VIII Lemma Let $\delta: \mathscr{A} \otimes \mathscr{A} \rightarrow \mathscr{A}$ be a duplicator on a von Neumann algebra $\mathscr{A}$. Then $\mathscr{A}$ is commutative and $\delta(a \otimes b)=a \cdot b$ for all $a, b \in \mathscr{A}$.
IX Proof To prove $\mathscr{A}$ is commutative we must show that all $a \in \mathscr{A}$ are central, but, of course, it suffices to show that all $p \in[0,1]_{\mathscr{A}}$ are central (by the usual reasoning). Similarly, we only need to prove that $\delta(a \otimes p)=a \cdot p$ for all $a \in \mathscr{A}$ and $p \in[0,1]_{\mathscr{A}}$. Given such $p \in[0,1]_{\mathscr{A}}$ define $f: \mathscr{A} \oplus \mathscr{A} \rightarrow \mathscr{A}$ by $f(a, b)=$ $\delta\left(a \otimes p+b \otimes p^{\perp}\right)$ for all $a, b \in \mathscr{A}$. Then $f$ is positive, unital, $f(1,0)=p$, and $f(a, a)=a$ for all $a \in \mathscr{A}$. Thus by VI $p$ is central, and $f(a, b)=a p+b p^{\perp}$ for all $a, b \in \mathscr{A}$. Then $a \cdot p=f(a, 0)=\bar{\delta}(a \otimes p)$.
X Remark The special case of VIII in which $\delta$ is completely positive can be found in the literature, see for example Theorem 6 of 45 (in which $\mathscr{A}$ is also finite dimensional).

Corollary Let $\mathscr{A}$ be a von Neumann algebra. Then $\mathscr{A}$ is duplicable iff there is an np-map $\delta: \mathscr{A} \otimes \mathscr{A} \rightarrow \mathscr{A}$ with $\delta(a \otimes b)=a \cdot b$ for all $a, b \in \mathscr{A}$, (and in that case $\mathscr{A}$ is commutative.)
Remark Thus for a non-commutative von Neumann algebra $\mathscr{A}$ multiplication $(a, b) \mapsto a b: \mathscr{A} \times \mathscr{A} \rightarrow \mathscr{A}$ is not a normal bilinear map in the sense of 112 II.
Corollary When the direct sum $\mathscr{A} \oplus \mathscr{B}$ of von Neumann algebras $\mathscr{A}$ and $\mathscr{B}$ is XIII duplicable, $\mathscr{A}$ and $\mathscr{B}$ are duplicable
Proof Let $\delta:(\mathscr{A} \oplus \mathscr{B}) \otimes(\mathscr{A} \oplus \mathscr{B}) \longrightarrow \mathscr{A} \oplus \mathscr{B}$ be a duplicator on $\mathscr{A} \oplus$ XIV $\mathscr{B}$. By VIII $\mathscr{A} \oplus \mathscr{B}$ is commutative and $\delta\left(\left(a_{1}, b_{1}\right) \otimes\left(a_{2}, b_{2}\right)\right)=\left(a_{1} a_{2}, b_{1} b_{2}\right)$ for all $a_{1}, a_{2} \in \mathscr{A}$ and $b_{1}, b_{2} \in \mathscr{B}$. Let $\kappa_{1}: \mathscr{A} \rightarrow \mathscr{A} \oplus \mathscr{B}$ be the nmiumap given by $\kappa_{1}(a)=(a, 0)$ for all $a \in \mathscr{A}$. Let $\delta_{\mathscr{A}}$ be the composition of $\mathscr{A} \otimes \mathscr{A}-\kappa_{1} \otimes \kappa>(\mathscr{A} \oplus \mathscr{B}) \otimes(\mathscr{A} \oplus \mathscr{B}) \longrightarrow \delta \longrightarrow \mathscr{A} \oplus \mathscr{B} — \pi_{1} \rightarrow \mathscr{A}$. Then $\delta_{\mathscr{A}}$ is normal, positive, and $\delta_{\mathscr{A}}\left(a_{1} \otimes a_{2}\right)=\pi_{1}\left(\delta\left(\left(a_{1}, 0\right) \otimes\left(a_{2}, 0\right)\right)\right)=\pi_{1}\left(a_{1} a_{2}, 0\right)=$ $a_{1} a_{2}$ for all $a_{1}, a_{2} \in \mathscr{A}$. Thus, by XI $\mathscr{A}$ is duplicable.

We will now work towards the proof that if $C$ is a continuous complete finite measure space, then $L^{\infty}(C)$ cannot be duplicable unless $\mu(C)=0$, see Х, Let us first fix some more terminology from measure theory (see 51 and 16 ).
Definition Let $X$ be a finite complete measure space.

1. A measurable subset $A$ of $X$ is atomic if $0<\mu(A)$ and $\mu\left(A^{\prime}\right)=\mu(A)$ for all $A^{\prime} \in \Sigma_{X}$ with $A^{\prime} \subseteq A$ and $\mu\left(A^{\prime}\right)>0$.
2. $X$ is discrete if $X$ is covered by atomic measurable subsets. (This coincides with being "purely atomic" from 211K of 16.)
3. $X$ is continuous (or "atomless") if $X$ contains no atomic subsets.

The following lemma, which will be very useful, is a variation on Zorn's Lemma (that does not require the axiom of choice).

Lemma Let $\mathcal{S}$ be a collection of measurable subsets of a finite complete measure space $X$ such that for every ascending countable sequence $A_{1} \subseteq A_{2} \subseteq \cdots$ in $\mathcal{S}$ there is $A \in \mathcal{S}$ with $A_{1} \subseteq A_{2} \subseteq \cdots \subseteq A$.

Then each element $A \in \mathcal{S}$ is contained in some $B \in \mathcal{S}$ that is maximal in $\mathcal{S}$ in the sense that $\mu\left(B^{\prime}\right)=\mu(B)$ for all $B^{\prime} \in \mathcal{S}$ with $B \subseteq B^{\prime}$.

128, 129..
$\checkmark$ Proof The trick is to consider for every $C \in \mathcal{S}$ the quantity

$$
\beta_{C}=\sup \{\mu(D): C \subseteq D \text { and } D \in \mathcal{S}\}
$$

Note that $\mu(C) \leqslant \beta_{C} \leqslant \mu(X)$ for all $C \in \mathcal{S}$, and $\beta_{C_{2}} \leqslant \beta_{C_{1}}$ for all $C_{1}, C_{2} \in \mathcal{S}$ with $C_{1} \subseteq C_{2}$. To prove this lemma, it suffices to find $B \in \mathcal{S}$ with $A \subseteq B$ and $\mu(B)=\beta_{B}$.

Define $B_{1}:=A$. Pick $B_{2} \in \mathcal{S}$ such that $B_{1} \subseteq B_{2}$ and $\beta_{B_{1}}-\mu\left(B_{2}\right) \leqslant 1 / 2$. Pick $B_{3} \in \mathcal{S}$ such that $B_{2} \subseteq B_{3}$ and $\beta_{B_{2}}-\mu\left(B_{3}\right) \leqslant 1 / 3$. Proceeding in this fashion, we get a sequence $B \equiv B_{1} \subseteq B_{2} \subseteq \cdots$ in $\mathcal{S}$ with $\beta_{B_{n}}-\mu\left(B_{n+1}\right) \leqslant 1 / n$ for all $n$. By assumption there is a $B \in \mathcal{S}$ with $B_{1} \subseteq B_{2} \subseteq \cdots \subseteq B$. Note that

$$
\mu\left(B_{1}\right) \leqslant \mu\left(B_{2}\right) \leqslant \cdots \leqslant \mu(B) \leqslant \beta_{B} \leqslant \cdots \leqslant \beta_{B_{2}} \leqslant \beta_{B_{1}}
$$

Since for every $n \in \mathbb{N}$ we have both $\mu\left(B_{n+1}\right) \leqslant \mu(B) \leqslant \beta_{B} \leqslant \beta_{B_{n}}$ and $\beta_{B_{n}}-$ $\mu\left(B_{n+1}\right) \leqslant 1 / n$, we get $\beta_{B}-\mu(B) \leqslant 1 / n$, and so $\beta_{B}=\mu(B)$.

VI Lemma Each finite complete measure space $X$ contains a discrete measurable subset $D$ such that $X \backslash D$ is continuous.
VII Proof Since clearly the countable union of discrete measurable subsets of $X$ is again discrete, there is by $\mathbb{I V}$ a discrete measurable subset $D$ of $X$ which is maximal in the sense that $\mu\left(D^{\prime}\right)=\mu(D)$ for every discrete measurable subset $D^{\prime}$ of $X$ with $D \subseteq D^{\prime}$. To show that $X \backslash D$ is continuous, we must prove that $X \backslash D$ contains no atomic measurable subsets. If $A \subseteq X \backslash D$ is an atomic measurable subset of $X$, then $D \cup A$ is a discrete measurable subset of $X$ which contains $D$, and $\mu(D \cup A)=\mu(D) \cup \mu(A)>\mu(D)$. This contradicts the maximality of $D$. Thus $X \backslash D$ is continuous.

VIII Lemma Given a continuous finite complete measure space $X$, and $r \in[0, \mu(X)]$, there is a measurable subset $A$ of $X$ with $\mu(A)=r$.
IX Proof Let us quickly get rid of the case that $\mu(X)=0$. Indeed, then $r=0$, and so $A=\varnothing$ will do. For the remainder, assume that $\mu(X)>0$.

For starters, we show that for every $\varepsilon>0$ and $B \in \Sigma_{X}$ with $\mu(B)>0$ there is $A \in \Sigma_{X}$ with $A \subseteq B$ and $0<\mu(A)<\varepsilon$. Define $A_{1}:=B$. Since $\mu(B)>0$, and $A_{1}$ is not atomic (because $X$ is continuous) there is $A \in \Sigma_{X}$ with $A \subseteq A_{1}$ and $\mu(A) \neq \mu\left(A_{1}\right)$. Since $\mu(A)+\mu\left(A_{1} \backslash A\right)=\mu\left(A_{1}\right)$, either $0<\mu(A) \leqslant \frac{1}{2} \mu\left(A_{1}\right)$ or $0<\mu(X \backslash A) \leqslant \frac{1}{2} \mu\left(A_{1}\right)$. In any case, there is $A_{2} \subseteq A_{1}$ with $A_{2} \in \Sigma_{X}$ and $0<$ $\mu\left(A_{2}\right) \leqslant \frac{1}{2} \mu\left(A_{1}\right)$. Similarly, since $A_{2}$ is not atomic (because $X$ is continuous), there is $A_{3} \subseteq A_{2}$ with $A_{3} \in \Sigma_{X}$ and $0<\mu\left(A_{3}\right) \leqslant \frac{1}{2} \mu\left(A_{2}\right)$. Proceeding in a
similar fashion, we obtain a sequence $B \equiv A_{1} \supseteq A_{2} \supseteq \cdots$ of measurable subsets of $X$ with $0<\mu\left(A_{n}\right) \leqslant 2^{-n} \mu(X)$. Then, for every $\varepsilon>0$ there is $n \in \mathbb{N}$ such that $0<\mu\left(A_{n}\right) \leqslant \varepsilon$ and $A_{n} \subseteq B$.

Now, let us prove that there is $A \in \Sigma_{X}$ with $\mu(A)=r$. By $\mathbb{V}$ there is a measurable subset $A$ of $X$ with $\mu(A) \leqslant r$ and which is maximal in the sense that $\mu\left(A^{\prime}\right)=\mu(A)$ for all $A^{\prime} \in \Sigma_{X}$ with $\mu(A) \leqslant r$ and $A \subseteq A^{\prime}$. In fact, we claim that $\mu(A)=r$. Indeed, suppose that $\varepsilon:=r-\mu(A)>0$ towards a contradiction. By the previous discussion, there is $C \in \Sigma_{X}$ with $C \subseteq X \backslash A$ such that $\mu(C) \leqslant \varepsilon$. Then $A \cup C$ is measurable, and $\mu(A \cup C)=\mu(A)+\mu(C) \leqslant \mu(A)+\varepsilon \leqslant r$, which contradicts the maximality of $A$.

Lemma Let $X$ be a continuous finite complete measure space for which $L^{\infty}(X)$ is duplicable. Then $\mu(X)=0$.
Proof Suppose that $\mu(X)>0$ towards a contradiction. Let $\delta$ be a duplicator on $L^{\infty}(X)$. By 128 VIII $\delta(\mathfrak{f} \otimes \mathfrak{g})=\mathfrak{f} \cdot \mathfrak{g}$ for all $\mathfrak{f}, \mathfrak{g} \in L^{\infty}(X)$.

Let $\omega: L^{\infty}(X) \rightarrow \mathbb{C}$ be given by $\omega\left(f^{\circ}\right)=\frac{1}{\mu(X)} \int f d \mu$ for all $f \in \mathcal{L}^{\infty}(X)$. Then $\omega$ is normal, positive, unital and faithful (cf. 51 IX ). We'll use the product functional $\omega \otimes \omega: L^{\infty}(X) \otimes L^{\infty}(X) \rightarrow \mathbb{C}$, (which is also faithful, by 118 IV ) to tease out a contradiction, but first we need a second ingredient.

Since $X$ is continuous, we may partition $X$ into two measurable subsets of equal measure with the aid of VIII, that is, there are measurable subsets $X_{1}$ and $X_{2}$ of $X$ with $X=X_{1} \cup X_{2}, X_{1} \cap X_{2}=\varnothing$, and $\mu\left(X_{1}\right)=\mu\left(X_{2}\right)=$ $\frac{1}{2} \mu(X)$. Similarly, $X_{1}$ can be split into two measurable subsets, $X_{11}$ and $X_{12}$, of equal measure, and so on. In this way, we obtain for every word $w$ over the alphabet $\{1,2\}$ - in symbols, $w \in\{1,2\}^{*}$ - a measurable subset $X_{w}$ of $X$ such that $X_{w}=X_{w 1} \cup X_{w 2}, X_{w 1} \cap X_{w 2}=\varnothing$, and $\mu\left(X_{w 1}\right)=\mu\left(X_{w 2}\right)=\frac{1}{2} \mu\left(X_{w}\right)$. It follows that $\mu\left(X_{w}\right)=\frac{1}{2 \# w} \mu(X)$, where $\# w$ is the length of the word $w$.

Now, $p_{w}:=\mathbf{1}_{X_{w}}^{\circ}$ is a projection in $L^{\infty}(X)$, and $\omega\left(p_{w}\right)=2^{-\# w}$ for every $w \in$ $\{1,2\}^{*}$. Moreover, $p_{w}=p_{w 1}+p_{w 2}$, and so

$$
\begin{aligned}
p_{w} \otimes p_{w} & =p_{w 1} \otimes p_{w 1}+p_{w 1} \otimes p_{w 2}+p_{w 2} \otimes p_{w 1}+p_{w 2} \otimes p_{w 2} \\
& \geqslant p_{w 1} \otimes p_{w 1}+p_{w 2} \otimes p_{w 2} .
\end{aligned}
$$

Thus, if we define $q_{N}:=\sum_{w \in\{1,2\}^{N}} p_{w} \otimes p_{w}$ for every natural number $N$, where $\{1,2\}^{N}$ is the set of words over $\{1,2\}$ of length $N$, then we get a descending sequence $q_{1} \geqslant q_{2} \geqslant q_{3} \geqslant \cdots$ of projections in $L^{\infty}(X) \otimes L^{\infty}(X)$. Let $q$ be the infimum of $q_{1} \geqslant q_{2} \geqslant \cdots$ in the set of self-adjoint elements of $L^{\infty}(X) \otimes L^{\infty}(X)$. Do we have $q=0$ ?

On the one hand, we claim that $\delta(q)=1$, and so $q \neq 0$. Indeed, $\delta\left(p_{w} \otimes p_{w}\right)=$
$p_{w} \cdot p_{w}=p_{w}$ for all $w \in\{1,2\}^{N}$. Thus $\delta\left(q_{N}\right)=\sum_{w \in\{1,2\}^{N}} \delta\left(p_{w} \otimes p_{w}\right)=$ $\sum_{w \in\{1,2\}^{N}} p_{w}=1$ for all $N \in \mathbb{N}$. Hence $\delta(q)=\bigwedge_{n} \delta\left(q_{N}\right)=1$, because $\delta$ is normal. On the other hand, we claim that $(\omega \otimes \omega)(q)=0$, and so $q=0$ since $\omega \otimes \omega$ is faithful and $q \geqslant 0$. Indeed, $(\omega \otimes \omega)\left(q_{N}\right)=\sum_{w \in\{1,2\}^{N}} \omega\left(p_{w}\right) \cdot \omega\left(p_{w}\right)=$ $\sum_{w \in\{1,2\}^{N}} 2^{-N} \cdot 2^{-N}=2^{-N}$ for all $N \in \mathbb{N}$, and so $(\omega \otimes \omega)(q)=\bigwedge_{N}(\omega \otimes \omega)\left(q_{N}\right)=$ $\bigwedge_{N} 2^{-N}=0$. Thus, $q=0$ and $q \neq 0$, which is impossible.

130 This takes care of the continuous case. To deal with the discrete case we first need some simple observations.
II Lemma Let $A$ be an atomic measure space. Then $L^{\infty}(A) \cong \mathbb{C}$.
III Proof Let $f \in \mathcal{L}^{\infty}(A)$ be given. It suffices to show that there is $z \in \mathbb{C}$ such that $f(x)=z$ for almost all $x \in A$. Moreover, we only need to consider the case that $f$ takes its values in $\mathbb{R}$ (because we may split $f$ in its real and imaginary parts, and in turn split these in positive and negative parts).

Let $S$ be some measurable subset of $A$. Note that either $\mu(S)=0$ or $\mu(A \backslash S)$. Indeed, if not $\mu(S)=0$, then $\mu(S)>0$, and so $\mu(S)=\mu(A)$ (by atomicity of $A$ ), which entails that $\mu(A \backslash S)=0$.

In particular, for every real number $t \in \mathbb{R}$ one of the sets

$$
\{x \in A: t \leqslant f(x)\} \quad\{x \in A: f(x)<t\}
$$

must be negligible. Whence either $t \leqslant f^{\circ}$ or $f^{\circ} \leqslant t$. It follows that the two closed sets $L:=\left\{t \in \mathbb{R}: t \leqslant f^{\circ}\right\}$ and $U:=\left\{t \in \mathbb{R}: f^{\circ} \leqslant t\right\}$ cover $\mathbb{R}$. Since clearly $-\|f\| \in L$ and $\|f\| \in U$, the sets $L$ and $U$ can't be disjoint, because they would partition $\mathbb{R}$ into two clopen non-empty sets. For an element $t \in L \cap U$ in the intersection we have $t \leqslant f^{\circ} \leqslant t$, and so $t=f^{\circ}$. Hence $L^{\infty}(X) \cong \mathbb{C}$.
IV Exercise Let $X$ be a measure space with $\mu(X)<\infty$. Show that $L^{\infty}(X) \cong$ $\bigoplus_{A \in \mathcal{A}} L^{\infty}(A)$ for every countable partition $\mathcal{A}$ of $X$ consisting of measurable subsets.

V Corollary For every discrete measure space $X$ with $\mu(X)<\infty$ there is a set $Y$ with $L^{\infty}(X) \cong \ell^{\infty}(Y)$.

131 We are now ready to prove the main result of this section.
II Proof of $\mathbf{1 2 7}$ III We have already seen that $\ell^{\infty}(X)$ can be equipped with a commutative monoid structure in $\mathbf{W}_{\text {MIU }}^{*}$ for any set $X$, and is thus duplicable. Conversely, let $\delta: \mathscr{A} \otimes \mathscr{A} \rightarrow \mathscr{A}$ be a duplicator with unit $u$ on a von Neumann algebra $\mathscr{A}$. By 127 VI we know that $u=1$, and by 128 VIII we know that $\mathscr{A}$
is commutative and $\delta(a \otimes b)=a \cdot b$ for all $a, b \in \mathscr{A}$. Thus, the only thing that remains to be shown is that $\mathscr{A}$ is miu-isomorphic to $\ell^{\infty}(Y)$ for some set $Y$. By $70 \mathrm{III} \mathscr{A} \cong \bigoplus_{i} L^{\infty}\left(X_{i}\right)$ for some finite complete measure spaces $X_{i}$. So to prove that $\mathscr{A} \cong \ell^{\infty}(Y)$ for some set $Y$ it suffices to find a set $Y_{i}$ with $L^{\infty}\left(X_{i}\right) \cong \ell^{\infty}\left(Y_{i}\right)$ for each $i$, because then $\mathscr{A} \cong \bigoplus_{i \in I} \ell^{\infty}\left(Y_{i}\right) \cong \ell^{\infty}\left(\bigcup_{i \in I} Y_{i}\right)$.

Let $i \in I$ be given. Since $\mathscr{A} \cong L^{\infty}\left(X_{i}\right) \oplus \bigoplus_{j \neq i} L^{\infty}\left(X_{j}\right)$ is duplicable, $L^{\infty}\left(X_{i}\right)$ is duplicable by 128 XIII By 129 VI there is a measurable subset $D$ of $X_{i}$ such that $D$ is discrete, and $C:=X \backslash D$ is continuous. We have $L^{\infty}\left(X_{i}\right) \cong L^{\infty}(D) \oplus L^{\infty}(C)$ by 130 IV , and so $L^{\infty}(D)$ and $L^{\infty}(C)$ are duplicable (again by 128 XIII$)$. By $129 \mathrm{X}, L^{\infty}(C)$ can only be duplicable if $\mu(C)=0$, and so $L^{\infty}(C) \cong\{0\}$. On the other hand, since $D$ is discrete, we have $L^{\infty}(D) \cong$ $\ell^{\infty}(Y)$ for some set $Y$ (by 130 V . All in all, we have $L^{\infty}\left(X_{i}\right) \cong \ell^{\infty}(Y)$.

### 4.4.2 Monoids

We further justify our choice, $\llbracket!A \rrbracket=\ell^{\infty}(\operatorname{nsp}(\llbracket A \rrbracket))$, by proving that $\ell^{\infty}(\operatorname{nsp}(\mathscr{A}))$ is the free (commutative) monoid on $\mathscr{A}$ in $\mathbf{W}_{\text {MIU }}^{*}$. As a corollary, we also obtain that $\ell^{\infty}\left(\mathbf{W}_{\text {CPSU }}^{*}(\mathscr{A}, \mathbb{C})\right)$ is the free (commutative) monoid on $\mathscr{A}$ in $\mathbf{W}_{\text {CPSU }}^{*}$.
Let us first recall some terminology. Given a symmetric monoidal category (SMC) $\mathbf{C}$, a monoid in $\mathbf{C}$ is an object $A$ from $\mathbf{C}$ endowed with a multiplication map $m: A \otimes A \rightarrow A$ and a unit map $u: I \rightarrow A$ satisfying the associativity and the unit law, i.e. making the following diagrams commute.


Here $\alpha, \lambda, \rho$ respectively denote the associativity isomorphism, and the left and the right unit isomorphism. A monoid $A$ is commutative if $m \circ \gamma=m$, where $\gamma: A \otimes A \rightarrow A \otimes A$ is the symmetry isomorphism. A monoid morphism between monoids $A_{1}$ and $A_{2}$ is an arrow $f: A_{1} \rightarrow A_{2}$ that satisfies $m_{A_{2}} \circ(f \otimes f)=f \circ m_{A_{1}}$ and $u_{A_{2}}=f \circ u_{A_{1}}$. We denote the category of monoids and monoid morphisms in $\mathbf{C}$ by $\operatorname{Mon}(\mathbf{C})$. The full subcategory of commutative monoids is denoted by cMon(C). Recall that $\mathbf{W}_{\text {MUU }}^{*}$ and $\mathbf{W}_{\text {CPSU }}^{*}$ are symmetric monoidal categories with $\mathbb{C}$ as tensor unit (see 119 V ), and so we may speak about monoids in $\mathbf{W}_{\text {MIU }}^{*}$ and $\mathbf{W}_{\text {CPSU }}^{*}$.

III Exercise Let $\mathscr{A}$ be a von Neumann algebra.

1. Show that any monoid structure on $\mathscr{A}$ in $\mathbf{W}_{\text {CPSU }}^{*}$ is a duplicator on $\mathscr{A}$.
2. Deduce from this and 127 III that there is a monoid structure on $\mathscr{A}$ in $\mathbf{W}_{\text {MIU }}^{*}$ or in $\mathbf{W}_{\text {CPSU }}^{*}$ iff $\mathscr{A}$ is duplicable iff $\mathscr{A} \cong \ell^{\infty}(X)$ for some set $X$; and that, in that case the multiplication $m: \mathscr{A} \otimes \mathscr{A} \rightarrow \mathscr{A}$ of the monoid is commutative and uniquely being fixed by $m(a \otimes b)=a \cdot b$.
3. Show that the monoid morphisms in $\mathbf{W}_{\text {MIU }}^{*}$ and in $\mathbf{W}_{\text {CPSU }}^{*}$ are precisely the nmiu-maps.
4. Conclude that $\mathrm{cMon}\left(\mathbf{W}_{\mathrm{MIU}}^{*}\right)=\operatorname{Mon}\left(\mathbf{W}_{\mathrm{MIU}}^{*}\right)=\operatorname{cMon}\left(\mathbf{W}_{\mathrm{CPSU}}^{*}\right)=\operatorname{Mon}\left(\mathbf{W}_{\mathrm{CPSU}}^{*}\right)$.
5. Show that $\operatorname{Mon}\left(\mathbf{W}_{\text {MIU }}^{*}\right) \cong \mathbf{d W}_{\text {MIU }}^{*} \simeq$ Set $^{\text {op }}$, where $\mathbf{d W}_{\text {MIU }}^{*}$ denotes the full subcategory of $\mathbf{W}_{\text {MIU }}^{*}$ consisting of duplicable von Neumann algebras.
(Hint: $\ell^{\infty}$ : Set $\rightarrow\left(\mathbf{W}_{\text {MIU }}^{*}\right)^{\text {op }}$ is full and faithful by 122 VI .)

IV Theorem Let $\mathscr{A}$ be a von Neumann algebra, and let $\eta: \mathscr{A} \rightarrow \ell^{\infty}(\operatorname{nsp}(\mathscr{A}))$ be the nmiu-map given by $\eta(a)(\varphi)=\varphi(a)$. Then $\ell^{\infty}(\operatorname{nsp}(\mathscr{A}))$ is the free (commutative) monoid on $\mathscr{A}$ in $\mathbf{W}_{\text {MIU }}^{*}$ via $\eta$.
$\vee$ Proof Let $\mathscr{B}$ be a monoid on $\mathbf{W}_{\text {MIU }}^{*}$, and let $f: \mathscr{A} \rightarrow \mathscr{B}$ be a nmiu-map We must show that there is a unique monoid morphism $g: \ell^{\infty}(\mathrm{nsp}(\mathscr{A})) \rightarrow \mathscr{B}$ such that $g \circ \eta=f$. Since the monoid structure on $\mathscr{B}$ is a duplicator on $\mathscr{B}$ we may assume, by 127 III that $\mathscr{B}=\ell^{\infty}(Y)$ for some set $Y$. Since nsp: $\left(\mathbf{W}_{\text {MIU }}^{*}\right)^{\text {op }} \rightarrow$ Set is left adjoint to $\ell^{\infty}:$ Set $\rightarrow\left(\mathbf{W}_{\text {MIU }}^{*}\right)^{\text {op }}$ with unit $\eta$ (see 122 III), there is a unique $\operatorname{map} h: Y \rightarrow \operatorname{nsp}(\mathscr{A})$ with $\ell^{\infty}(h) \circ \eta=f$. Since $\ell^{\infty}$ is full and faithful by 122 VI , the only thing that remains to be shown is that $\ell^{\infty}(h)$ is a monoid morphism. Indeed it is, since the monoid multiplication on $\ell^{\infty}(\operatorname{nsp}(\mathscr{A}))$ and $\ell^{\infty}(Y)$ is given by ordinary multiplication, which is preserved by $\ell^{\infty}(h)$ being a miu-map.

VI Corollary Let $\mathscr{A}$ be a von Neumann algebra. Then $\ell^{\infty}\left(\mathbf{W}_{\text {CPSU }}^{*}(\mathscr{A}, \mathbb{C})\right)$ is the free (commutative) monoid on $\mathscr{A}$ in $\mathbf{W}_{\text {cPSU }}^{*}$.
VII Proof By $\mathbb{I V} \ell^{\infty} \circ$ nsp is a left adjoint to the forgetful functor $\operatorname{Mon}\left(\mathbf{W}_{\text {MIU }}^{*}\right) \rightarrow$ $\mathbf{W}_{\text {MIU }}^{*}$. Note that by III the forgetful functor $\operatorname{Mon}\left(\mathbf{W}_{\text {CPSU }}^{*}\right) \rightarrow \mathbf{W}_{\text {CPSU }}^{*}$ factors through $\mathbf{W}_{\text {MIU }}^{*}$ as:

where $\mathcal{F}$ is from 124 III . Thus the free monoid on $\mathscr{A}$ in $\mathbf{W}_{\text {CPSU }}^{*}$ is given by:

$$
\left(\ell^{\infty} \circ \operatorname{nsp} \circ \mathcal{F}\right)(\mathscr{A})=\ell^{\infty}\left(\mathbf{W}_{\mathrm{MIU}}^{*}(\mathcal{F} \mathscr{A}, \mathbb{C})\right) \cong \ell^{\infty}\left(\mathbf{W}_{\mathrm{CPSU}}^{*}(\mathscr{A}, \mathbb{C})\right)
$$

as was claimed.

Conclusion Here ends this thesis, but not the entire story. There's much more to133 be said about self-dual Hilbert $\mathscr{A}$-modules, about dilations and their relation to purity, and about the abstract theory of corners, filters, and $\diamond$-positivity. You'll see all this, and more, in the sequel, "Dagger and dilations in the category of von Neumann algebras" 74], brought to you by my twin brother.
(Paragraphs numbered 134 and up can be found in 74 .)

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## Z

$Z(\mathscr{A})$, centre of $\mathscr{A}, 65 \mathrm{II}$

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[^0]:    *Though other methods of modelling infinite dimensional quantum computing have been proposed as well e.g. using non-standard analysis 20], pre-sheaves 48], the geometry of inter-

[^1]:    ${ }^{\dagger}$ An excerpt from Eugene P. Wigner's writings, see page 130 of 75.

[^2]:    *Although clearly related to the Wedderburn-Artin theorem, see e.g. 51, this description of finite-dimensional $C^{*}$-algebras does not seem to be an immediate consequence of it.

[^3]:    $\dagger$ In case you've never seen the argument: the limit $b:=\lim _{N}\|a\|^{N}$ exists, because $\|a\| \geqslant$ $\|a\|^{2} \geqslant \cdots \geqslant 0$, and is zero because $\|a\| b=\lim _{N}\|a\|^{N+1}=b$ and $\|a\|<1$.

[^4]:    *In other words, the collection of normal positive functionals should be faithful (see 21 II . Interestingly, it's already enough for the normal positive functionals to be centre separating, but since we have encountered no example of a von Neumann algebra where it wasn't already clear that the normal positive functionals are faithful instead of just centre separating we did not use this weaker albeit more complex condition.

[^5]:    ${ }^{\dagger}$ Note that every element of $\mathcal{L}^{\infty}(X)$ being bounded is integrable by 122 P of 16.

[^6]:    $\ddagger$ Indeed, one may take $X_{1}$ to be a measure space consisting of a single non-negligible point * (so $X_{1}=\{*\}$ and $\mu\left(X_{1}\right) \neq 0$ ), while letting $X_{2}$ be a measure space on an uncountable set formed by taking for the measurable subsets of $X_{2}$ the countable subsets and their complements, by making the countable subsets negligible, and by giving all cocountable subsets the same non-zero measure. Then all measurable functions on $X_{1}$ and on $X_{2}$ are constant almost everywhere, (because in $X_{1}$ and $X_{2}$ there are no two non-negligible disjoint measurable subsets, ) so that $L^{\infty}\left(X_{1}\right) \cong \mathbb{C} \cong L^{\infty}\left(X_{2}\right)$.

